Notes on inconsistencies with established models of choice under uncertainty*

1 Introduction

In this short set of notes two papers are discussed. One is Kahneman and Tversky's (KT) 1979 Econometrica paper titled Prospect Theory: An Analysis of Decision under Risk. The other is Majumdar and Radner's (MR) 1991 Economic Theory paper titled Linear Models of Economic Survival under Production Uncertainty (be careful when reading this paper – it is the first ever issue of Economic Theory and it appears the proofreaders were not the best). Both argue that individuals may make choices inconsistent with established models of choice behavior. In KT they provide more evidence than a formal model. The MR paper is essentially a formal model of behavior with only anecdotal evidence.

2 KT Evidence

KT document some cases that violate expected utility theory (EUT). They begin with the Allais paradox, which we have already seen. Some quick notation. A gamble in KT is denoted $(x_1, p_1; x_2, p_2; ...; x_N, p_N)$ where the x_i are the outcomes of the gamble and the p_i are the probabilities associated with the outcomes, with $\sum_{i=1}^{N} p_i = 1$. They use (x, p) to denote $(x, p; 0, (1-p))$ but I have written all the gambles in their problems out using the full format with fractions denoting the probabilities rather than decimals so it is easier to see what the gamble is. They begin my stating 3 key assumptions of expected utility theory:

1. Expectation: $U(x_1, p_1; ...; x_N, p_N) = p_1 u(x_1) + ... + p_N u(x_N)$ – this is that the vNM utility is linear or that it has the expected utility form

2. Asset integration: $(x_1, p_1; ...; x_N, p_N)$ is acceptable at asset position w if and only if $U(w + x_1, p_1; ...; w + x_N, p_N) > u(w)$ – in other words, a prospect (gamble) is acceptable if the utility resulting from integrating the prospect with one's own assets exceeds the utility of the assets alone

*Based on Kahneman and Tversky (1979) Econometrica and Majumdar and Radner (1991) Economic Theory.

3. Risk aversion: u is concave – an individual is risk averse if he prefers x with certainty to a prospect that has an expected value of x.

Problem 1
A: $(2500, \frac{33}{100};\ 2400, \frac{66}{100};\ 0, \frac{1}{100})$
*B: $(2400, \frac{100}{100})$
Problem 2
*C: $(2500, \frac{33}{100};\ 0, \frac{67}{100})$
D: $\left(2400, \frac{34}{100};\ 0, \frac{66}{100}\right)$
We can obtain Problem 2 from Problem 1 by subtracting 0.66 from the 2400 outcome.
Problem 3
A: $\left(4000, \frac{80}{100};\ 0, \frac{20}{100}\right)$
*B: $\left(3000, \frac{100}{100}\right)$
Problem 4
*C: $\left(4000, \frac{20}{100};\ 0, \frac{80}{100}\right)$
D: $\left(3000, \frac{25}{100},\ 0, \frac{75}{100}\right)$
We can obtain problem 4 from problem 3 by multiplying the positive outcomes by $\frac{1}{4}$
Problem 5
A: 50% chance to win a 3-week tour of England, France, and Italy
*B: 1-week tour of England with certainty
Problem 6
*C: 5% chance to win a 3-week tour of England, France, and Italy
D: 10% chance to win a 1-week tour of England
We can obtain problem 6 from problem 5 by taking 10%
Problem 7
A: $\left(6000, \frac{45}{100};\ 0, \frac{55}{100}\right)$
*B: $\left(3000, \frac{90}{100};\ 0, \frac{10}{100}\right)$
Problem 8
*C: $\left(6000, \frac{1}{1000};\ 0, \frac{999}{1000}\right)$
D: $\left(3000, \frac{2}{1000};\ 0, \frac{998}{1000}\right)$
In problems 7 and 8 the probability of getting 3000 is 2x the probability of getting 6000 in both. It seems as if the people see "high probability" and then gravitate towards that (especially if the probabilities are "different") and if the two probabilities are approximately equal, then gravitate towards the greater of A or B
Problem 11
A: $\left(1000, \frac{50}{100};\ 0, \frac{50}{100}\right)$
*B: $\left(500, \frac{100}{100}\right)$
Problem 12
*C: $\left(-1000, \frac{50}{100};\ 0, \frac{50}{100}\right)$
D: $\left(-500, \frac{100}{100}\right)$
We can obtain problem 12 from problem 11 by subtracting 1000 from each outcome

All of these B/C choices by individuals shows a preference reversal. It should be noted that they are looking at aggregate numbers, so the results may not be as strong as they claim. For instance, it is possible that 30 of the 72 people who participated in problems 1 and 2 adhere to the theory. Also note that they are not paid subjects. But they do consistently document the Allais paradox well.

