Introduction to Proofs*

Many times in economics we will need to prove theorems to show that our "theories" can be supported by specific assumptions. While economics is an observational science, we use mathematics to simplify reality. Thus, we make some basic assumptions about how humans behave and then derive theories based on those assumptions. These notes are designed to give you an overview of various methods that can be used to prove theorems. Hopefully these notes are a review.

1 The Conditional Statement

A conditional statement is simply an "if, then" statement. Typically we will write, if p, then q. There are many other details and plenty of terminology that I will skip. You can consult Epp, Susanna. *Discrete Mathematics with Applications*. 2^{nd} ed. PWS Publishing Company, Boston, MA, 1995. That is the source of the bulk of these notes.

Related to our conditional statement, which we state as *if p, then q*, (we will write this as $p \rightarrow q$, or "p implies q") we have three other statements:

Converse: $q \rightarrow p$

Inverse: $\sim p \rightarrow \sim q$ (the symbol $\sim$ is defined as "not" in this context; later we will define it as "indifferent to")

Contrapositive: $\sim q \rightarrow \sim p$

Of these 3 statements, only the contrapositive is *logically equivalent* to the conditional statement. For two items to be logically equivalent, they need to have the same *truth table*. A truth table is a table that lists the truth-values of a proposition that result from all the possible combinations of the truth-values of its components. Table 1 shows the truth table for the conditional statement, $p \rightarrow q$. Note that the conditional statement is only false when p is true and q is false. If both are true then the statement is true. However, the statement says nothing about whether or not q will occur if p does NOT occur, so the statement is still true in those instances when p does not occur regardless of whether or not q occurs. Thus, if the conditional statement is: If you show up to class, then you will receive an A, the only time we can say that the statement is false is when you show up to class and do not receive an A. If you fail to show up to class then all bets are off – you may receive an A, you may not,

*Based on Epp (1995).

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

Table 1: Truth table for conditional statement

p	q	$\sim p$	$\sim q$	$p \to q$	$q \to p$	$\sim p \to \sim q$	$\sim q \to \sim p$
T	T	F	F	T	T	T	T
T	F	F	T	F	T	T	F
F	T	T	F	T	F	F	T
F	F	T	T	T	T	T	T

Table 2: Truth table for conditional statement, converse, inverse, and contrapositive

but the conditional statement does not tell us anything about what happens in those instances.

Table 2 shows a truth table for the statement, the converse, the inverse, and the contrapositive. Note that we will need the truth values for $\sim p$ and $\sim q$ as well. We can see that the truth values for the contrapositive are identical to those of the statement, so that the two are logically equivalent. This will be useful information momentarily.

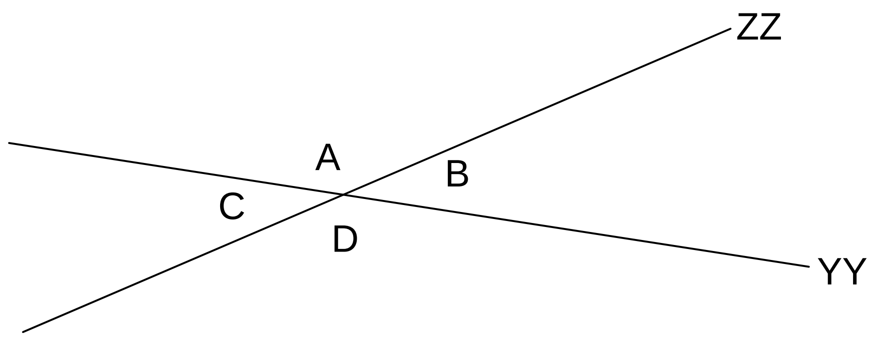
2 Three Methods of Proof

We will discuss three specific methods of proving theorems that may prove useful to you. Those three methods are direct proof, proof by contradiction, and proof by contraposition. Proofs can be written out in paragraph form, with correct grammatical structure (and should be for journal articles). However, that sometimes obscures the thought process, and I will write proofs in a t-table.

2.1 Direct Proof

Direct proof is straightforward – if we have a conditional statement and want to prove that q is true, we assume that p is true and deduce, using "what we know", that q is true. "What we know" typically consists of previously proven theorems and results, definitions, and other common knowledge.

Suppose we have the following diagram:



And you are told that segments ZZ and YY are straight lines. The following statement is then made: If ZZ and YY are straight lines, then $\angle B$ and $\angle C$ are congruent. You are then asked to prove this:

Proof. If ZZ and YY are straight lines, then angle B and angle C are congruent

Statement	Reason
1. $\angle A + \angle B = 180°$; $\angle A + \angle C = 180°$	1. Definition of a straight line
2. $\angle B = 180° - \angle A$ $\angle C = 180° - \angle A$	2. Subtraction
3. $\angle B = \angle C$	3. Substitution

$\therefore \angle B$ and $\angle C$ are congruent by the definition of congruent. ■

2.2 Proof by Contradiction

A second method of proof is proof by contradiction. Here are the steps.

