# 1 Simultaneous move games of complete information<sup>1</sup>

One of the most basic types of games is a game between 2 or more players when all players choose strategies simultaneously. While the word simultaneously is used, it does not necessarily mean they choose strategies at the exact same instance – all we need is for one player to be unaware of the strategy choice made by the other player(s). To further reduce the complexity of the game, we assume that there is complete information. This means that there is no uncertainty about any player's type in the game. Simultaneous games are NOT games of perfect information, as players do not know which point they are at in the game because they do not observe the other player's strategy choice prior to making their own strategy choice. In one last step to reduce the complexity, we assume (for now) that these are one-shot games. A one-shot game is simply a game that is played only once.

The simplest of these games are those with only 2 players, where each player only has 2 strategies. If there was only 1 player we know this would technically be a decision, not a game, and if one of the two players only had one strategy then it would not be a very interesting game.<sup>2</sup> Rock, Paper, Scissors is an example of a simultaneous move game. An oligopoly market where the demand curve and all firm cost functions are known could be a simultaneous move game if firms have to make their production choices without knowledge of the other firms' production choices. These types of games will be discussed later in the course. For now we focus on the following:

Story: You are one of two producers in a market. Each of you can produce either 10 or 20 widgets. If you both produce 20 widgets then you each earn a profit of \$5. If you both produce 10 widgets then you each earn a profit of \$11. If one of you produces 20 widgets and the other produces 10 widgets, then the one who produces 20 widgets will receive \$16 while the one who produces 10 widgets will receive \$3. How many widgets do you produce?

There are a few things to remember. This is only a one-shot game, so there is no repetition. While one could, and quite possibly should, add more detail to make the game better reflect the "real world", one is restricted to choosing a strategy based upon the rules of the game. This is something many beginning game theory students fail to grasp - do not add more detail to the game than is there.

Since this is our first formal game of study, it would be useful to identify the components of the game. The players are 2 producers in the market. The rules are that the production choices are made simultaneously and that each producer is limited to choosing a quantity of 10 or a quantity of 20. There are 4 outcomes:

Producer A chooses 10, Producer B chooses 10

Producer A chooses 10, Producer B chooses 20 Producer A chooses 20, Producer B chooses 10

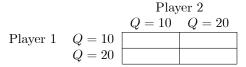
Producer A chooses 20, Producer B chooses 20

And then there are the payoffs associated with these

outcomes, given by the profits of each producer for each outcome. So all 4 components of a game are present here.

#### 1.1 Constructing the strategic form of the game

The strategic form of the game goes by different names: the normal form and the matrix (or bi-matrix) form being the most common other names. The strategic form is the best description though, which will become clear (hopefully) when we move to sequential games. Note that all the components of the game are present in the strategic form of the game. Begin with making a table with the number of rows equal to the number of strategies player 1 has and the number of columns equal to the number of strategies player 2 has. So if both have 2 strategies, it will be a 2x2 "table" (we will call it a 2x2 matrix). Then for each row list one of player 1's strategies and for each column list one of player 2's strategies. So the initial construction for the quantity choice game above should look like:



<sup>&</sup>lt;sup>1</sup>These notes are similar to what is in the Harrington text in the second half of chapter 2 and chapter 3.

<sup>&</sup>lt;sup>2</sup>It would be even less interesting if all players only had 1 strategy.

We would call Player 1 the row player and Player 2 the column player. It really does not matter which one is the row player and which is the column player, or what order the strategies are in, as long as the strategies are correct for each player. Now there are 4 empty cells, one corresponding to each outcome. So the outcomes are already present in the strategic form of the game, as each cell in the matrix represents an outcome. The last thing to do is to add the payoffs. The key here is to look at each outcome cell and determine what the payoff for each player would be at that outcome. Then simply list the payoffs in the outcome cell. **IMPORTANT:** The convention is to list the row player's payoff first, and then list the column player's payoff second. This is the convention used throughout the study of game theory, much like listing quantity on the x-axis and price on the y-axis is the convention for a supply and demand graph. So the finished matrix would look like:

Player 2 Q = 10 Q = 20Player 1 Q = 10 \$11, \$11 \$3, \$16Q = 20 \$16, \$3 \$5, \$5

Now all 4 components of the game are present.

## 2 Solving games

There are a variety of different techniques one could use to solve games. The key is that a solution to a game is a set of strategies. Let me repeat that: A solution to a game is a Set of strategies. A solution is NOT a set of payoffs. Now on to how to solve games.