Now, what happens when gains (all of the first 8 problems are in the domain of gains) are replaced by losses? Now, instead of the modal choices being the B & C pairs the modal choices are the A & D pairs. Thus, there is essentially a "reflection effect" in that the inequalities are switched (see Table 1). They have 3 empirical results from these problems:

1. Risk aversion in the positive domain is accompanied by risk seeking in the negative domain, which yields an S-shaped utility function
2. Choices under positive outcomes violated expected utility theory, and so do those under negative outcomes
3. Certainty is not generally desirable (not in the loss domain)

They also note that two prospects that are equivalent in probabilities and outcomes could have different values depending on their formulations They then decompose what people may be doing in order to evaluate the prospects. They first discuss the isolation effect, which is that individuals disregard the components that the alternatives share and focus on the differences.

3 Theory

Their theory is an attempt to understand how individuals might simplify the complex prospects proposed to them. They suggest that prospects undergo two phases, the editing phase and the phase of evaluation. In the editing phase there is preliminary analysis of the offered prospects and simpler representation of the prospects. In te evaluation phase the edited prospects are evaluated.

3.1 Editing phase

KT discuss a few possible methods of editing the prospects.

1. Coding – gains and losses vs. final states – relative to a reference point
2. Combination – Simplify by combining probabilities with identical outcomes
3. Segregation – riskless component is segragated, so that $\left(300, \frac{80}{100};\ 200, \frac{20}{100}\right)$ becomes $\left(200, \frac{100}{100}\right)$ and $\left(100, \frac{80}{100};\ 0, \frac{20}{100}\right)$
4. Cancellation (2 or more prospects) – $\left(200, \frac{20}{100};\ 100, \frac{50}{100};\ -50, \frac{30}{100}\right)$ and $\left(200, \frac{20}{100};\ 150, \frac{50}{100};\ -100, \frac{30}{100}\right)$ is reduced to $\left(100, \frac{50}{100};\ -50, \frac{30}{100}\right)$

5. Simplification – $\left(101, \frac{49}{100};\ 0, \frac{51}{100}\right)$ may be recorded as $\left(100, \frac{50}{100};\ 0, \frac{50}{100}\right)$. Also, that people discard extremely unlikely events.

6. Dominance – Seem to reject dominated alternatives, so they prefer $\left(200, \frac{50}{100};\ 0, \frac{50}{100}\right)$ to $\left(100, \frac{50}{100};\ 0, \frac{50}{100}\right)$

It is assumed that these operations are performed whenever possible, so that $\left(500, \frac{20}{100};\ 101, \frac{49}{100};\ 0, \frac{31}{100}\right)$ appears to dominate $\left(500, \frac{15}{100};\ 99, \frac{51}{100};\ 0, \frac{34}{100}\right)$ if $\left(101, \frac{49}{100}\right)$ and $\left(99, \frac{51}{100}\right)$ are both simplified to $\left(100, \frac{50}{100}\right)$

There is a drawback in that edited prospects could depend on the order of editing. However, in this paper they discuss those prospects which have no room for editing or no ambiguity in the editing process. After editing, the decision-maker is assumed to evaluate the edited prospects and choose the one with the higher value.

3.2 Some notation

Let V be the overall value of the prospect. This is akin to U (the vNM utility) in EUT. There are 2 scales, π and v. The scale π associates with each probability a decision weight $\pi(p)$ which reflects the impact of p on the overall value of the prospect. The scale π does NOT measure probability, so it is typically the case that $\pi(p) + \pi(1-p) < 1$. The scale v assigns to each outcome x a number $v(x)$, which is the subjective value of that outcome. This is akin to $u(x)$ (the Bernoulli utility) in EUT. Note that in prospect theory the value $v(x)$ is defined relative to a reference point.