1. Suppose the statement to be proved is false.
2. Show that this supposition leads naturally to a contradiction.
3. Conclude that the statement to be proved is true.

Why does this work? Essentially, when you contradict the negation of the statement (which is what you are assuming is true in step 1), you are proving it false. If the negation is false, then the statement is true. We will use proof by contradiction to show that there is no greatest integer.

Proof. *There is no greatest integer.*

By contradiction. Assume there is a greatest integer, M.

Statement	Reason
1. $M > m$, $\forall$ integers m	1. Definition of greatest
2. Let $N = M + 1$	2. Defining a new number by addition
3. N is an integer	3. The sum of integers is an integer
4. $M + 1 > M$	4. Definition of greater
5. $N > M$	5. Substitution

Step 5 contradicts the original assumption, that M is the greatest integer, because we have found an integer that is greater than M. $\therefore$ There is no greatest integer. $\blacksquare$

Although there are no set rules as to when to use proof by contradiction, here are two guidelines as to when you might find proof by contradiction useful.

1. When you want to show that there is no object with a certain property.
2. When you want to show that a certain object does not have a certain property.

2.3 Proof by Contraposition

A third method of proof is proof by contraposition. Here are the steps:

1. State the contrapositive of the statement.
2. Prove the contrapositive by direct proof.

Why does this work? Recall that the contrapositive and the conditional statement are logically equivalent. Thus, if the contrapositive is true, then the conditional statement is also true. We will show that for all integers n, if n^2 is even, then n is even.

Proof. $\forall$ integers n, if n^2 is even, then n is even

By contraposition. $\forall$ integers n, if n is odd, then n^2 is odd

Statement	Reason
1. k is an integer	1. assumption
2. $2k+1$ is an odd integer	2. definition of an odd integer
3. $n = 2k+1$	3. we have now defined what is given to us
4. $n^2 = (2k+1)(2k+1)$	4. Definition of square
5. $n^2 = 4k^2 + 4k + 1$	5. Multiplication (FOIL)
6. $n^2 = 2\left(2k^2 + 2k\right) + 1$	6. Distribution
7. $x = 2k^2 + 2k$	7. Defining a new number
8. x is an integer	8. Addition and multiplication of integers results in an integer
9. $n^2 = 2x + 1$	9. Substitution

$\therefore n^2$ is odd by definition of odd. Since we have proven the contrapositive true, we know the statement is also true, so that if n^2 is even, then n is even $\forall$ integers n $\blacksquare$

Note that the proof in the book skips some steps because they “know” that the product of two odd integers is odd, so they can invoke what they “know” to conclude that n^2 is odd after step 4. However, we did not know that.

Again, there are no set rules for using proof by contraposition. One benefit of using proof by contraposition is that we know exactly what we need to prove, so there is some guidance. With proof by contradiction, we do not necessarily know what contradiction we are looking for. However, the benefit of proof by contradiction is that once we have shown any contradiction then we are finished.

Also, if a theorem is proved using proof by contraposition, then it can also be proved using proof by contradiction. However, the converse of that statement is NOT true (but you already knew that).

2.4 NOT Methods of Proof

Here is a list of things that people might want to use to prove a theorem. However, they are NOT methods of proof.

1. Proof by converse – we know that a statement and its converse are NOT logically equivalent, so proving the converse is true does not necessarily prove the statement is true.

2. Proof by inverse – replace "converse" with "inverse" in the statement above.

   Note: I have been thinking about this, and although I have no evidence, I believe that people like to use "proof by converse" and "proof by inverse" as methods of proving the truth of a conditional statement is because if p and q are both true then the conditional, converse, inverse, and contrapositive are all true (just look at the truth table). However, one needs to consider ALL the possible combinations of the truth of p and q, and the truth table shows us that the conditional and the inverse (and converse) are not logically equivalent.

3. Proof by example – one million examples may suggest that something is true; however, all it takes is one counterexample to prove that something is NOT true. And if you give one million examples of something and state that it is true when it really is not true, someone with time on his or her hands will find that one counterexample.

4. Proof by appealing to someone else's proof for a case that is related to their theory, but not exactly as it is specified – economists will typically appeal to someone else's theory (which will typically be done in a continuous space for mathematical tractability) and say that it suggests certain results in their example (which will typically be a discrete space if they are writing about something that actually happened). If you have a specific example about which you are writing, take a little time to make sure simple counterexamples to the theory do not exist.

5. As a final note, realize that just because a theorem is published that does not necessarily mean that it is correct. People have made their livings off of correcting proofs that are published.