

# 1 Describing Mixed Strategy Nash Equilibria

Consider the following two games.

The first game is one you might be familiar with: Rock, Paper, Scissors. In case you are not, in this game there are 2 players who simultaneously determine which object to form with their fingers. Each player has 3 strategies – form a Rock, form Paper, or form Scissors. If both players form the same object then they tie and receive 0. If one player forms a Rock and the other forms Scissors then Rock wins and receives a payoff of 1 while Scissors loses and receives a payoff of  $-1$ . If one player forms Scissors and the other forms Paper then Scissors wins and receives a payoff of 1 while Paper loses and receives a payoff of  $-1$ . If one player forms Paper and the other forms Rock then Paper wins and receives a payoff of 1 and Rock loses and receives a payoff of  $-1$ . Essentially, Rock smashes Scissors, Scissors cut Paper, and Paper covers Rock. The normal form version of the game is:

		Player 2		
		Rock	Paper	Scissors
Player 1	Rock	0, 0	$-1, 1$	$1, -1$
	Paper	$1, -1$	0, 0	$-1, 1$
	Scissors	$-1, 1$	$1, -1$	0, 0

Now consider a slightly simpler game, Matching Pennies. In this game there are two players who move simultaneously. Each player places a penny on the table. If the pennies match (both heads or both tails) then Player 1 receives a payoff of 1 and Player 2 receives a payoff of  $(-1)$ . If the pennies do not match (one heads and one tails), then Player 1 receives a payoff of  $(-1)$  and player 2 receives a payoff of 1. The matrix representation of the game is here:

		Player 2	
		Heads	Tails
Player 1	Heads	$1, -1$	$-1, 1$
	Tails	$-1, 1$	$1, -1$

Note that neither player has a dominant strategy nor a dominated strategy in either Rock, Paper, Scissors or Matching Pennies. We can then look at the best response correspondences for the players by enclosing the payoffs in a square. For Rock, Paper, Scissors we have:

		Player 2		
		Rock	Paper	Scissors
Player 1	Rock	0, 0	$-1, \boxed{1}$	$\boxed{1}, -1$
	Paper	$\boxed{1}, -1$	0, 0	$-1, \boxed{1}$
	Scissors	$-1, \boxed{1}$	$\boxed{1}, -1$	0, 0

For Matching Pennies we have:

		Player 2	
		Heads	Tails
Player 1	Heads	$\boxed{1}, -1$	$-1, \boxed{1}$
	Tails	$-1, \boxed{1}$	$\boxed{1}, -1$

There is now a problem. Nash's theory says that an equilibrium exists if there are a finite number of players and if each of those players has a finite number of strategies. There are 2 players (fairly finite) and each player has 3 strategies or 2 strategies (also fairly finite). But there are no outcome cells that have both payoffs enclosed. Is Nash wrong?

## 1.1 Playing strategies with probabilities

Thus far in the course we have considered only strategies which are played 100% of the time. A strategy that is played 100% of the time (like Confess in the Prisoner's Dilemma) is known as a *pure strategy*. All the strategy choices listed in the strategic form of the game are pure strategies. However, players may choose to select a strategy randomly from among their set of strategies – this is how some of the "professional" Rock, Paper, Scissors players make their strategy choices.<sup>1</sup> Intuitively this should make sense – one could never rationally ALWAYS choose Rock and have this choice be part of a Nash equilibrium to the game.<sup>2</sup> If one

<sup>1</sup>The "professional" players being professional poker players who play Rochambeau (Rock, Paper, Scissors) in addition to playing poker.

<sup>2</sup>Simpson's Episode:

ALWAYS chose Rock, then opposing players would ALWAYS choose Paper and would win every single time. But this cannot be a Nash equilibrium to the game, because if opposing players ALWAYS choose Paper, then the player who chooses Rock would like to switch to Scissors. Then the opposing players would like to switch to Rock, then the player would like to switch to Paper, now the opposing players are choosing Scissors, and now the original player chooses Rock. Thus, there is no equilibrium if we consider only strategies which must have a 100% weight on them – basically, someone always wants to switch strategies, which violates the notion of equilibrium meaning "at rest".

The question is how do we find the probabilities so that no player wishes to switch strategies? There is a theorem – I will spare you all the gory details of this theorem – that essentially states that *a set of strategies is a mixed strategy Nash equilibrium if and only if the players are indifferent among their pure strategies*. Think about what this means in the Rock, Paper, Scissors game. It means that Player 2 would have the same expected value if Rock were chosen or if Scissors were chosen or if Paper were chosen. This is only true for strategies that a player would play as a best response, so that a strictly dominated strategy would NOT have to meet this requirement (it would be impossible for a strictly dominated strategy to meet this requirement and still satisfy the laws of probability).

## 1.2 A digression on probability and expected value (or expected utility)

Before beginning the discussion on how to find a mixed strategy Nash equilibrium (MSNE) there needs to be a short refresher on Kolmogorov's axioms of probability and expected value.

### 1.2.1 Axioms of Probability

These are Kolmogorov's three axioms:

1. The probability that an event will occur is greater than or equal to 0. No negative probabilities, although some probabilities can be 0.
2. The probability that some event will occur is 1. Basically, something happens.
3. The sum of the probabilities of the events is 1. Taken together with the first two axioms, this means that there can be no events with a probability greater than 1. Basically, something cannot occur with 110% probability.

