Chapter 6 Notes

These notes correspond to chapter 6 of Mas-Colell, Whinston, and Green, on choice under uncertainty.

1 Introduction

Up to now we have discussed the consumer's problem as a maximization problem with known bundles of goods – 10 units of x_1 and 12 units of x_2, 15 units of x_1 and 9 units of x_2, etc. However, many decisions in life have uncertain outcomes – you can think about choosing a career path, choosing a spouse, or something as seemingly simple as choosing whether or not to eat spinach (given the recent E. Coli situation). What we want to do now is to consider a setting where alternatives with uncertain outcomes are describable by means of objectively known probabilities defined on an abstract set of choices. The focus of this chapter will be to discuss the consumer's *expected utility function*, as opposed to the consumer's utility function.

2 Expected Utility Theory

A decision-maker chooses between a number of risky alternatives. Each risk alternative has a number of possible outcomes, but the decision-maker is uncertain which will occur when he makes his choice. We will denote the set of all outcomes as C. This could be the consumption set or a set of monetary payoffs – it is just a general set of outcomes right now. We will assume that the set of all possible outcomes is finite and index them by $n = 1, ..., N$. Also, assume the probabilities on the outcomes are *objectively* known, meaning that there is no uncertainty about the probabilities.

Definition 1 *A simple lottery L is a list $L = (p_1, ..., p_N)$ with $p_n \geq 0$ for all n and $\sum_n p_n = 1$, where p_n is interpreted as the probability of the outcome occurring.*

In the case of a simple lottery, there is no choice – something either happens or does not happen. Think about the weather – the simple lottery L could be that there is a 50% chance of rain and a 50% chance of no rain. The decision-maker has no control over this. What the decision-maker would have control over is whether or not to bring an umbrella, and then he would need to calculate the expected utility of bringing an umbrella with the expected utility of not bringing an umbrella based on the probability of rain and then make a decision.

We can represent the simple lottery geometrically in the $(N-1)$ dimensional simplex, $\Delta = \left\{p \in \mathbb{R}^N : p_1 + ... + p_N = 1\right\}$. With a 2 outcome lottery, we can represent this lottery in the 1-dimensional simplex. Consider a line segment with length equal to 1 as in the top of Figure 1. At one end of the segment we have outcome A and at the other end outcome B. Any point along the segment will represent a lottery over A and B, with the distance from point A to a specific point equal to the probability of outcome B, and the distance from point B to a specific point equal to the probability of outcome A.

With a 3 outcome lottery, we can represent any lottery in the 2-dimensional simplex. This will be an equilateral triangle with altitude equal to 1 as in the bottom of Figure 1. With this representation, any lottery over the three outcomes can be constructed by a point inside (or along the edge) the triangle. Each vertex represents one of the outcomes occurring with probability 1. Any point along the edge of the triangle is simply a lottery over the outcomes forming the endpoints of that edge, thus assigning 0 probability to the outcome at the vertex opposite the side. To find the actual lottery for any given point, take the point in the triangle, and draw a perpendicular line to each side of the triangle. The length of each of those respective perpendicular lines represents the probability of the outcome represented by the vertex opposite the side of the triangle.

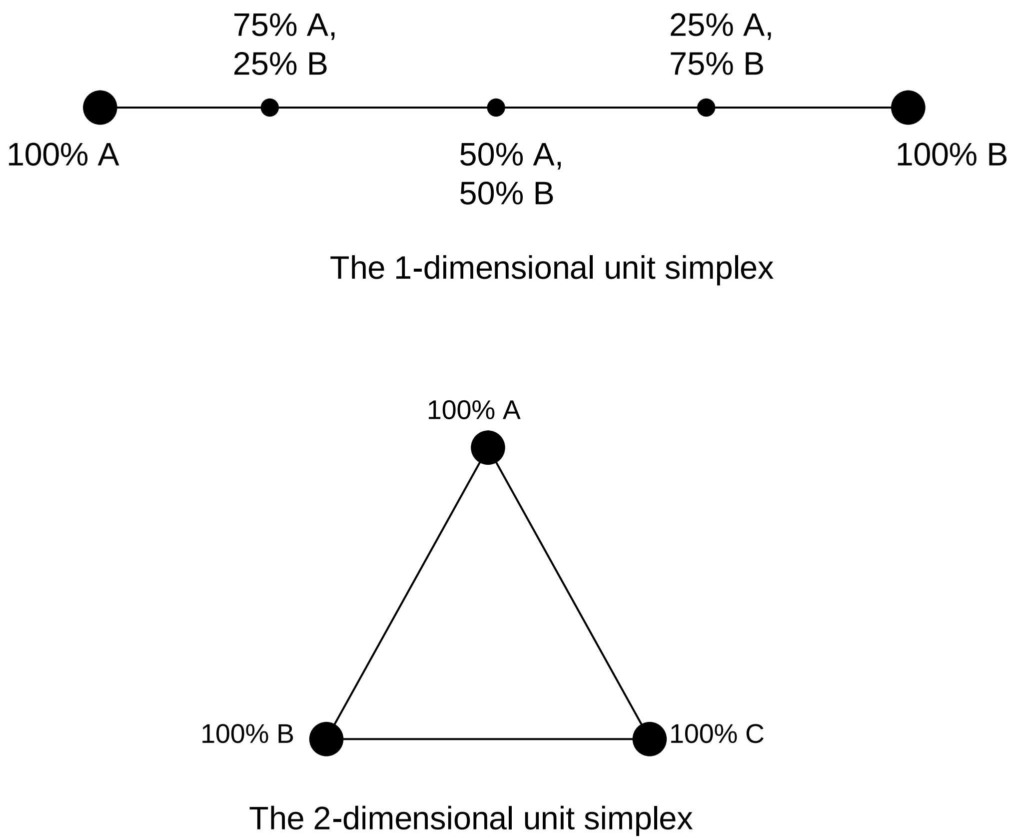

Figure 1: The 1 and 2 dimensional unit simplexes.