KT consider prospects with at most 2 non-zero outcomes, so their general form is $(x, p;\ y, q;)$, with the outcome 0 having probability $(1-p-q)$ and $1-p-q+p+q=1$ (these are the true probabilities). An offered prospect is strictly positive if its outcomes are all positive, so that $x, y > 0$ and $p+q=1$ (there is no 0 outcome). It is strictly negative if $x, y < 0$ and $p+q=1$. Otherwise it is a regular prospect. If $(x, p;\ y, q)$ is a regular prospect, then $V(x) = \pi(p)\, v(x) + \pi(q)\, v(y)$. Let $v(0) = 0$, $\pi(0) = 0$, and $\pi(1) = 1$. Typically, $V(x) \neq v(x)$, although with certain outcomes $V(x) = v(x)$.

With strictly positive and strictly negative prospects decision-makers use a different rule. They break the prospect into 2 pieces in the editing phase:

1. riskless component (minimum gain or loss)

2. risky component (additional gain or loss)

If $p+q=1$ and either $x > y > 0$ or $x < y < 0$, then

$$V(x, p;\ y, q) = v(y) + \pi(p)\,[v(x) - v(y)] \quad (1)$$

Note that in this case the outcome y is the "sure" gain or loss since there are only two possible outcomes and $|x| > |y|$. Thus, the value of the prospect equals the value of the riskless component, $v(y)$, plus the difference in the value of the risky component multiplied by the weight of the more extreme outcome (x in this example). So a prospect such as $V\left(400, \frac{1}{4};\ 100, \frac{3}{4}\right)$ would have a $V(\cdot)$ of:

$$V\left(400, \frac{1}{4};\ 100, \frac{3}{4}\right) = v(100) + \pi\left(\frac{1}{4}\right)[v(400) - v(100)]. \tag{2}$$

Thus there is a decision weight applied to the risky component but not the riskless component. Note that if $\pi(p) + \pi(1-p) = 1$, then:

$$v(y) + \pi(p)[v(x) - v(y)] = \pi(p)v(x) + \pi(q)v(y). \tag{3}$$

Note the 2 key differences between prospect theory and EUT:

1. Values are attached to changes in states rather than final states

2. Decision weights do not coincide with actual probabilities

Note that this may lead the decision-maker to have inconsistent choices. If the inconsistencies are pointed out to the decision-maker they may correct them, but the decision-maker may never know about the inconsistencies so that in the "real-world" the decision-maker never corrects these inconsistencies and they are observed.

3.3 Value function

The value function $V(x)$ focuses on changes in states rather than final states. This is consistent with other biological and physiological responses to changes, such as brightness, loudness, and temperature. Consider a class of water straight from the faucet (I know, no one drinks water from the faucet anymore). Compared to a piece of ice, the water is warm. Compared to a cup of boiling water, the water is cold. So it is relative states that matter. Also, the initial asset position matters. Thus, the value function depends on the initial position and on the magnitude of the change relative to the initial position. Also, many times similar absolute changes in magnitude (say a change of 100 in wealth) appear different when viewed from different initial positions. For instance, KT suggest that a difference in gain from 100 to 200 in wealth seems larger than a change from 1100 to 1200, and similarly the difference between a loss of 100 and a loss of 200 seems larger than the difference between a loss of 1100 and a loss of 1200. Hypothesize that the shape of the value function is is normally concave above the reference point and convex below it (the S-shaped utility function or value function). The problem they use as evidence is:

Problem 13
A: $\left(6000, \frac{25}{100};\ 0, \frac{75}{100}\right)$
*B: $\left(4000, \frac{25}{100};\ 2000, \frac{25}{100};\ 0, \frac{50}{100}\right)$
Problem 13':
*C: $\left(-6000, \frac{25}{100};\ 0, \frac{75}{100}\right)$
D: $\left(-4000, \frac{25}{100};\ -2000, \frac{25}{100};\ 0, \frac{50}{100}\right)$

What we then see is that:

$$\pi(.25)\, v(6000) < \pi(.25)\left[v(4000) + v(2000)\right] \quad (4)$$
$$\pi(.25)\, v(-6000) > \pi(.25)\left[v(-4000) + v(-2000)\right] \quad (5)$$

Or $v(6000) < v(4000) + v(2000)$ and $v(-6000) > v(-4000) + v(-2000)$.