#### 2.1 Strictly and weakly dominant strategies

When solving games, one should first check to see if a player has a strictly dominant strategy. A strictly *dominant strategy* is a strategy that does strictly better (provides a strictly higher payoff) than any other strategy choice by the player regardless of the strategy chosen by the other player(s). If a player has a strictly dominant strategy, he should simply use that strategy – why use any other strategy if there is one strategy that always does best? In looking at our quantity choice game, if Player 2 were to choose Q = 10, and Player 1 knew this, Player 1 would choose Q = 20 because \$16 > \$11. If Player 2 were to choose Q = 20, and Player 1 knew this, Player 1 would choose Q = 20 because 5 > 3. Thus, regardless of what Player 2 could choose, Player 1 would always choose Q = 20. So Q = 20 is a strictly dominant strategy for Player 1. We can conduct the same analysis for Player 2 to see that regardless of the choice made by Player 1, Player 2 would always want to choose Q = 20. So that for this particular game, Player 1 choosing Q = 20 and Player 2 choosing Q = 20 is the solution to the game. A closely related concept is that of a weakly dominant strategy. A weakly dominant strategy does at least as well as any other strategy regardless of the strategy chosen by the other player(s). The at least as well part simply means that there may be ties between the payoffs of the weakly dominant strategy and other strategies. However, if a player has a weakly dominant strategy that player should still play the weakly dominant strategy, making it relatively easy to solve the game.<sup>3</sup> Consider the modified quantity choice game: Plaver 2

Player 2  

$$Q = 10$$
  $Q = 20$   
Player 1  $Q = 10$   $\$11, \$11$   $\$5, \$16$   
 $Q = 20$   $\$16, \$3$   $\$5, \$5$ 

The payoff for Player 1 if she chooses Q = 10 and Player 2 chooses Q = 20 is now \$5 instead of \$3. Notice now that Q = 20 is not a strictly dominant strategy for Player 1 (because the payoff to Player 1 when Player 2 chooses Q = 20 is the same for both Q = 10 and Q = 20) but it is weakly dominant. Player 1 choosing Q = 20 and Player 2 choosing Q = 20 is still a solution to the game, but now Player 1 has a weakly dominant strategy and Player 2 has a strictly dominant strategy. Note that it is possible that there are more solutions to the game, as we will discuss shortly.

<sup>&</sup>lt;sup>3</sup>Note that if one player only has a *weakly* dominant strategy and the other has a strictly dominant strategy that there may be more than one solution to the game. See the example a few sections below.

#### 2.2 Strictly and weakly dominated strategies

A related concept to strictly and weakly dominant strategies is the concept of strictly dominated strategies. Note the slight difference in terminology. A strictly dominated strategy is a strategy that does strictly worse than some other strategy. Suppose there are 7 strategies, labelled A-G. Now suppose that strategy D always does better than strategy F. Then we would say that strategy F is strictly dominated by strategy D. Note that strategy D does not have to be strictly dominant to strictly dominate strategy F. If D were strictly dominant this would mean that it always does better than A, B, C, E, F, and G. All that is being said about strategy D if strategy F is strictly dominated by strategy D is better than strategy F. In the 2-player quantity choice game, strategy Q = 10 is strictly dominated by strategy Q = 20for both players. Hopefully this makes sense – if one strategy (Q = 20 in this case) is strictly dominant, then all other strategies will be strictly dominated by it. A strictly dominated strategy should NEVER be part of the solution to the game because there is always some strategy that does better than it.

The last type of strategy to discuss is the weakly dominated strategy. A weakly dominated strategy does no better than some other strategy. Again, the difference between a weakly dominated strategy and a strictly dominated strategy is subtle. With a weakly dominated strategy there may be ties between the weakly dominated strategy and the other strategy. When thinking of the difference between strictly and weakly think of strictly as being a greater than sign, >, while weakly is a greater than or equal to sign,  $\geq$ . Recall from one paragraph ago that a strictly dominated strategy is NEVER part of a solution to the game, while a weakly dominated strategy MAY BE part of a solution to a game.