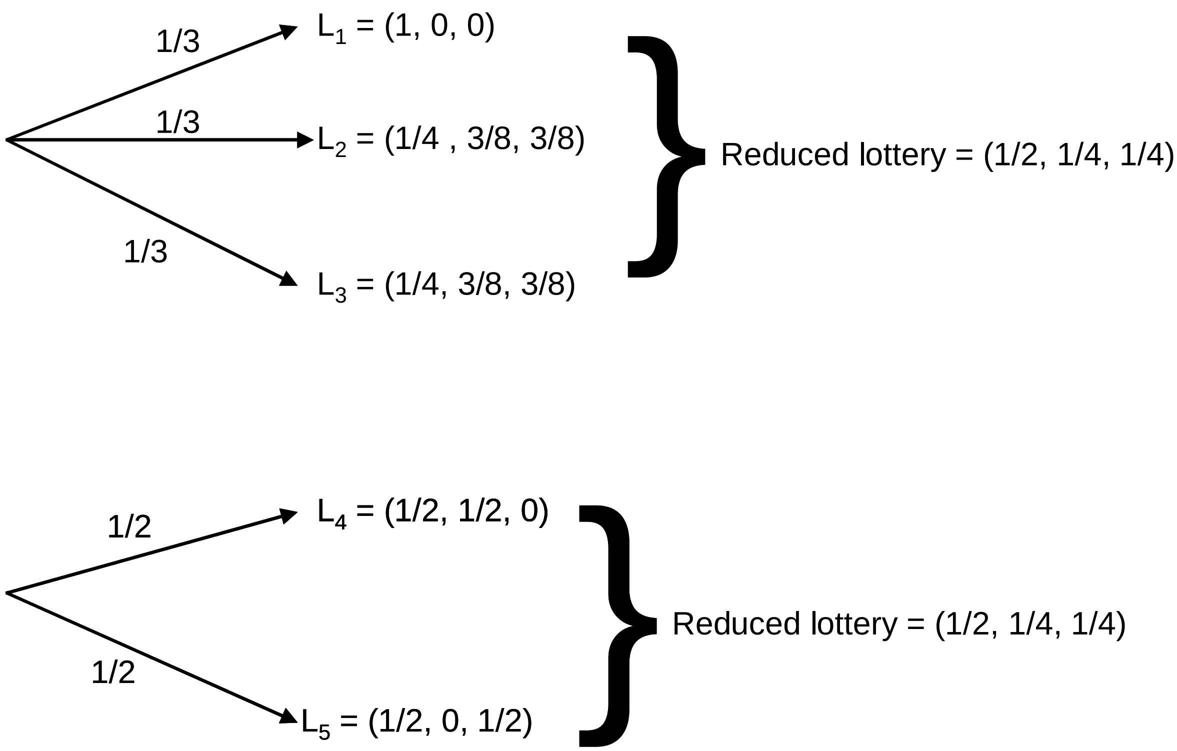

Figure 2: Two compound lotteries with the same reduced lottery.

Definition 2 *Given* K *simple lotteries* $L_k = \left(p_1^k, ..., p_N^k\right)$, $k = 1, ..., K$ *and probabilities* $\alpha_k \geq 0$ *with* $\sum_k \alpha_k = 1$, *the compound lottery* $(L_1, ..., L_K; \alpha_1, ..., \alpha_K)$ *is the risky alternative that yields the simple lottery* L_k *with probability* α_k *for* $k = 1, ..., K$.

We want to transform the compound lottery into a simple lottery. We can do this by using the *reduced lottery*, which is the simple lottery $L = (p_1, ..., p_N)$ that generates the same ultimate distribution of outcomes. For our compound lottery, the reduced lottery has each $p_n = \alpha_1 p_n^1 + ... + \alpha_k p_n^k$. As an example, consider the simple lotteries $L_1 = \left(\frac{1}{2}, \frac{1}{4}, \frac{1}{4}\right)$ and $L_2 = \left(\frac{3}{4}, \frac{1}{8}, \frac{1}{8}\right)$ that occur with $\alpha_1 = \frac{1}{4}$ and $\alpha_2 = \frac{3}{4}$. Our compound lottery is $\left(L_1, L_2; \frac{1}{4}, \frac{3}{4}\right)$. The reduced lottery would be $L = \left(\frac{11}{16}, \frac{5}{32}, \frac{5}{32}\right)$.

2.1 Preferences over lotteries

We have now modeled risky alternatives, and we need to discuss the decision-maker's preferences over them. In doing so, we assume that only the reduced lottery over final outcomes is of relevance to the decision-maker. The reduced lottery could derive from two or more distinct compound lotteries, but only the reduced lottery will be of concern to the decision-maker. Thus, if two distinct compound lotteries have the same reduced lottery, then the decision-maker will view them equivalently. Figure 2 shows two sets of distinct compound lotteries that yield the same reduced lottery.

Take the set of alternatives, denoted $\mathcal{L}$, and define it as the set of all simple lotteries over the set of outcomes C. We assume that the decision maker has a rational and continuous preference relation $\succsim$ over $\mathcal{L}$. Rationality is slightly stronger now than initially, because now we are assuming decision-makers can compare lotteries with only minor differences in outcomes. Continuity tells us that if $L \succ L'$, then adding some small probability of a third lottery, L'', to the first lottery will not change the nature of the ordering of the two lotteries. If $C = \{\$1000, \$10, "death"\}$, most decision-makers would likely rank the lottery $L = (1, 0, 0)$ as preferred to $L' = (0, 1, 0)$, which is preferred to $L'' = (0, 0, 1)$. However, continuity says that for some positive probability α, if you prefer L to L' then you must prefer $(1-\alpha)L + \alpha L''$ to L'. Is this unreasonable? Not really – it's like saying that you have the choice of \$10 magically (and in an unharmful manner) appearing in your pocket, or that you have to drive downtown to pick up \$1000. There is a very small positive probability of death associated with driving downtown to pick up \$1000, but most decision-makers would likely take that chance to get the \$1000. The continuity assumption also implies the existence of a utility function representing $\succsim$, a function $U : \mathcal{L} \to \mathbb{R}$ such that $L \succsim L'$ if and only if $U(L) \geq U(L')$.