That is it for the axioms of probability. They are fairly simple and intuitive but necessary when determining MSNE.

### 1.2.2 Expected value (or expected utility)

First, note that there is (generally) a difference between the terms *expected value* and *expected utility*.<sup>3</sup> For expected value simply find the weighted average of the events. As an example, consider a game which a fair coin (one that lands on heads 50% of the time and tails the other 50% of the time) pays an individual \$2 if it lands on Heads and (−\$3) if it lands on Tails. The *expected value* is then the weighted average (product of the probability an event occurs and its value) or  $\frac{1}{2} * \$2 + \frac{1}{2} * (-\$3) = (-\$0.5)$ . Thus an individual would, on average, lose 50 cents each time this game was played. Not a very attractive proposition, but this is why Vegas makes money.

As for *expected utility*, this is the weighted average of the UTILITY values of the outcomes. It is generally assumed that individuals have utility functions denoted by  $U(\cdot)$ . This utility function will take either bundles of goods or amounts of money or ... anything really and translate it into a "utility value". If we let  $U(\cdot) = x - 1$  then  $U(2) = 1$  and  $U(-3) = -4$ , so that the expected utility is  $\frac{1}{2} * 1 + \frac{1}{2} * (-4) = (-\frac{3}{2})$ .

The difference between the two terms is subtle, as the calculation is basically the same. For our purposes we can assume that the payoffs in the strategic form version of the game represent utility values. Alternatively, we can assume, at least for now, that the individuals are all risk-neutral and have a utility function such that  $U(x) = x$ .

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<sup>3</sup>Unless the economic agent happens to be risk-neutral.

## 2 Finding MSNE

The following describes the process to find MSNE. For the games we have seen it relies heavily on algebra. It also relies on that theorem about players being indifferent among their pure strategies. I promise one thing – it will get messy.

### 2.1 Matching Pennies MSNE

Consider the Matching Pennies game since it has less strategies. I will add one more row and column.<sup>4</sup> Let  $p_{1H}$  denote the probability that Player 1 assigns to playing Heads and  $p_{1T}$  denote the probability that Player 1 assigns to playing Tails. Let  $q_{2H}$  denote the probability that Player 2 assigns to playing Heads and  $q_{2T}$  denote the probability that Player 2 assigns to playing Tails.

		Player 2		
		$q_{2H}$	$q_{2T}$	
		Heads	Tails	
Player 1	$p_{1H}$	Heads	1, -1	-1, 1
	$p_{1T}$	Tails	-1, 1	1, -1

Now, we need Player 1 to be indifferent among his pure strategies. All this means is that the expected utility for Player 1 of playing Heads has to equal the expected utility for Player 1 of playing Tails. Rather than continually writing out "expected utility for Player 1 of playing Heads" I will use some shorthand. Let  $E_1 [Heads]$  mean "expected utility for Player 1 of playing Heads" and let  $E_1 [Tails]$  mean "expected utility for Player 1 of playing Tails". So what is needed is:

$$E_1 [Heads] = E_1 [Tails]$$

So far so good. Now we need to find  $E_1 [Heads]$  and  $E_1 [Tails]$ . If Player 1 always chooses Heads (this is what it means to play a pure strategy), then he will receive 1 with probability  $q_{2H}$  and -1 with probability  $q_{2T}$ . Thus, we have:

$$E_1 [Heads] = 1 * q_{2H} + (-1) * q_{2T}$$

If Player 1 always chooses Tails then he will receive -1 with probability  $q_{2H}$  and 1 with probability  $q_{2T}$ . Thus:

$$E_1 [Tails] = (-1) * q_{2H} + 1 * q_{2T}$$

Now set  $E_1 [Heads] = E_1 [Tails]$  and simplify to find:

$$\begin{aligned} 1 * q_{2H} + (-1) * q_{2T} &= (-1) * q_{2H} + 1 * q_{2T} \\ q_{2H} - q_{2T} &= -q_{2H} + q_{2T} \\ 2q_{2H} &= 2q_{2T} \\ q_{2H} &= q_{2T} \end{aligned}$$

So we find that  $q_{2H} = q_{2T}$  but cannot simplify any farther without adding one more equation. We have 2 unknowns but 1 equation so we are not going to be able to solve for both unknowns with only 1 equation. But there is a second equation that is given to us by the probability axioms. This equation is:

$$q_{2H} + q_{2T} = 1$$

because we know that probabilities sum to 1. So now we have:

$$\begin{aligned} q_{2H} &= q_{2T} \\ q_{2H} + q_{2T} &= 1 \end{aligned}$$

and we can solve for  $q_{2H}$  and  $q_{2T}$ . By substitution we find that:

$$\begin{aligned} q_{2H} + q_{2H} &= 1 \\ 2q_{2H} &= 1 \\ q_{2H} &= \frac{1}{2} \end{aligned}$$

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<sup>4</sup>This is just for illustrative purposes and is not typically done.