Consider the following 3x3 game:

		Player 1			
		Left	Center	Right	
	Top	7, 4	6,3	4,11	
Player 2	Middle	8, 8	10, 4	6,7	
	Bottom	18, 7	11,9	4, 6	

Again, first check for strictly dominant strategies. The easiest way to check for a strictly (or weakly) dominant strategy is to identify the strategy for each player that gives the highest amount in the game. This would have to be the strictly dominant strategy. For Player 2 that payoff is 18 and the strategy is Bottom. Note that if Player 1 chooses Left, then Player 2 would choose Bottom. If Player 1 would choose Center Player 2 would choose Bottom. But if Player 1 would choose Right Player 2 would want to choose Top. So Player 2 does not have a strictly or weakly dominant strategy. For Player 1 we need to check if Right is the strictly dominant strategy because 11 is Player 1's highest payoff. Clearly if Player 2 were to choose Top Player 1 would choose Right. But if Player 2 were to choose Middle then Player 1 would prefer to choose Left, so Right is NOT a strictly or weakly dominant strategy. Thus, this game cannot be solved by considering strictly or weakly dominant strategies.

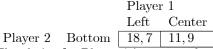
Next one can turn to looking for strictly dominated strategies. Compare Top and Middle for Player 2. If Player 1 chooses Left Player 2 chooses Middle (8 > 7). If Player 1 chooses Center Player 2 chooses Middle (10 > 6). If Player 1 chooses Right Player 2 chooses Middle (6 > 4). So Top is strictly dominated by Middle. Thus, Top can be removed from consideration in the game. The reason is that Player 2 would NEVER choose Top because it is strictly dominated by Middle. The strategy Top can be removed for both players because of the assumption of common knowledge – not only does Player 2 know that he will never use Top, but Player 1 knows this as well. So we can reduce the game to look like:

		Player 1				
		Left	$\operatorname{Center}$	Right		
Player 2	Middle	8,8	10, 4	6,7		
	Bottom	18,7	11,9	4, 6		

Now in this reduced game we can check to see if Middle is strictly dominated by Bottom or vice versa. It turns out that neither is strictly dominated. But if we now look at Player 1 and compare the strategies Left and Right we find that, in this reduced 2x3 game, Right is strictly dominated by Left. If Player 2 were to play Middle Player 1 would choose Left (8 > 7). If Player 2 were to choose Bottom Player 1 would choose Left (7 > 6). So now we can eliminate Right from the game, reducing the game to a 2x2 game.

		Player	r 1	
		Left	Center	
Player 2	Middle	8,8	10,4	1
	Bottom	18,7	11,9	

Again we can check to see if Left or Center is strictly dominant and we can see that neither are. So we return to Player 2 now. In the reduced 2x2 game, is either Bottom or Middle strictly dominant? The answer is yes, Bottom is strictly dominant because 18 > 8 and 11 > 10. So now we know that Player 2 would choose Bottom. This leads to further reducing the game to a 2x1 game:



The choice for Player 1 is now to choose Left and get 7 or choose Center and get 9. So Player 1 chooses Center and the resulting solution is that Player 2 chooses Bottom and Player 1 chooses Center.

The process that we just performed to find the solution to the game is called *iterated elimination of dominated strategies* (IEDS for short). Yes there are a lot of multisyllabic words strung together, but just think about what the phrase means. One strictly dominated strategy is eliminated for one player, then we turn to the other player and eliminate strictly dominated strategies, then we go back to the original player, etc., etc., until we either (1) reach a solution or (2) reach a point where no more dominated strategies exist in the reduced game (it is possible).

#### 2.3 Finding "solutions" without dominant or dominated strategies

Most games do not have dominant or dominated strategies. However, there are still solutions to these games. Consider the following game:

Two people wish to attend either a boxing match or an opera. Unfortunately, they have lost their cell phones and all other devices that allow for communication. If they both go to the boxing match, then Player 1 receives a payoff of 2 and Player 2 receives a payoff of 1. If they both go to the opera, then Player 1 receives a payoff of 1 and Player 2 receives a payoff of 2. However, if they show up at either event and the other person is not there they are deeply saddened and receive a payoff of 0.

This game is known as the "Battle of the Sexes" and it belongs to the general class of coordination games. In coordination games, it is usually the case that players are better off if they choose the same action than if they choose different actions. The strategic form of the game is:

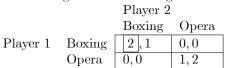
		Player 2		
		Boxing	Opera	
Player 1	Boxing	2,1	0,0	
	Opera	0,0	1, 2	

The easiest types of solutions are for those games with strictly or weakly dominant strategies. However, note that neither player has a strictly or weakly dominant strategy in this game. Since the game is only a 2x2 matrix, it should (hopefully) be clear that there are no strictly dominated strategies either. So the question then becomes, How do we solve games without dominant or dominated strategies?