Another assumption that we will make is the independence assumption. The independence assumption means that if one lottery is at least as good as another lottery, and we mix a third lottery with each of the first two in the same proportion, then the preference ordering of the resulting mixtures does not depend on the third lottery. Formally,

Definition 3 *The preference relation $\succsim$ on the space of simple lotteries $\mathcal{L}$ satisfies the independence axiom if for all $L, L', L'' \in \mathcal{L}$ and $\alpha \in (0,1)$ we have*

$$L \succsim L' \text{ if and only if } \alpha L + (1-\alpha) L'' \succsim \alpha L' + (1-\alpha) L''$$

Note how this is different than consumer theory – if we have two bundles of goods, bundle A $\left(x_1^A, x_2^A\right) = (7,5)$ and bundle B $\left(x_1^B, x_2^B\right) = (9,1)$, it may be the case that $A \succsim B$. Now, if we mix bundle C $\left(x_1^C, x_2^C\right) = (0,10)$ in with A and B at a 50% rate, we get $\frac{1}{2}A + \frac{1}{2}C = (3.5, 7.5)$ and $\frac{1}{2}B + \frac{1}{2}C = (4.5, 5.5)$. It could be the case that the mixture of B and C is actually now at least as good as the mixture of A and C. If we chose a different C bundle, C', we may find that the mixture of A and C' is better than the mixture of B and C'. With lotteries this is not the case. Consider $L \succsim L'$ and a third lottery L''. Now, consider the mixtures of $\frac{1}{2}L + \frac{1}{2}L''$ and $\frac{1}{2}L' + \frac{1}{2}L''$. One-half of the time in the first mixture the decision-maker will face lottery L, and in the second lottery will face lottery L', and it is known that $L \succsim L'$ so in that case the decision-maker should choose the first mixture. The other half of the time the decision-maker will face lottery L'' in the first mixture, and lottery L'' in the second mixture, and will be indifferent. Thus, the decision-maker should find the first mixture at least as good as the second mixture if $L \succsim L'$. The difference between mixtures of bundles and mixtures of lotteries is that we consume the mixed bundle itself, whereas with mixtures of lotteries we do not consume the mixture of the lotteries, but one lottery or the other one.

Definition 4 *The utility function $U : \mathcal{L} \to \mathbb{R}$ has an expected utility form if there is an assignment of numbers $(u_1, ..., u_N)$ to the N outcomes such that for every simple lottery $L = (p_1, ..., p_N) \in \mathcal{L}$ we have*

$$U(L) = u_1 p_1 + ... + u_N p_N$$

A utility function $U : \mathcal{L} \to \mathbb{R}$ with the expected utility form is called a von Neumann-Morgenstern (vN-m) expected utility function.

Consider the lottery L^n with outcome n that occurs with probability 1. This is a degenerate lottery of the form $L = (0, ..., 1, ..., 0)$, where the 1 is in the position of the n^{th} outcome. Define $U(L^n) = u_n$. Essentially, expected utility is the expected value of the utilities u_N of the N outcomes. The expression $U(L) = \sum_n u_n p_n$ is a general form for functions linear in the probabilities.

Proposition 5 *A utility function $U : \mathcal{L} \to \mathbb{R}$ has an expected utility form if and only if it is linear, that is, if and only if it satisfies the property that*

$$U\left(\sum_{k=1}^{K} \alpha_k L_k\right) = \sum_{k=1}^{K} \alpha_k U(L_k)$$

for any k lotteries $L_k \in \mathcal{L}$, $k = 1, ..., K$ and probabilities $(\alpha_1, ..., \alpha_k) \geq 0$ and $\sum_k \alpha_k = 1$.

Note what this proposition states. It states that the utility function has an expected utility form if and only if it is linear, which means that the utility of the weighted average of the lotteries is equal to the weighted average of the utility of the lotteries.

Proof. If $U\left(\sum_{k=1}^{K} \alpha_k L_k\right) = \sum_{k=1}^{K} \alpha_k U(L_k)$, then U has an expected utility form, $\sum_n u_n p_n$.

Statement	Reason
1. $U\left(\sum_{k=1}^{K}\alpha_k L_k\right)=\sum_{k=1}^{K}\alpha_k U\left(L_k\right)$	1. Given
2. $L=\left(p_1,...,p_N\right)$	2. Definition of lottery
3. L^n is the degenerate lottery for outcome n	3. Definition of L^n
4. $L=\sum_n p_n L^n$	4. Definition of reduced lottery and degenerate lottery
5. $U\left(L\right)=U\left(\sum_n p_n L^n\right)$	5. Substitution
6. $U\left(\sum_n p_n L^n\right)=\sum_n p_n U\left(L^n\right)$	6. From 1
7. $\sum_n p_n U\left(L^n\right)=\sum_n p_n u_n$	7. Definition of u_n
8. $U\left(L\right)=\sum_n p_n u_n$	8. From 5 and 7
9. U has the expected utility form	9. Definition of expected utility form

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Now, starting with the expected utility form and working towards linearity.

Proof. If U has an expected utility form, then $U\left(\sum_{k=1}^{K}\alpha_k L_k\right)=\sum_{k=1}^{K}\alpha_k U\left(L_k\right)$.

Statement	Reason
1. $\left(L_1,...,L_K;\alpha_1,...,\alpha_K\right)$ is a compound lottery with $L_k=\left(p_1^k,...,p_N^k\right)$	1. Definition of compound lottery
2. $L'=\sum_k \alpha_k L_k$ is a reduced lottery of the compound lottery in 1	2. Definition of reduced lottery
3. $U\left(\sum_k \alpha_k L_k\right)=\sum_n u_n\left(\sum_k \alpha_k p_n^k\right)$	3. U has the expected utility form
4. $\sum_n u_n\left(\sum_k \alpha_k p_n^k\right)=\sum_k \alpha_k\left(\sum_n u_n p_n^k\right)$	4. Summation is commutative
5. $\sum_k \alpha_k\left(\sum_n u_n p_n^k\right)=\sum_k \alpha_k U\left(L_k\right)$	5. U has the expected utility form
6.$U\left(\sum_k \alpha_k L_k\right)=\sum_k \alpha_k U\left(L_k\right)$	6. From 3 and 5

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Since U must be linear in order to have the expected utility form, then U is preserved only under positive linear transformations. This proposition is stated without proof.