and then we know that  $q_{2T} = \frac{1}{2}$  as well. Thus, we have found the probabilities that Player 2 would use to make Player 1 indifferent among his pure strategies. Basically, Player 2 flips the coin to make Player 1 indifferent. You can note that with  $q_{2H} = q_{2T} = \frac{1}{2}$  that if Player 1 chooses to play Heads all the time his expected utility is 0, and if Player 1 chooses to play Tails all the time his expected utility is also 0. Actually, if Player 1 were to choose to play Heads 75% of the time and Tails 25% of the time his expected utility is STILL 0. Now, we are only half done. We still need to solve for  $p_{1H}$  and  $p_{1T}$ . We can do this by setting the expected utility for Player 2 of playing Heads equal to the expected utility for Player 2 of playing Tails or:

$$E_2 [Heads] = E_2 [Tails]$$

Now we have to find  $E_2 [Heads]$  and  $E_2 [Tails]$ . If Player 2 always chooses Heads then he will receive  $-1$  with probability  $p_{1H}$  and  $1$  with probability  $p_{1T}$ . Or:

$$E_2 [Heads] = (-1) * p_{1H} + 1 * p_{1T}$$

If Player 2 always chooses Tails then he will receive  $1$  with probability  $p_{1H}$  and  $-1$  with probability  $p_{1T}$ . Or:

$$E_2 [Tails] = 1 * p_{1H} + (-1) * p_{1T}$$

Putting the two together we find:

$$\begin{aligned} E_2 [Heads] &= E_2 [Tails] \\ (-1) * p_{1H} + 1 * p_{1T} &= 1 * p_{1H} + (-1) * p_{1T} \\ -p_{1H} + p_{1T} &= p_{1H} - p_{1T} \\ 2p_{1T} &= 2p_{1H} \\ p_{1T} &= p_{1H} \end{aligned}$$

Again, we need a second equation,  $p_{1H} + p_{1T} = 1$ . Hopefully this looks familiar, as we will find  $p_{1T} = p_{1H} = \frac{1}{2}$ . So if Player 1 chooses to play a 50/50 mix of Heads and Tails Player 2 will be indifferent among his pure strategies. We have now found our first MSNE. To write this equilibrium out it would be:

MSNE: Player 1 chooses to play Heads 50% of the time and Tails 50% of the time. Player 2 chooses to play Heads 50% of the time and Tails 50% of the time.

## 2.2 Rock, Paper, Scissors MSNE

This is another problem of algebra, only now each player has 3 strategies. Let  $p_{1rock}$  be the probability that Player 1 plays Rock,  $p_{1paper}$  be the probability that Player 1 plays Paper, and  $p_{1scissors}$  be the probability that Player 1 plays Scissors. Let  $p_{2rock}$ ,  $p_{2paper}$ , and  $p_{2scissors}$  be the respective probabilities for Player 2. Player 1's goal is to make Player 2 indifferent among his pure strategies. We know that if Player 1 uses his mixed strategy and Player 2 ALWAYS chooses Rock, then Player 2 will receive 0 with probability  $p_{1rock}$ ,  $-1$  with probability  $p_{1paper}$ , and  $1$  with probability  $p_{1scissors}$ . If Player 2 ALWAYS chooses Paper, then Player 2 will receive  $1$  with probability  $p_{1rock}$ ,  $0$  with probability  $p_{1paper}$ , and  $-1$  with probability  $p_{1scissors}$ . If Player 2 ALWAYS chooses Scissors, then Player 2 will receive  $-1$  with probability  $p_{1rock}$ ,  $1$  with probability  $p_{1paper}$ , and  $0$  with probability  $p_{1scissors}$ . We can say that the expected value for Player 2 of playing each of these strategies is then:

$$\begin{aligned} E_2 [Rock] &= 0 * p_{1rock} + (-1) * p_{1paper} + 1 * p_{1scissors} \\ E_2 [Paper] &= 1 * p_{1rock} + 0 * p_{1paper} + (-1) * p_{1scissors} \\ E_2 [Scissors] &= (-1) * p_{1rock} + 1 * p_{1paper} + 0 * p_{1scissors} \end{aligned}$$

We now have 3 unknowns -  $p_{1rock}$ ,  $p_{1paper}$ , and  $p_{1scissors}$ . It must be that Player 2 has:

$$\begin{aligned} E_2 [Rock] &= E_2 [Paper] \\ E_2 [Paper] &= E_2 [Scissors] \end{aligned}$$

By transitivity, this gives that  $E_2[Rock] = E_2[Scissors]$ . However, there are only 2 equations and 3 unknowns. The third equation is that probabilities must sum to 1, so that our 3 equations are now:

$$\begin{aligned} 0 * p1_{rock} + (-1) * p1_{paper} + 1 * p1_{scissors} &= 1 * p1_{rock} + 0 * p1_{paper} + (-1) * p1_{scissors} \\ 1 * p1_{rock} + 0 * p1_{paper} + (-1) * p1_{scissors} &= (-1) * p1_{rock} + 1 * p1_{paper} + 0 * p1_{scissors} \\ p1_{rock} + p1_{paper} + p1_{scissors} &= 1 \end{aligned}$$

We can now solve the 3 equations for  $p1_{rock}$ ,  $p1_{paper}$ , and  $p1_{scissors}$ . Rewrite  $p1_{scissors} = 1 - p1_{rock} - p1_{paper}$  and substitute into the first two equations. We get:

$$\begin{aligned} (-1) * p1_{paper} + 1 * (1 - p1_{rock} - p1_{paper}) &= 1 * p1_{rock} + (-1) * (1 - p1_{rock} - p1_{paper}) \\ 1 * p1_{rock} + (-1) * (1 - p1_{rock} - p1_{paper}) &= (-1) * p1_{rock} + 1 * p1_{paper} \end{aligned}$$

Now it is just a simple matter of solving the system.