For these strategic form games with a small number of players who each have a small number of strategies, the idea is to find a set of strategies (one for each player) such that neither player would like to change strategies given what the other player is choosing. Thus, each player would be playing a best response to the other player's strategy and would not be able to receive a higher payoff by changing his strategy. One way to find whether or not a set of strategies is a "solution" to the game is to look at an outcome cell and determine if either player could, by himself or herself, earn a higher payoff by switching his or her strategy (but the other player would keep the same strategy).<sup>4</sup> If either player would like to change his strategy, then that set of strategies cannot be a solution because someone wants to change. In essence, we are defining the concept of equilibrium here – the basic definition of equilibrium is that it is a state of rest or balance. The concept of Nash equilibrium will be formalized shortly.

 $<sup>^{4}</sup>$ Note that this method will work for ALL games, including those with strictly or weakly dominant strategies as well as those with strictly or weakly dominated strategies.

Another method of finding the equilibrium in a strategic form game is to consider what the best response of one player is to another player's choice of strategy is. Consider the "Battle of the Sexes" game. If Player 2 were to choose Boxing, Player 1's best response would be to choose Boxing because the payoff to Player 1 to choosing Boxing (2) is greater than the payoff to Player 1 of choosing Opera (0). If we could somehow make a note of this on the matrix so that we did not forget this it would be useful. Well, the matrix is ours to do what we want with it, so just circle (or square, or triangle, or enclose) Player 1's payoff of 2 in the game. This denotes that Boxing is Player 1's best response to Player 2's choice of Boxing. By "circling" I mean something like the following:



Now, what is Player 1's best response if Player 2 chooses Opera? It is Opera, because Player 1's payoff is 1, whereas Player 1's payoff is 0 if he chooses Boxing. Now we can enclose the 1 so that we have a complete set of best responses for Player 1:



The process needs to be repeated for Player 2. If Player 1 were to choose Boxing, Player 2's best response would be Boxing, because 1 > 0. So we enclose the 1 for Player 2. If Player 1 were to choose Opera, Player 2 would choose Opera because 2 > 0. After all the best responses are found, the matrix now looks like:



Any outcome cell where both (or all if there are more than 2 players) payoffs are enclosed is a solution to the game. As the Battle of the Sexes game shows, it is possible for multiple solutions to exist in games. To convince yourself of this, look at the outcome cells. If the players are in the Boxing, Boxing outcome, can either player unilaterally deviate by changing his strategy to make himself better off? No, if either changes then the player who changes will end up with 0, which is less than either 2 or 1. In the Opera, Opera outcome, if either player changes then that player will end up with 0, which is less than either 2 or 1. Now consider either the Boxing, Opera outcome or the Opera, Boxing outcome. If either player changes, then that Player will end up with either 2 or 1 instead of 0, so the Boxing, Opera outcome and the Opera, Boxing outcome are NOT solutions.

We can use this "enclosing the payoff method" for the other games we have seen, just to see that it delivers the same solution. Consider the first Prisoner's Dilemma game where both players have strictly dominant strategies:

		Player 2				
		Q =	10	Q = 20		
Player 1	Q = 10	\$11,	\$11	\$3,	\$16	$\square$
	Q = 20	\$16	, \$3	\$5	, \$5	Π

Note that the solution found is the same as when we found that both players had a strictly dominant strategy, where both players choose Q = 20. Now consider the modified version of this game where Player 1 only had a weakly dominant strategy:

		Player 2					
		Q = 10	Q = 20				
Player 1	Q = 10	\$11, \$11	\$5, \$16				
	Q = 20	\$16, \$3	\$5, \$5				
	.1 .	1	$\Sigma$ $+$ $\cdot$ $\cdot$	1			

There are two things to note here. First, if the highest payoff a player receives is the same for more than one strategy, then BOTH (or all) those payoffs should be enclosed. This is why both \$5 payoffs are enclosed for Player 1 – if Player 2 chooses Q = 20 then it does not matter which strategy Player 1 chooses (both are best responses). Second, as mentioned in an earlier footnote, if one player only has a weakly dominant strategy then that may add solutions to the game. This is what is seen here, as there are now two solutions to the game. Player 1 chooses Q = 20, Player 2 chooses Q = 20 is one solution. The other solution is Player 1 chooses Q = 10 and Player 2 chooses Q = 20. You can check to see that this is true by looking at the outcome cells – does either player receive a higher payoff by unilaterally deviating from those outcomes? Since the answer is no, they are both solutions to the game. Now consider the 3x3 game discussed above. Recall that we used IEDS to solve this game.