They then discuss “special circumstances”. Saving money for buying a house (say needing \$60,000), or not losing money to be forced to move from one’s house. In short, they consider utility theory, not money theory. Also, losses loom larger than gains – utility decreases by more when you lose \$1 than when you gain \$1. So the value function is:

1. defined on deviations from the reference point
2. generally concave for gains and convex for losses
3. steeper for losses than gains

3.4 Decision weights

How do individuals derive these decision weights? They don’t have any explicit mapping from the actual probabilities to the decision weights, but propose the following, in part based on problem 14 and 14’.

Problem 14
*A: $\left(5000, \frac{1}{1000};\ 0, \frac{999}{1000}\right)$
B: $\left(5, \frac{100}{100}\right)$
Problem 14’
C: $\left(-5000, \frac{1}{1000};\ 0, \frac{999}{1000}\right)$
*D: $\left(-5, \frac{100}{100}\right)$

In this case KT believe the individuals are putting more weight on the small probability event occurring than it really will occur.[1] They distinguish this from overestimation of rare events because the probabilities are known in the problems, but state that overestimation of rare events and overweighting of small probabilities may work hand in hand in the "real world". So the rules that they propose for individuals assigning decision weights are:

1. Small probabilities are overweighted (see problem 14)
2. For $0 < p < 1$ $\pi(p) + \pi(1-p) < 1$ (subcertainty)
3. For a fixed ratio of probabilities, the ratio of the corresponding decision weights is closer to unity when the probabilities are low than when they are high (subproportionality).

[1] Personally, with problems 14 and 14’ I take the view that \$5 is basically no change for anyone, or that \$5 $\approx$ \$0. There is an “old” song by Arrested Development, called Mr. Wendell, with a line that is something to the effect of, "Here brother man, have a dollar, no in fact have two; two dollars means a snack to me it means a big deal to you". Basically, \$5 is meaningless to most people, but \$5000 is NOT.

For subcertainty KT look at problems 1 and 2 to see:

$$\begin{aligned}
v(2400) &> \pi(0.66)\, v(2400) + \pi(0.33)\, v(2500) \text{ so} && (6)\\
[1-\pi(0.66)]\, v(2400) &> \pi(0.33)\, v(2500) \text{ and} && (7)\\
\pi(0.33)\, v(2500) &> \pi(0.34)\, v(2400) \text{ so} && (8)\\
1-\pi(0.66) &> \pi(0.34) \text{ or} && (9)\\
\pi(0.66)+\pi(0.34) &< 1. && (10)
\end{aligned}$$

Subcertainty suggests that preferences are generally less sensitive to variations in probability than the expectation principle would dicatate.

For subproportionality, see problems 7 and 8. Essentially, when considering two prospects with a fixed ratio of probabilities, if the probabilities are very low then the decision-maker is likely to view them as equal, but if the actual probabilities are "large" ("close" to 1) then the decision-maker will view them differently, despite the fact that the ratio is the same.

KT provide a possible mapping of probabilities and decision weights in figure 4. Note that this decision weight function may be discontinuous. However, this is only one possible mapping. KT also note that low probability events may be either overweighted or discarded (in the editing phase) and that high probability events may either be treated as certain events (in the editing phase) or neglected.

3.5 Quick conclusion

In some sense, this is not a proper theory because its assumptions cannot be tested. People "may or may not" behave this way. So if someone attempted to test this theory its assumptions could NEVER be refuted because it allows essentially everything. The question would then become how one could document what makes people (or what types of people) either discard or overweight small probabilities, etc.

4 Majumdar and Radner

The basic idea behind this paper is to determine what kind of investment decisions individuals will make when they have a survival constraint. What is ultimately shown is that when the individuals are close to "ruin" they choose options that are risky – that is, if given options with the same mean but different variances they choose the option with the higher variance.[2] However, prior to discussing the case where the investor gets to choose among investments, they first set out the conditions under which the investor will certainly face "ruin" or will "survive forever". This is a classic example of building a baseline model and then extending it.