Proposition 6 *Suppose that $U:\mathcal{L}\rightarrow\mathbb{R}$ is a vN-M utility function for the preference relation $\succsim$ on $\mathcal{L}$. Then $\widetilde{U}:\mathcal{L}\rightarrow\mathbb{R}$ is another vNM utility function for $\succsim$ if and only if there are scalars $\beta>0$ and γ such that $\widetilde{U}\left(L\right)=\beta U\left(L\right)+\gamma$ for every $L\in\mathcal{L}$.*

The key is that the vNM utility function, which is a utility function on the space of lotteries, is only preserved under positive linear transformations. We will soon be discussing the concept of a Bernoulli utility function, defined over wealth, which is slightly different than the vNM utility function.

Expected Utility Theorem

If the decision-maker's preferences over lotteries satisfy continuity and independence, then his preferences are representable by a utility function with the expected utility form. A formal statement, again without proof, follows.

Proposition 7 *(Expected Utility Theorem) Suppose that the rational preference relation $\succsim$ on the space of lotteries $\mathcal{L}$ satisfies the continuity and independence axioms. Then $\succsim$ admits a utility representation of the expected utility form. That is, we can assign a number u_n to each outcome $n=1,..,N$ in such a manner that for any two lotteries $L=\left(p_1,...,p_N\right)$ and $L'=\left(p_1',...,p_N'\right)$ we have*

$$L\succsim L' \text{ if and only if } \sum_n u_n p_n\geq\sum_n u_n p_n'$$

Thus, a lottery L is at least as good as another lottery L' if and only if the expected utility of L is greater than the expected utility of L'. We will forgo the proof, but discuss some implications of continuity and independence. Consider a case with 3 outcomes, so that we can represent the lotteries in the 2-dimensional simplex. Continuity guarantees that preferences are representable by a continuous utility function. Since the expected utility form is linear in the probabilities, representability by the expected utility form is equivalent to indifference curves in the simplex being straight, parallel lines. Consider two lotteries, L_A and L_B, such that $L_A\sim L_B$. If continuity and independence are satisfied, then $L_A\sim L_B$ implies that $U\left(L_A\right)=U\left(L_B\right)$.

Now consider any convex combination of lotteries A and B, $L_C = \alpha L_A + (1-\alpha) L_B$ for all $\alpha \in [0,1]$. Note that the convex combination is simply the line between L_A and L_B. The utility of L_C is

$$U(L_C) = U(\alpha L_A + (1-\alpha) L_B) = \alpha U(L_A) + (1-\alpha) U(L_B)$$

But $U(L_B) = U(L_A)$ since $L_A \sim L_B$, so $U(L_C) = \alpha U(L_A) + (1-\alpha) U(L_A) = U(L_A)$. If $U(L_C) = U(L_A)$, then $L_C \sim L_A$. Thus, the indifference curves as represented in the simplex will be linear. Moreover, they will be parallel. If they were NOT parallel, then you could have the case of one lottery that lies at the intersection of 2 indifference curves. If that lottery is indifferent to all the lotteries on BOTH of those indifference curves, but all of the lotteries on BOTH indifference curves are not indifferent, then we would have something to the effect of $L_A \sim L_B$ and $L_B \sim L_C$, but that $L_A \succ L_C$, which as we proved earlier in the course cannot be because $\sim$ is transitive.

3 Money Lotteries and Risk Aversion

Now, we will formalize a concept of risk because people tend to have different attitudes towards risk. We will focus on risky alternatives over money, and represent money as a continuous variable. Decision-makers will now face lotteries over money, where money is denoted by x. A monetary lottery can be described by a cumulative distribution function (CDF) $F : \mathbb{R} \to [0,1]$. For any x, $F(x)$ gives the probability that a realized value $\widetilde{x} \leq x$, so $F(x) = \Pr(\widetilde{x} \leq x)$. Note that the distribution function preserves the linear structure of lotteries. The final distribution, $F(\cdot)$, induced by the compound lottery $(L_1, ..., L_K; \alpha_1, ..., \alpha_K)$ is just the weighted average of the distributions induced by each of the lotteries that contribute to it, $F(x) = \sum_k \alpha_k F_k(x)$, where F_k is the distribution of the payoff under lottery L_k.

We now define $\mathcal{L}$ as the set of all distributions over nonnegative amounts of money. The decision-maker has rational preferences $\succsim$ defined over $\mathcal{L}$. If we apply the expected utility theorem to continuous variables, we get

$$U(F) = \int u(x)\, dF(x)$$

This is the continuous version of $\sum_n u_n p_n$. Also, note that expected utility makes the utility of monetary lotteries sensitive to the mean and higher moments of the distribution.

Some notation. We have $U(\cdot)$ defined as the vNM utility function defined over lotteries. We now have $u(\cdot)$ defined over sure amounts of money. Although this is not necessarily standard in the literature, we will call $u(\cdot)$ the Bernoulli utility function. The general axioms we have discussed place restrictions on $U(\cdot)$, the vNM utility function, but not on $u(\cdot)$, the Bernoulli utility function. At a minimum we typically assume that $u(\cdot)$ is increasing and continuous.

3.1 Risk Aversion and Measurement

Definition 8 *A decision-maker exhibits risk aversion if for any lottery $F(\cdot)$, the degenerate lottery that yields the amount $\int x dF(x)$ with certainty is at least as good as the lottery itself. If the decision-maker is always (for any $F(\cdot)$) indifferent between these two lotteries, we say that he is risk-neutral. He is strictly risk averse if indifference only holds for degenerate lotteries.*

From the definition of risk aversion

$$U(F) = \int u(x)\, dF(x) \leq u\left(\int x dF(x)\right) \text{ for all } F(\cdot).$$

From the mathematical appendix we can see that this is Jensen's Inequality, and that it defines a concave function ($u(x)$ in our case). Note that $u(x)$ is concave IF the decision-maker is risk averse. If the decision-maker is strictly risk averse, then we have a strictly concave Bernoulli utility function.