		Player 1				
		Left	Center	Right		
	Top	7,4	6, 3	4, 11		
Player 2	Middle	8, 8	10,4	[6], 7		
	Bottom	18,7	11,9	4, 6		
· · /1	4	11. 1	/ E	<u>· · · · · · · · · · · · · · · · · · · </u>		

Again, there are two things to note. First, this method finds the same solution to the game as IEDS: Player 2 chooses Bottom, Player 1 chooses Center. Second, we can easily identify a strategy that would not be played – Player 2 never uses Top as a best response to any strategy choice by Player 1. In this case it turns out that Top is a strictly dominated strategy, but that does not necessarily have to be the case (at least not as we have defined strictly dominated so far – but that will have to wait for a discussion of mixed strategies).

So now there is a general method for finding solutions to games. But I grow weary from using the term solution, and would like to use its proper name: Nash equilibrium.

## 3 Nash equilibrium

What we have been calling a solution is really a Nash equilibrium. Again, consider the term equilibrium, which means at rest or in balance. This just means that nothing or no one should be changing (or wanting to change) anything they do. For all the solutions that we have found, that is the similarity between them all. Technically, a Nash equilibrium is a set of strategies such that no player can unilaterally deviate from that set of strategies and make himself or herself strictly better off. The key points are that (1) it is a set of strategies (2) no player can unilaterally deviate (meaning by himself or herself) and (3) strictly better off (a player may be able to choose a different strategy and receive the same payoff, as in that modified quantity choice game, but not one that is STRICTLY greater).

Economists who do theoretical work are generally concerned with two concepts. The first is the notion of existence, as in: Does an equilibrium exist? Once we know that an equilibrium exists, we then turn to the notion of uniqueness, as in: Is the equilibrium unique? What we would really like is for the equilibrium to exist and be unique, and in much of the microeconomics that you might study (such as a well-specified consumer's choice with strictly convex preferences) we find this existence and uniqueness result. We have already seen, even with simple games, that uniqueness might be a problem in game theory. As we progress throughout the course we will make refinements to this notion of Nash equilibrium, with the idea behind those refinements being that we would like to eliminate certain Nash equilibria because they seem implausible given the structure of the game. In some cases, we are not able to arrive at a unique solution even with these refinements, and then a third question (beyond those of existence and uniqueness) arises, which is how is one equilibrium selected over the other. For now, here are two theorems about existence of (at least one, maybe more) Nash equilibria in games that we would be able to represent using the tools we have already developed.

**Theorem 1** Consider a normal form game with I players, where I is a finite number, and where each player has a finite number of strategies. If a game meets these criteria, then there exists at least one Nash equilibrium to the game.

So this theorem gives us two conditions needed for existence of a Nash equilibrium. The number of players needs to be finite and the number of strategies each player has also needs to be finite. These seem like reasonable assumptions, and all the games we have studied so far meet these two criteria.

Here is a more advanced (and useful when showing a picture) version of the theorem. I give this to you for two reasons. First, if you are considering going to graduate school for economics (particularly a PhD program), then you should know now that you will see many, many symbols and terms like this. Second,

it's a useful statement of existence for Nash equilibrium in games that we will discuss later in class (such as the Cournot quantity choice game). There is some notation here that I will define. The term  $\Gamma_N$  simply means "normal form game". The term I simply refers to the number of players. The term  $\{S_i\}$  simply means the set of available strategies for each player i and the term  $\{u_i\}$  simply means the set of payoff (or utility) functions for each player i. A normal form game is just all of these things, as we initially defined it in words.