$$\begin{aligned} -p1_{paper} + 1 - p1_{rock} - p1_{paper} &= p1_{rock} - 1 + p1_{rock} + p1_{paper} \\ p1_{rock} - 1 + p1_{rock} + p1_{paper} &= -p1_{rock} + p1_{paper} \end{aligned}$$

Simplifying:

$$\begin{aligned} -3 * p1_{paper} + 2 &= 3 * p1_{rock} \\ 3 * p1_{rock} - 1 &= 0 \end{aligned}$$

We have  $p1_{rock} = \frac{1}{3}$  from the last equation. Substituting that into the first equation we get:

$$-3 * p1_{paper} + 2 = 3 * \frac{1}{3}$$

Solving for  $p1_{paper}$  gives  $p1_{paper} = \frac{1}{3}$ . Now, using  $p1_{scissors} = 1 - p1_{rock} - p1_{paper}$  we find that  $p1_{scissors} = \frac{1}{3}$ . So if Player 1 plays Rock  $\frac{1}{3}$  of the time, Paper  $\frac{1}{3}$  of the time, and Scissors  $\frac{1}{3}$  of the time this will make Player 2 indifferent over his pure strategies. The expected value for Player 2 of playing Rock is 0, of playing Paper is 0, and of playing Scissors is 0. We can also check some mixed strategies for Player 2. If Player 2 plays Rock 50% of the time and Paper 50% of the time his expected value of playing that strategy is 0. If Player 2 plays Rock 50% of the time, Paper 25% of the time, and Scissors 25% of the time his expected value is 0. This is what is required when finding a MSNE.

Now, we have only found the probabilities for Player 1. We know that all strategies (pure or mixed) by Player 2 provide the same expected value when Player 1 chooses Rock, Paper, and Scissors  $\frac{1}{3}$  of the time each. Can Player 2 then just choose any strategy? No, because not any strategy will make Player 1 indifferent over his strategies. We then have to find  $p2_{rock}$ ,  $p2_{paper}$ , and  $p2_{scissors}$  using the same methodology that we just used to find  $p1_{rock}$ ,  $p1_{paper}$ , and  $p1_{scissors}$ . Luckily, in the Rock, Paper, Scissors game the two players have symmetric payoffs and strategies, so that the probabilities for Player 2 that make Player 1 indifferent between his strategies are  $p2_{rock} = \frac{1}{3}$ ,  $p2_{paper} = \frac{1}{3}$  and  $p2_{scissors} = \frac{1}{3}$ . The mixed strategy Nash equilibrium for this game (and the only Nash equilibrium of this game) is that Player 1 chooses Rock, Paper, and Scissors each with  $\frac{1}{3}$  probability, and Player 2 chooses Rock, Paper, and Scissors each with  $\frac{1}{3}$  probability. Note that the expected value for both players of playing this game is 0.

Now, suppose that Player 2 uses a different strategy, like  $p2_{rock} = \frac{1}{3}$ ,  $p2_{paper} = \frac{1}{2}$  and  $p2_{scissors} = \frac{1}{6}$ . Should Player 1 respond with the exact same strategy? NO. If Player 1 uses the exact same strategy as Player 2 then his expected value of using that strategy is also 0. Player 1 can do BETTER than 0 if he uses a strategy like "Always choose Scissors". If Player 1 uses this strategy against  $p2_{rock} = \frac{1}{3}$ ,  $p2_{paper} = \frac{1}{2}$  and  $p2_{scissors} = \frac{1}{6}$  then Player 1 will have an expected value of  $\frac{1}{6}$  because he will earn  $(-1)$  with  $\frac{1}{3}$  probability (when Player 2 picks Rock), 1 with  $\frac{1}{2}$  probability (when Player 2 picks Paper) and 0 with  $\frac{1}{6}$  probability (when Player 2 chooses Scissors). Of course, if Player 1 always chooses Scissors then Player 2 would always choose Rock. But then Player 1 would always choose Paper. And the cycle would go on. The only time it stops is when  $p1_{rock} = \frac{1}{3}$ ,  $p1_{paper} = \frac{1}{3}$  and  $p1_{scissors} = \frac{1}{3}$  and  $p2_{rock} = \frac{1}{3}$ ,  $p2_{paper} = \frac{1}{3}$  and  $p2_{scissors} = \frac{1}{3}$ .

## 2.3 Coordination game MSNE

Finally, we return to the coordination game to find its MSNE. Here is the game again:

		Player 2	
		Boxing	Opera
Player 1	Boxing	2, 1	0, 0
	Opera	0, 0	1, 2

There were two PSNE – Player 1 choose Boxing, Player 2 choose Boxing; and Player 1 choose Opera, Player 2 choose Opera. There is also a MSNE to this game. Let  $p1_{boxing}$  and  $p1_{opera}$  be the probabilities with which Player 1 chooses Boxing and Opera respectively. Let  $p2_{boxing}$  and  $p2_{opera}$  be the probabilities with which Player 2 chooses Boxing and Opera respectively. Player 1 must make Player 2 indifferent over his 2 pure strategies, so:

$$E_2 [Boxing] = E_2 [Opera]$$

Or:

$$1 * p1_{boxing} + 0 * p1_{opera} = 0 * p1_{boxing} + 2 * p1_{opera}$$

We also have that  $p1_{boxing} + p1_{opera} = 1$ . Using these two equations we find that:

$$p1_{boxing} = 2(1 - p1_{boxing})$$

Or:

$$p1_{boxing} = \frac{2}{3}$$

This means that  $p1_{opera} = \frac{1}{3}$ . Now we need to find Player 2's mixed strategy. Note that while this game looks symmetric it is not. We need:

$$E_1 [Boxing] = E_1 [Opera]$$

Or:

$$2 * p2_{boxing} + 0 * p2_{opera} = 0 * p2_{boxing} + 1 * p2_{opera}$$

Using  $p2_{boxing} + p2_{opera} = 1$  we have:

$$2p2_{boxing} = 1 - p2_{boxing}$$

Or:

$$p2_{boxing} = \frac{1}{3}$$

This means that  $p2_{opera} = \frac{2}{3}$ . Thus, the MSNE for the Boxing-Opera game is that Player 1 chooses Boxing with  $\frac{2}{3}$  probability and Opera with  $\frac{1}{3}$  probability and Player 2 chooses Boxing with  $\frac{1}{3}$  probability and Opera with  $\frac{2}{3}$  probability. Note that the expected value of either player from playing this set of strategies is  $\frac{2}{3}$ , which is lower (for both players) than following either one of the two pure strategy Nash equilibria. But with these mixed strategies neither player can do any better by changing his strategy.

### 2.3.1 It's your opponents payoffs that matter

In the Boxing-Opera game it looks fairly reasonable – each player chooses the venue that he prefers with a higher probability than the other venue. But what if the game looked like:

		Player 2	
		Boxing	Opera
Player 1	Boxing	100, 1	0, 0
	Opera	0, 0	1, 2

In this game, Player 1 REALLY likes going to the Boxing match with Player 2. Would the probabilities that Player 1 used for his mixed strategy in the earlier version (where the 100 was a 2) change?

No.

Player 1's probabilities depend on Player 2's payoffs. They have nothing whatsoever to do with his own payoffs. In fact, we could turn the 100 into a  $\frac{1}{2}$  and Player 1's probabilities would not change. But Player

2's probabilities would change because Player 1's payoffs had changed. In this new game (with the 100 payoff), Player 2's Nash equilibrium mixed strategy would be  $p_{2_{boxing}} = \frac{1}{101}$  and  $p_{2_{opera}} = \frac{100}{101}$ . Because Player 1 has a larger payoff of going to the Boxing match this actually reduces the amount of times the 2 players end up at the boxing match.

### 3 MSNE when a player has strictly dominated strategies

In order to find a MSNE both players must choose probabilities that will make the other player indifferent over playing any of their other strategies that they would play with positive probability. Now we discuss the last part of that statement, the part about "that they would play with positive probability". Consider a modified version of the Prisoner's Dilemma:

		Player 2	
		A	B
Player 1	A	5, 5	16, 3
	B	3, 16	11, 11

For both players, strategy B is strictly dominated by strategy A. It is impossible to make either player indifferent between A and B unless the laws of probability are violated.<sup>5</sup> In games where players have a strictly dominated strategy those strictly dominated strategies need to be eliminated before finding the MSNE. So with this Prisoner's Dilemma there would be no MSNE because strategy B would be eliminated for both players and ... well, there won't be any strategies over which a player could mix because only strategy A remains.

Consider the following game:

		P2		
		D	E	F
P1	A	3, 2	4, 1	0, 4
	B	1, 3	3, 1	4, 5
	C	5, 5	3, 1	1, 4

First note that there are PSNE at: (1) P1 choose C, P2 choose D and (2) P1 choose B, P2 choose F. But you should also note that strategy E (which always plays P2 a payoff of 1) is strictly dominated by both strategy D and strategy F. So P2 would NEVER choose E and this strategy can be removed from the matrix. Once this strategy is removed then strategy A is strictly dominated by strategy C and so A can also be removed leaving:

		Player 2	
		D	F
Player 1	B	1, 3	4, 5
	C	5, 5	1, 4

Now there are no strategies which are strictly dominated so the process of finding the MSNE can begin. And since this is a 2x2 game it is not that difficult. Let  $P_{2D}$  be the probability that P2 chooses D and  $P_{2F} = 1 - P_{2D}$  be the probability that P2 chooses F. Setting P1's expected values equal to each other we have:

$$\begin{aligned}
 E_1 [B] &= E_1 [C] \\
 1 * P_{2D} + 4 * (1 - P_{2D}) &= 5 * P_{2D} + 1 * (1 - P_{2D}) \\
 P_{2D} + 4 - 4P_{2D} &= 5P_{2D} + 1 - P_{2D} \\
 3 - 3P_{2D} &= 4P_{2D} \\
 3 &= 7P_{2D} \\
 \frac{3}{7} &= P_{2D}
 \end{aligned}$$

So P2 would choose strategy D with probability  $\frac{3}{7}$  and strategy F with probability  $\frac{4}{7}$ . Now to find P1's probabilities let  $P_{1B}$  be the probability that P1 chooses B and  $P_{1C} = 1 - P_{1B}$  be the probability that P1

<sup>5</sup>Technically, if Player 2 chooses A with probability  $\frac{5}{3}$  and B with probability  $-\frac{2}{3}$  then Player 1 will be indifferent between A and B (and his expected value will be  $-\frac{7}{3}$ ). But hopefully it is obvious that these probabilities, while they sum to 1, violate the rules that probabilities must be greater than or equal to zero and less than or equal to 1.

chooses  $C$ . Setting P2's expected values equal to each other we have:

$$\begin{aligned} E_2[D] &= E_2[F] \\ 3 * P_{1B} + 5 * (1 - P_{1B}) &= 5 * P_{1B} + 4 * (1 - P_{1B}) \\ 3P_{1B} + 5 - 5P_{1B} &= 5P_{1B} + 4 - 4P_{1B} \\ 1 &= 3P_{1B} \\ \frac{1}{3} &= P_{1B} \end{aligned}$$

So P1 would choose strategy  $B$  with probability  $\frac{1}{3}$  and strategy  $C$  with probability  $\frac{2}{3}$ .