Figure 3 shows the Bernoulli utility function $u(x) = \sqrt{x}$. The figure shows that the utility of a certain amount is greater than the utility of a lottery that gives that amount on average. The lottery in the picture is a lottery over the outcomes \$1 and \$3 with probability $\frac{1}{2}$ on each. Given that $u(1) = 1$, $u(2) = \sqrt{2}$, and $u(3) = \sqrt{3}$, we can see that the lottery $L = \left(\frac{1}{2}, \frac{1}{2}\right)$ over the outcomes \$1 and \$3 has an expected value

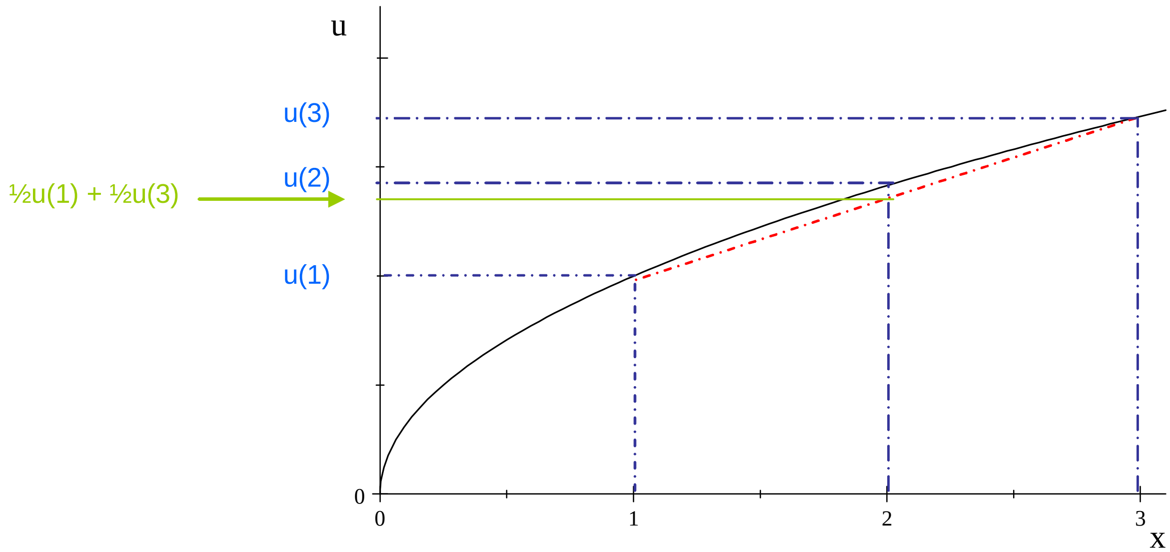

Figure 3: An individual with a Bernoulli utility function $u(x) = \sqrt{x}$.

of \$2, yet the individual's expected utility is only $\frac{1+\sqrt{3}}{2} < \sqrt{2}$. Strict concavity implies that the marginal utility of money is decreasing, so that if an individual has \$2, the utility gain from an additional dollar (to \$3) is less than the utility loss of an additional dollar (to \$1). If an individual is risk-neutral, then $\int u(x)\, dF(x) = u\left(\int x dF(x)\right)$.

Definition 9 *Given a Bernoulli utility function* $u(\cdot)$

1. *The certainty equivalent of* $F(\cdot)$ *denoted* $c(F,u)$ *is the amount of money for which the individual is indifferent between the gamble* $F(\cdot)$ *and the certain amount* $c(F,u)$*, that is,* $u(c(F,u)) = \int u(x)\, dF(x)$.

2. *For any fixed amount of money* x *and positive number* ε*, the probability premium denoted by* $\pi(x, \varepsilon, u)$ *is the excess in winning probability over fair odds that makes the individual indifferent between the certain outcome* x *and a gamble between the two outcomes* $x+\varepsilon$ *and* $x-\varepsilon$*. That is,* $u(x) = \left(\frac{1}{2} + \pi(x, \varepsilon, u)\right) u(x+\varepsilon) + \left(\frac{1}{2} - \pi(x, \varepsilon, u)\right) u(x-\varepsilon)$.

Figure 4 shows the Bernoulli utility function $u(x) = \sqrt{x}$ for the lottery $L = \left(\frac{1}{2}, \frac{1}{2}\right)$ over the outcomes \$1 and \$3, although the certainty equivalent, $c(F,u)$, has been added to this figure. The certainty equivalent is the sure amount of money that yields the same utility as the expected value of the gamble. To find this, we need to set $u(x) = u(\textit{gamble})$. In the example, the expected utility of the gamble is given by $\frac{1}{2}u(1) + \frac{1}{2}u(3)$, or $\frac{1+\sqrt{3}}{2}$. So $\sqrt{x} = \frac{1+\sqrt{3}}{2}$. Squaring both sides we find that $x = 1 + \frac{\sqrt{3}}{2} \approx 1.866$, and $u\left(1 + \frac{\sqrt{3}}{2}\right) = \frac{1+\sqrt{3}}{2}$. Be careful to distinguish between the terms *expected value* and *expected utility* in this context, as it is easy to gloss over the particular terms. The *expected value* is simply the weighted average (with the weights given by the specific lottery) of the actual outcomes, while the *expected utility* is the weighted average (with the weights given by the specific lottery) of the UTILITY of those outcomes.