**Theorem 2** A Nash Equilibrium exists in game  $\Gamma_N = [I, \{S_i\}, \{u_i\}]$  if for all i = 1, ..., I

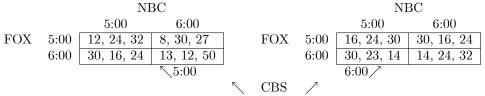
- 1.  $S_i$  is a nonempty, convex, and compact subset of some Euclidean space  $\mathbb{R}^M$
- 2.  $u_i(s_1,...,s_I)$  is continuous in  $(s_1,...,s_I)$  and quasiconcave in  $s_i$

For the first part think of a strategy space that is something like the closed interval from [0, 1]. Alternatively, think about a firm determining how much of a good to produce. The lowest amount they can produce is 0, while the largest amount (call it  $\overline{C}$ ) they can produce is constrained by their available technology, the amount of money they can spend, and the prices of inputs (for simplicity we assume that firms can produce any real number between 0 and  $\overline{C}$ ). Thus, each firm would have a strategy space of  $[0, \overline{C}]$ , which will satisfy the nonempty, convex, and compact portions of the  $S_i$ . The Euclidean space  $\mathbb{R}^M$  just means some space of real numbers – the Euclidean space  $\mathbb{R}^2$  is something you all are familiar with, it is the Cartesian plane on which you draw all of your graphs. For the second part, think about the fact that there are no "large jumps" in payoffs when moving from one strategy to another that is "close" to it (if the firm changes from producing 1 unit to producing 1.00001 units there is not a large change in payoff). The quasiconcave part simply means that the utility function has a single maximum (or a supremum). The idea behind this theorem relies on a fixed-point theorem. An overview of a fixed-point theorem is that if we take a function f(x)with domain and range of [0, 1] then there exists at least one fixed-point, which is a point where  $f(x^*) = x^*$ . In the case of the games we are discussing, there are points in the best response correspondences of players that map back into themselves – essentially, like the rest of economics, there is a point where two lines cross (except in game theory those lines may cross multiple times, meaning there is more than one equilibrium). Please note that neither version of the theorem makes any statement about uniqueness of the equilibrium.

### 4 **3-player games**

Once these simultaneous games get beyond 3 players it becomes a little unwieldy to write them down in the strategic form. Even the 3 player games are a little unwieldy. Nonetheless, here is an example of a 3-player simultaneous move game.

Consider the problem faced by three major network affiliate television stations in the western Wisconsin area: Fox Channel 25, NBC Channel 13, and CBS Channel 8. All three stations have the option of airing the evening network news program at 5:00 P.M. or in a delayed broadcast at 6:00 P.M. Each station's' objective is to maximize its viewing audience in order to maximize its advertising revenue. The following representation describes the share of western Wisconsin's total population that is "captured" by each station as a function of the times at which the new programs are aired. The stations make their choice simultaneously. The payoffs are listed according to the order Fox, NBC, and CBS.<sup>5</sup>

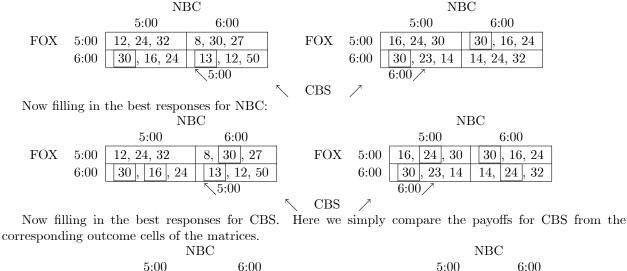


Now what should be done to find the Nash equilibrium (solution) to this game? Instead of holding one player's strategy constant, now we need to hold the other two players' strategies constant. So we need to answer the following questions for FOX:

 $<sup>^{5}</sup>$  The ordering of payoffs is generally: row, column, "matrix choice" player, but this convention is less stable than the one with only 2 players.

- 1. What would FOX choose if NBC chose 5 and CBS chose 5?
- 2. What would FOX choose if NBC chose 6 and CBS chose 5?
- 3. What would FOX choose if NBC chose 5 and CBS chose 6?
- 4. What would FOX choose if NBC chose 6 and CBS chose 5?

Then we would have to answer the same questions for CBS (holding FOX and NBC's strategies constant) and NBC (holding FOX and CBS' strategies constant). So there would be 12 best responses that we would need to find, four for each player. Filling in the best response for FOX.:



		5:00	6:00				5:00	6:00
FOX	5:00	12, 24, 32	8, 30, 27		FOX	5:00	16, 24, 30	30, 16, 24
	6:00	30, 16, 24	13, 12, 50			6:00	30, 23, 14	14, 24, 32
			5:00				6:00	
				5	CBS	7		

Note that in this game CBS has a strictly dominant strategy of choosing 5:00pm. The only Nash equilibrium to the game is FOX chooses 6pm, NBC chooses 5pm, and CBS chooses 5pm.