Now let's look at the actual expected values of the strategies of both players given the probabilities used by the other player. Start with P1's expected values first, realizing that (1) we are looking at all three strategies (A, B, and C) and (2) that P2 is choosing strategy E with a probability of 0.

$$\begin{aligned} E_1[A] &= \frac{3}{7} * 3 + 0 * 4 + \frac{4}{7} * 0 = \frac{9}{7} \\ E_1[B] &= \frac{3}{7} * 1 + 0 * 3 + \frac{4}{7} * 4 = \frac{19}{7} \\ E_1[C] &= \frac{3}{7} * 5 + 0 * 3 + \frac{4}{7} * 1 = \frac{19}{7} \end{aligned}$$

Now, P1 receives the same expected value from choosing  $B$  or  $C$ , which is good. They receive a lower expected value from choosing strategy  $A$  (which we did not use in our calculation because it was strictly dominated), which is also good because  $A$  should give a lower value because it was strictly dominated. The reason I did the check was to show you that, given the mixed strategy probabilities used by P2, P1 would not switch to strategy  $A$ .

Now consider P2's expected values:

$$\begin{aligned} E_2[D] &= 0 * 2 + \frac{1}{3} * 3 + \frac{2}{3} * 5 = \frac{13}{3} \\ E_2[E] &= 0 * 1 + \frac{1}{3} * 1 + \frac{2}{3} * 1 = \frac{3}{3} \\ E_2[F] &= 0 * 4 + \frac{1}{3} * 5 + \frac{2}{3} * 4 = \frac{13}{3} \end{aligned}$$

Again, P2 receives the same expected value if he uses  $D$  or  $F$  (which we used in our calculation) but a lower expected value if he were to switch to  $E$ . P2 receives a lower expected value from  $E$  because it was strictly dominated. Again, the purpose of this is to show you that P2 would not switch to strategy  $E$ .

## 4 Randomize

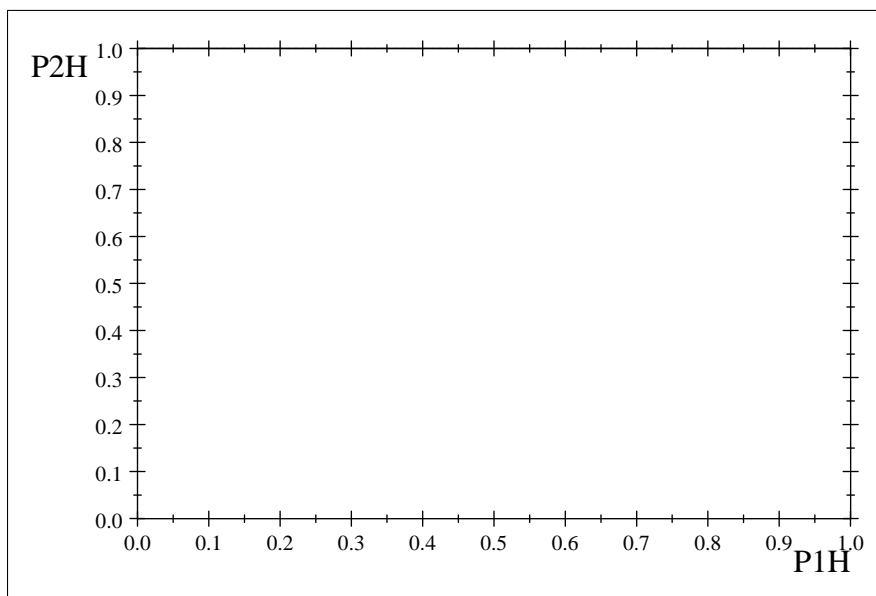
One last note on mixed strategies. If you only play a game that requires mixed strategies once in your life then it can never be shown that you did not correctly calculate the mixed strategy (there is only observation after all). However, consider playing Rock, Paper, Scissors repeatedly. It would be easy to devise a statistical test that determines whether or not your play is consistent with the probabilities of the MSNE. However, the key to using a MSNE is to *RANDOMIZE*. If you played 99 games of Rock, Paper, Scissors and you followed the strategy of "Choose Rock in the first game", "Choose Paper in the second game", and "Choose Scissors in the third game", then repeat (Rock in 4<sup>th</sup>, 7<sup>th</sup>, 10<sup>th</sup> etc. games, Paper in 5<sup>th</sup>, 8<sup>th</sup>, 11<sup>th</sup> etc. games, Scissors in 6<sup>th</sup>, 9<sup>th</sup>, 12<sup>th</sup> etc. games) it would not be very difficult to beat you every time because the pattern is predictable. Even though you would be playing the strategies in the correct proportions you would not end up earning 0 on average if you were playing someone with any shred of intelligence. The person would beat you practically every time, and certainly every time after the 10<sup>th</sup> round or so. Sometimes when ESPN shows the World Series of Poker they will cut away to the game of Rochambeau (which is the fancy way of saying Rock, Paper, Scissors). Some people when playing the game will use a randomizing device to determine their strategy for the game (one person in particular uses the first character of a dollar



bill to determine what to do – I am still not quite certain what the mechanics of the process are). The key is to have the randomizing device yield the probabilities of  $\frac{1}{3}$  for each strategy. Thus, throwing a single die and assigning the rolls of 1 and 2 to choosing Rock, 3 and 4 to choosing Paper, and 5 and 6 to choosing Scissors would be one method of randomizing with  $\frac{1}{3}$  probability for each strategy.