It is also just a matter of algebra to calculate the probability premium, $\pi(x, \varepsilon, u)$. In our gamble, we have $x = 2$ (the expected value of the gamble), $\varepsilon = 1$ (since $3 = 2 + 1$ and $1 = 2 - 1$), and $u = \sqrt{x}$. We

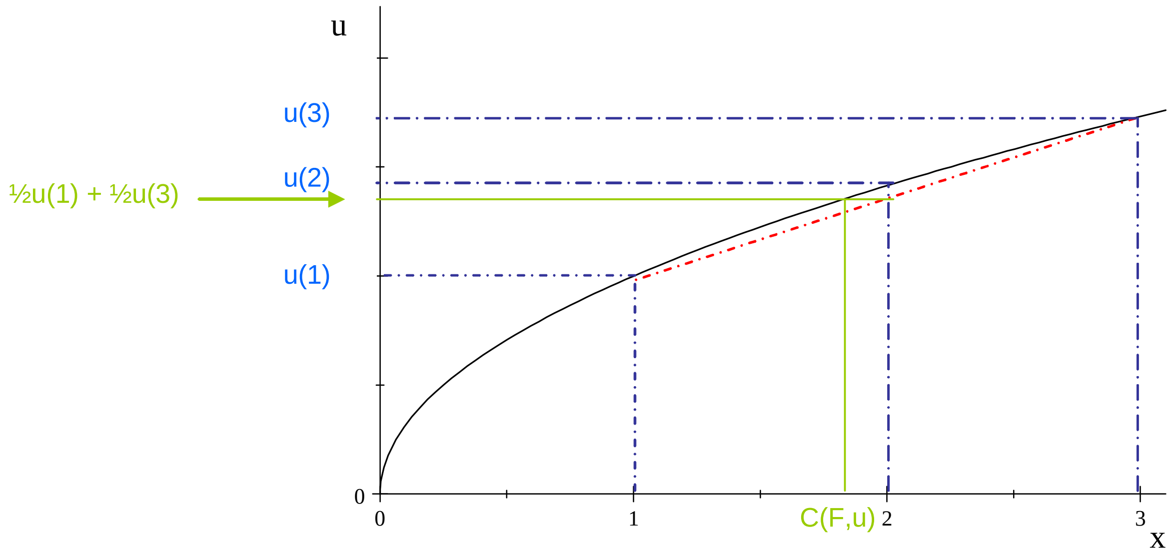

Figure 4: An individual with a Bernoulli utility function $u(x) = \sqrt{x}$, with the certainty equivalent added.

want to solve for $\pi(x, \varepsilon, u)$ given these parameters, so just plug these into the formula:

$$\begin{aligned}
u(2) &= \left(\frac{1}{2} + \pi(x,\varepsilon,u)\right) u(3) + \left(\frac{1}{2} - \pi(x,\varepsilon,u)\right) u(1) \\
u(2) &= \frac{1}{2}u(3) + \pi(x,\varepsilon,u)\, u(3) + \frac{1}{2}u(1) - \pi(x,\varepsilon,u)\, u(1) \\
\sqrt{2} &= \frac{1}{2}\sqrt{3} + \pi(x,\varepsilon,u)\sqrt{3} + \frac{1}{2} - \pi(x,\varepsilon,u) \\
2\sqrt{2} &= \sqrt{3} + 2\sqrt{3}\pi(x,\varepsilon,u) + 1 - 2\pi(x,\varepsilon,u) \\
2\sqrt{2} - \sqrt{3} - 1 &= 2\sqrt{3}\pi(x,\varepsilon,u) - 2\pi(x,\varepsilon,u) \\
\pi(x,\varepsilon,u) &= \frac{2\sqrt{2} - \sqrt{3} - 1}{2\sqrt{3} - 2} \approx 0.0658
\end{aligned}$$

So the individual would be indifferent between \$2 and the gamble which pays \$1 with probability $\left(\frac{1}{2} - \left(\frac{2\sqrt{2}-\sqrt{3}-1}{2\sqrt{3}-2}\right)\right)$ (about 0.4342) and \$3 with probability $\left(\frac{1}{2} + \left(\frac{2\sqrt{2}-\sqrt{3}-1}{2\sqrt{3}-2}\right)\right)$ (about 0.5658).

Proposition 10 *Suppose a decision-maker is an expected utility maximizer with a Bernoulli utility function $u(\cdot)$ on amounts of money. Then the following properties are equivalent:*

1. *the decision-maker is risk averse*
2. *$u(\cdot)$ is concave*
3. *$c(F,u) \leq \int x dF(x)$ for all $F(\cdot)$*
4. *$\pi(x,\varepsilon,u) \geq 0$ for all x, ε*

3.2 Measuring Risk Aversion

Now that we have defined risk aversion we need some method of measuring it.

Definition 11 *Given a (twice-differentiable) Bernoulli utility function $u(\cdot)$ for money, the Arrow-Pratt coefficient of absolute risk aversion at x is defined as $r_A(x) = -\frac{u''(x)}{u'(x)}$.*

Why use a measurement that depends on $u''(x)$ and $u'(x)$? Consider a risk-neutral individual. His Bernoulli utility function is linear, as in the earlier picture. The risk averse individual has a concave Bernoulli utility function, so looking at the curvature of the Bernoulli utility function is a natural place to begin. Why use $-\frac{u''(x)}{u'(x)}$ as opposed to simply $u''(x)$? The second derivative of the Bernoulli utility function is not invariant to positive linear transformations. Dividing by the first derivative is the easiest way to make this measure invariant to positive linear transformations. Why use a negative sign in front of the ratio? The first derivative will be positive if $u(x)$ is increasing in x and the second derivative will be negative if $u(x)$ is concave so that the individual is risk averse, so the result without the negative sign will be negative if the individual is risk averse. Changing the sign means that we will obtain a positive number for an increasing and concave Bernoulli utility function.

Now that we have a measure of risk aversion we can compare

1. risk attitudes across individuals
2. risk attitudes for one individual at different levels of wealth

3.2.1 Comparing across individuals

We have individuals 1 and 2 with Bernoulli utility functions $u_1(\cdot)$ and $u_2(\cdot)$. When can we say that individual 2 is more risk averse than individual 1?