[2]If you've ever seen the movie Rounders, it's like the end of the movie when Mike (Matt Damon) has very few options left and faces certain "ruin" and must make a risky move in order to survive.

4.1 Some notation

The individual begins "life" with an initial endowment of capital y_0. The individual must consume (or pay out) at least c each period in order to survive. If during any period t it is ever the case that $y_t - c < 0$, then the individual is ruined in period t. To begin, we assume $y_0 > c > 0$, otherwise if $y_0 \leq c$ then the individual is ruined in the first period and this is an uninteresting model. For periods when $y_t - c > 0$, the individual invests the additional amount $(y_t - c)$ in some asset position. In the baseline model, where the individual does not get to choose what investments to make, the return on the first period investment is given by:

$$y_1 = (\exp R_1)(y_0 - c) \tag{11}$$

where R_1 is a random variable representing the rate of return for period 1. Thus, period 1's "capital stock" is y_1. If $y_1 > c$ the individual continues on to period 2, where:

$$y_2 = (\exp R_2)(y_1 - c). \tag{12}$$

This process continues until the individual is ruined or forever if the individual survives. Initially R_t is a sequence of iid (independently and identically distributed) random variables. Later in the paper they discuss that y_t is a Markov chain (or process). In short, this means that the path taken to reach consumption level y_t is unimportant, and that the only thing that matters at time t is the actual value of y_t. Essentially, all the relevant information from the past is summed up in the single state variable y_t. This is the basic structure of the baseline model – although there is a lot of math to show results, the basic model can be explained in incredible simple terms to non-mathematicians. Note that these are usually the best types of models – very simple structure explaining an interesting economic phenomenon, and then some good old-fashioned 'rithmetic to prove a theorem.

4.1.1 Baseline case

In the baseline case they show that the important variable is the ratio $\frac{y_0}{c}$, and then if this ratio is "high enough" the individual should expect to survive forever. In section 3 of the paper they remove the iid assumption and prove results about ruin and survival for a general class of random variables. In section 4 they begin with a discrete time model and then move to a continuous time model and show how an explicit formula for calculating the probability of survival is derived – this is equation 4.28 or 4.29. It is important to remember what is trying to be shown and to NOT GET LOST in the mathematics when reading through a paper.

4.1.2 Active agent

Section 5 is where the really interesting phenomenon occurs – agents choosing among investments with equal expected values will choose an investment with a higher variance if the individual is close to ruin. The agent is allowed to choose an investment with an (m, v) pair (mean, variance) each period. The objective of the agent is to choose (m, v) so as to maximize the probability of survival. This is the agent's stated goal – note that it is not to "maximize utility" or any such thing that we have discussed. Obviously there has to be some restrictions on the (m, v) pair that the agent can choose each period – otherwise the agent would choose the following pair (really big number, 0) because receiving a "really big number" for certain each period is a pretty good deal. Note that the set of feasible acts, A, is restricted by Assumption 5.1. This is an exogenous assumption, although the amount of capital stock influences how the fortune will evolve over time (as shown in equation 5.1). Figure 1 shows a potential available set of mean/variance choices (top of page 23). Note that choices should only be on the upper bound of this set – why anyone would take an investment with the same variance but a lower mean makes no sense. Given that choices are along the upper bound of this set, it is easy to see that someone could choose an (m, v) pair with the same m but a higher v. And that is what the remainder of the math in section 5 establishes.

The intuition behind this choice is that if one is very close to "ruin" then it makes no sense to go with the "safe" alternative that may be unlikely to stave off "ruin". For example, consider a choice between two alternatives, L_1 and L_2, which both pay out 0 on average. L_1 is a 50/50 chance at \$1 and $-\$1$, while L_2 is a 50/50 chance at $\$10,000$ and $-\$10,000$. The variance of L_1 is 2 and the variance of L_2 is 200,000,000 (using $N-1$ in the denominator for the variance). So the safe play in terms of a mean/variance choice is L_1, but if the individual has \$4 and needs \$10 to survive next period, then choosing L_1 is not going to do a whole lot of good. In this model, having \$5 when you need \$10 is the same as having $-\$9,996$ when you need \$10.