## 5 Digression on plotting best response "functions"

Note that I am using the term "function" loosely here. Some of these best response "functions" will not pass a vertical line test, and so they are not really "functions". But the term "best response function" is taken, in this context, to mean a plot of the player's best responses. There is an alternative method of thinking about this particular game which gets back to the "more rigorous" theorem on the existence of Nash equilibria in these simple games. We can consider that instead of each player having a strategy space of  $\{Heads, Tails\}$  the player now has a strategy space of assigning a probability from 0 to 1 to Heads (call it  $p_H$ ), with the realization that the probability assigned to Tails will be  $1 - p_H$ . We can depict this space for both players in a graph of the unit square (a square that has a side length equal to one). The basic graph will look like:



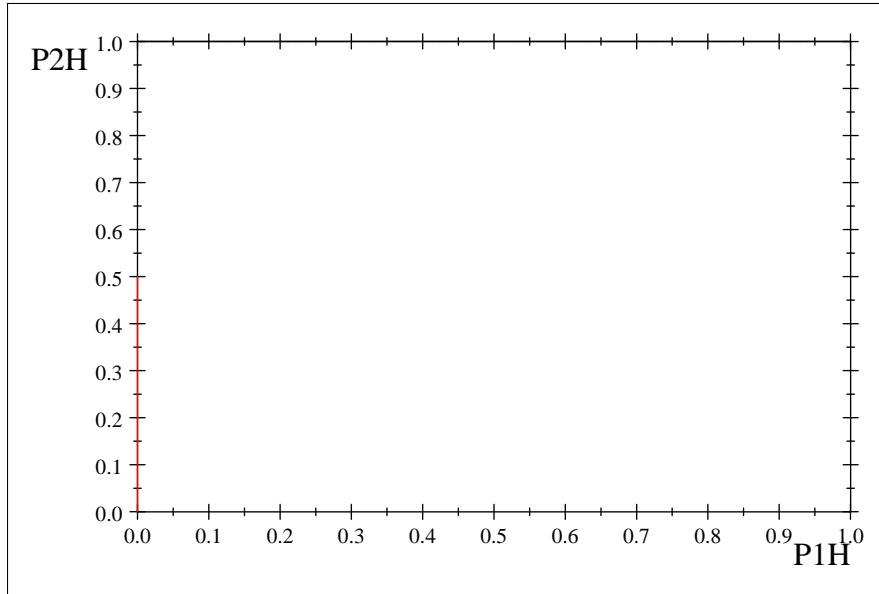
Note that EVERY possible outcome is in this graph. The origin represents both players choosing a pure strategy of Tails. The point (1,1) corresponds to both players choosing a pure strategy of Heads. The point (1,0) corresponds to Player 1 choosing a pure strategy of Heads and Player 2 choosing a pure strategy of Tails. The point (0,1) corresponds to Player 1 choosing a pure strategy of Tails and Player 2 choosing a pure strategy of Heads. At every point along the axes at least one player is using a pure strategy, and at every point inside the square both players are using some mixed strategy. This is basically a matrix, only with a lot of strategies.

What we want to do is to figure out what the best responses are for each player and graph them. For this we will use no math beyond simple addition (and multiplication, but that's really just a shortcut for addition) and some intuition. I suppose the intuition part might make it harder.

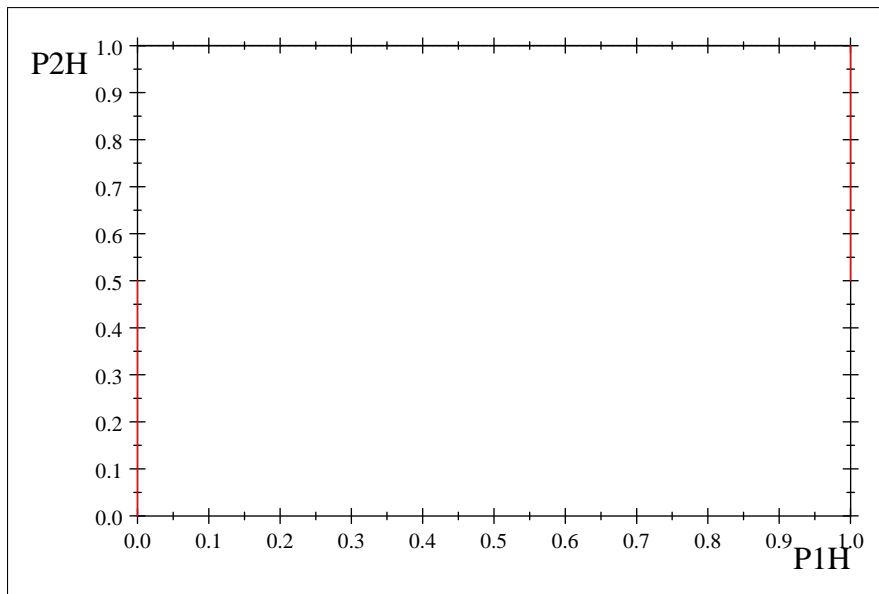
Suppose that Player 2 used a pure strategy of Tails, meaning Player 2 is choosing the point 0 on the y-axis. What strategy would maximize Player 1's expected utility? Choosing Tails of course because Player 1 would always "win" and receive 1.