Proposition 12 *Given two individuals with Bernoulli utility functions $u_1(\cdot)$ and $u_2(\cdot)$ on amounts of money, then the following properties are equivalent if individual 2 is more risk averse than individual 1:*

1. *$r_A(x, u_2) \geq r_A(x, u_1)$ for all x*
2. *There exists an increasing concave function $\psi(\cdot)$ such that $u_2(x) = \psi(u_1, x)$ at all x; that is, $u_2(\cdot)$ is a concave transformation of u_1.*
3. *$c(F, u_2) \leq c(F, u_1)$ for any $F(\cdot)$*
4. *$\pi(x, \varepsilon, u_2) \geq \pi(x, \varepsilon, u_1)$ for any x and ε*
5. *Whenever $u_2(\cdot)$ finds a lottery $F(\cdot)$ at least as good as the riskless outcome $\overline{x}$, then $u_1(\cdot)$ also finds $F(\cdot)$ at least as good as $\overline{x}$.*

Thus, if any of these hold then individual 2 is more risk averse than individual 1.

3.2.2 Comparisons across wealth levels

Typically, as wealth increases for an individual, that individual is more willing to take chances.

Definition 13 *The Bernoulli utility function $u(\cdot)$ for money exhibits decreasing absolute risk aversion if $r_A(x, u)$ is a decreasing function of x.*

Individuals who satisfy decreasing absolute risk aversion take more risk as they become wealthier.

Proposition 14 *The following properties are equivalent*

1. *The Bernoulli utility function exhibits decreasing absolute risk aversion*
2. *Whenever $x_2 < x_1$, $u_2(z) = u(x_2 + z)$ is a concave transformation of $u_1(z) = u(x_1 + z)$*
3. *For any risk $F(z)$, the certainty equivalent of the lottery formed by adding z to wealth level x, given by the amount c_x at which $u(c_x) = \int u(x+z)\, dF(z)$ is such that $(x - c_x)$ is decreasing in x. The higher x is, the less the individual is willing to pay to get rid of risk.*

4. *The probability premium $\pi(x, \varepsilon, u)$ is decreasing in x.*

5. *For any $F(z)$, if $\int u(x_2 + z)\, dF(z) \geq u(x_2)$ and $x_2 < x_1$, then $\int u(x_1 + z)\, dF(z) \geq u(x_1)$.*

Note that we are focusing on ABSOLUTE risk aversion, which is based on absolute movements of wealth. There are also concepts of relative risk aversion, where the Arrow-Pratt measure of absolute risk aversion is weighted by wealth, so that: $r_R(x) = -x\frac{u''(x)}{u'(x)} = -x r_A(x)$.

3.3 Comparison of Payoff Distributions in Terms of Return and Risk

Our goal now is to compare payoff distributions, not utility levels. We will look at payoff distributions in terms of their levels of returns and the dispersion of their returns. We say that a payoff distribution $F(\cdot)$ first-order stochastically dominates payoff distribution $G(\cdot)$ if $F(\cdot)$ yields unambiguously higher returns than $G(\cdot)$. We say that a payoff distribution $F(\cdot)$ second-order stochastically dominates payoff distribution $G(\cdot)$ if $F(\cdot)$ is unambiguously less risky than $G(\cdot)$. There are also concepts of third-order and fourth-order stochastic dominance. However, a Google search (10/11/2006) shows that there are 31,000 hits for "first order stochastic dominance" (32,500 on 10/10/2007); 19,000 hits for "second order stochastic dominance" (20,200 on 10/10/2007); 400 hits for "third order stochastic dominance" (1250 on 10/10/2007); and 2 hits for "fourth order stochastic dominance" (33 on 10/10/2007). Thus, we will discuss first and second order stochastic dominance, and leave the cutting edge of research in third and fourth order stochastic dominance for you to peruse on your own.

For first-order stochastic dominance, we could

1. test to see if whether every expected utility maximizer prefers $F(\cdot)$ to $G(\cdot)$

2. verify whether, for every amount of money x, the probability of getting at least x is higher under $F(\cdot)$ than under $G(\cdot)$

Definition 15 *The distribution $F(\cdot)$ first order stochastically dominates $G(\cdot)$ if, for every nondecreasing function $u : \mathbb{R} \to \mathbb{R}$ we have*

$$\int u(x)\, dF(x) \geq \int u(x)\, dG(x)$$

Proposition 16 *The distribution of monetary payoffs $F(\cdot)$ first order stochastically dominates $G(\cdot)$ if and only if $F(x) \leq G(x)$ for every x.*

There are two points to take away from this proposition. First, it is not necessary for every possible return of $F(\cdot)$ to be higher than every possible return of $G(\cdot)$ in order for $F(\cdot)$ to first order stochastically dominate $G(\cdot)$. Thus, we do not need something like $F(\cdot)$ to be defined on outcomes over $[50, 100]$ and $G(\cdot)$ to be defined on outcomes over $[1, 10]$, although $F(\cdot)$ would first order stochastically dominate $G(\cdot)$ in this case. Second, it is NOT sufficient to show that the mean of $F(\cdot)$ is greater than the mean of $G(\cdot)$, as it is possible for the mean of $F(\cdot)$ to be larger than the mean of $G(\cdot)$ even though $F(\cdot)$ does not first order stochastically dominate $G(\cdot)$. Thus, it is the entire distribution of payoffs that matters, not just the mean.

As an example of first-order stochastic dominance, consider lotteries $L_1 = \left(\frac{1}{2}, 0, \frac{1}{2}, 0\right)$ and $L_2 = \left(0, \frac{1}{2}, 0, \frac{1}{2}\right)$ over the outcomes 1, 3, 4, and 10. The distribution functions for the risky alternatives are in Figure 5.Note that L_2 is always equal to or below L_1, which is exactly what proposition 16 states. If we were to rearrange the lotteries, so that $L_1' = \left(0, \frac{1}{2}, \frac{1}{2}, 0\right)$ and $L_2' = \left(\frac{1}{2}, 0, 0, \frac{1}{2}\right)$, then neither would first-order stochastically dominate the other despite the fact that the mean of L_1' is 3.5 and the mean of L_2' is 5.5.

When using the concept of second order stochastic domination, we now compare the riskiness of distributions with the *same mean* and ask the question of which one is more risky. Given $F(\cdot)$ and $G(\cdot)$ with $\int x dF(x) = \int x dG(x)$, $G(\cdot)$ is riskier than $F(\cdot)$ if every risk averter prefers $F(\cdot)$ over $G(\cdot)$.

Definition 17 *For any two distributions $F(\cdot)$ and $G(\cdot)$ with the same mean, $F(\cdot)$ second order stochastically dominates $G(\cdot)$ if for every nondecreasing concave function $u : \mathbb{R} \to \mathbb{R}$ we have*

$$\int u(x)\, dF(x) \geq \int u(x)\, dG(x)$$

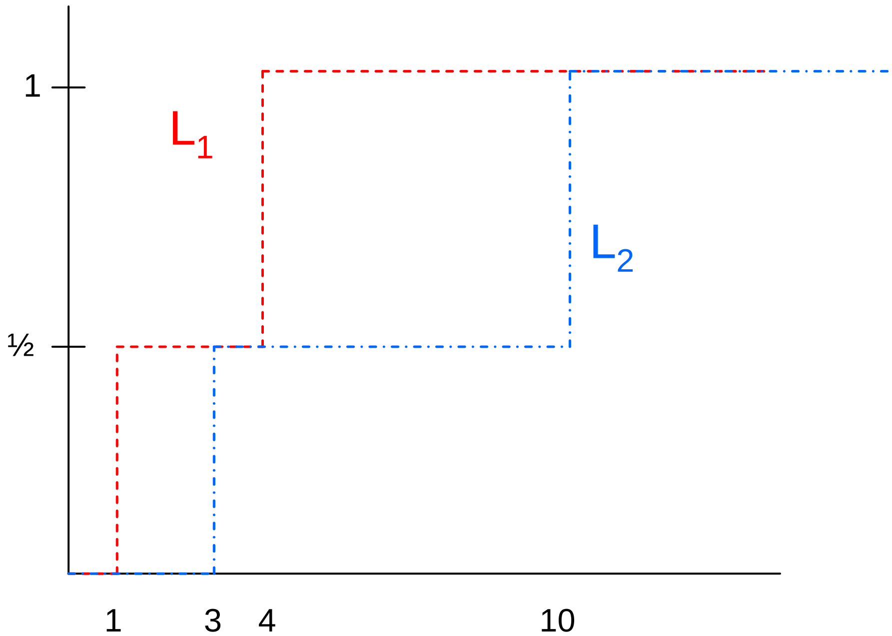

Figure 5: An example where L_2 first-order stochastically dominates L_1.

Note that the equation in the definition for first-order stochastic dominance is the same as the equation in the definition for second-order stochastic dominance. However, with second-order stochastic dominance we are assuming that the mean of $F(x)$ and the mean of $G(x)$ are the SAME.

Proposition 18 *Consider two distributions $F(\cdot)$ and $G(\cdot)$ with the same mean. Then the following statements are equivalent.*

1. *$F(\cdot)$ second order stochastically dominates $G(\cdot)$*
2. *$G(\cdot)$ is a mean-preserving spread of $F(\cdot)$*
3. *$\int_0^x G(t)\,dt \geq \int_0^x F(t)\,dt$ for all x*

Again, we will use an example to discuss these concepts. Consider the lotteries $L_3 = \left(0, \frac{1}{2}, \frac{1}{2}, 0\right)$ and $L_4 = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right)$ over the outcomes 1, 2, 3, and 4. Note that the mean of both lotteries is 2.5, so we can compare them using the concept of second order stochastic dominance. The idea of a mean-preserving spread is that one lottery is simply more spread out than the other, but the mean remains the same. Thus, L_4 illustrates one potential mean-preserving spread of L_3 because it has the same mean but the probabilities are more spread out over the outcomes. Figure 6 shows the two distributions. Note that neither L_3 nor L_4 is always below the other, so neither first-order stochastically dominates the other. However, the third part of proposition 18 states that as long as the area under one distribution is always less than or equal to the area under another distribution up to ANY POINT in the distributions, then the one with the area always less than or equal to the area of the other second-order stochastically dominates it. In the figure, the area under L_3 is always less than or equal to the area under L_4, or at least it would be if I could draw the distributions accurately, so L_3 second-order stochastically dominates L_4.

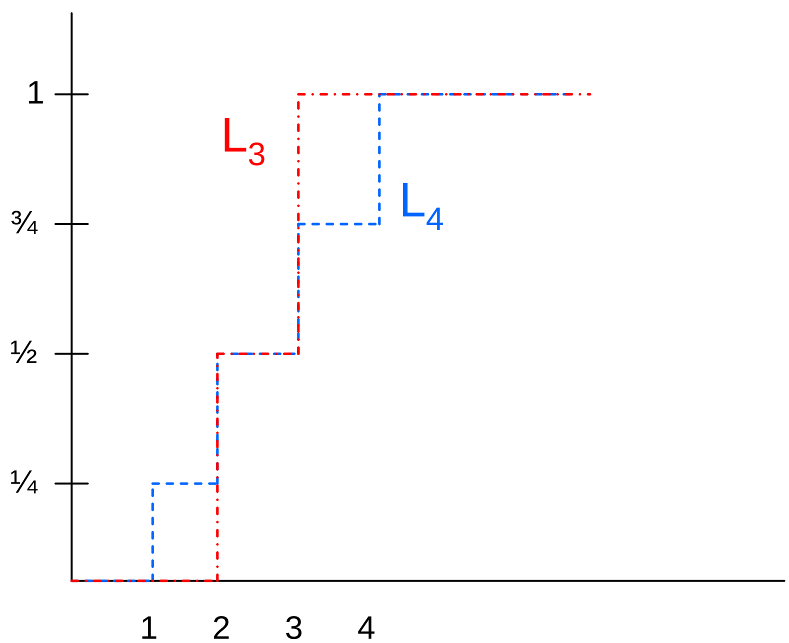

Figure 6: An example where L_3 second-order stochastically dominates L_4.