Suppose that Player 2 uses a mixed strategy of 25% Heads and 75% Tails, meaning that Player 2 is choosing a "strategy" of 0.25 on the y-axis. What strategy would maximize Player 1's utility? It may seem odd, but that strategy is Player 1 ALWAYS choosing Tails. If Player 1 always chooses Tails, then 75% of the time Player 1 will receive 1 and 25% of the time Player 1 will receive  $-1$ , so Player 1's expected utility is:  $1 * .75 + (-1) * .25 = .75 - .25 = .5$ . There is no other strategy that will yield a higher expected utility

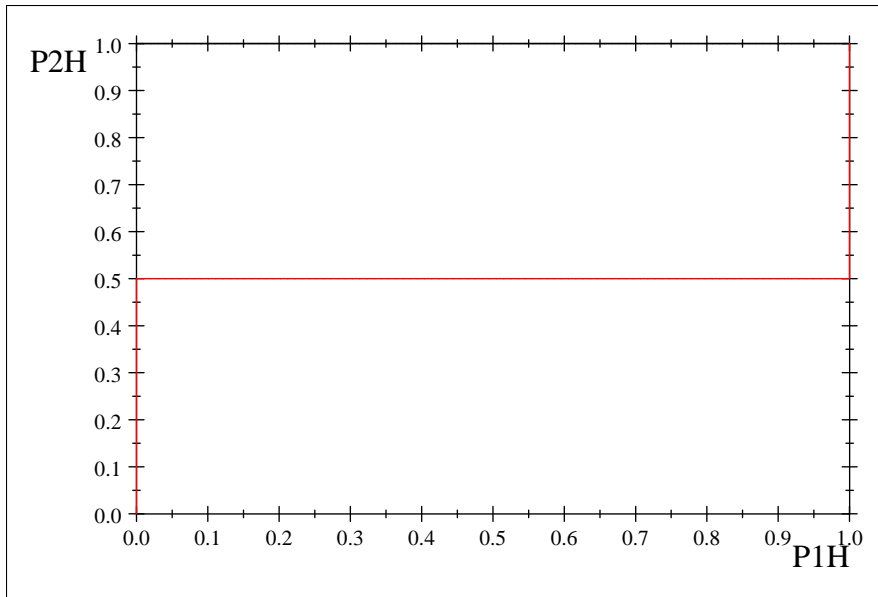
for Player 1 than choosing Tails 100% of the time. In fact, this is true for ANY mixed strategy choice by Player 2 up to Player 2 choosing 50% Heads and 50% Tails. Thus, if Player 2 is weighted more heavily towards Tails, Player 1 should always choose Tails, even if Player 2 is only slightly weighted towards Tails. By slightly weighted towards Tails I mean that Player 2 is using a strategy like 49.8% Heads and 50.2% Tails. Even with this strategy choice by Player 2 we would still see Player 1 choosing Tails 100% of the time. So we have figured out part of Player 1's best response correspondence. We know if Player 2 chooses anything weighted towards Tails Player 1 will choose Tails. We can graph this:



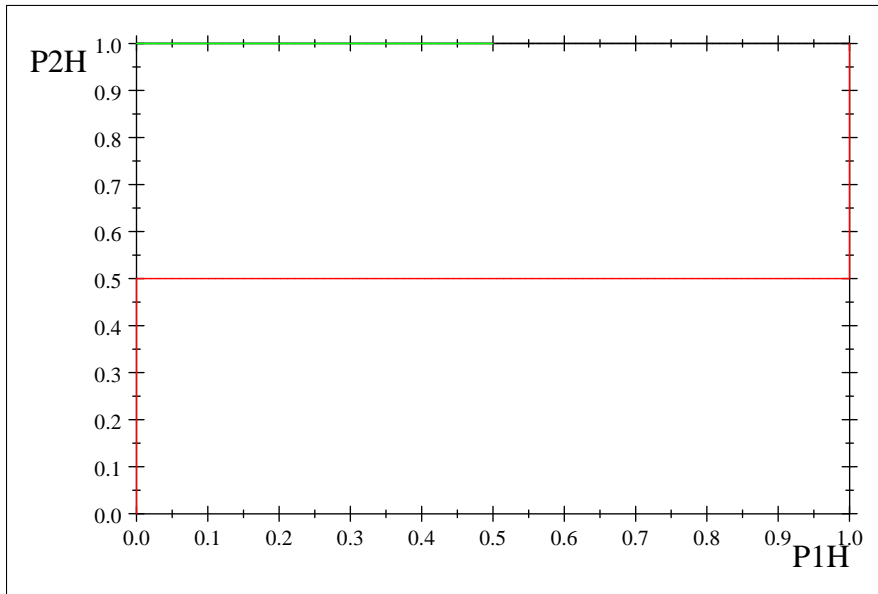
By similar intuition, if Player 2 is weighted more towards Heads, even slightly, then Player 1 should always choose Heads. We can graph this:



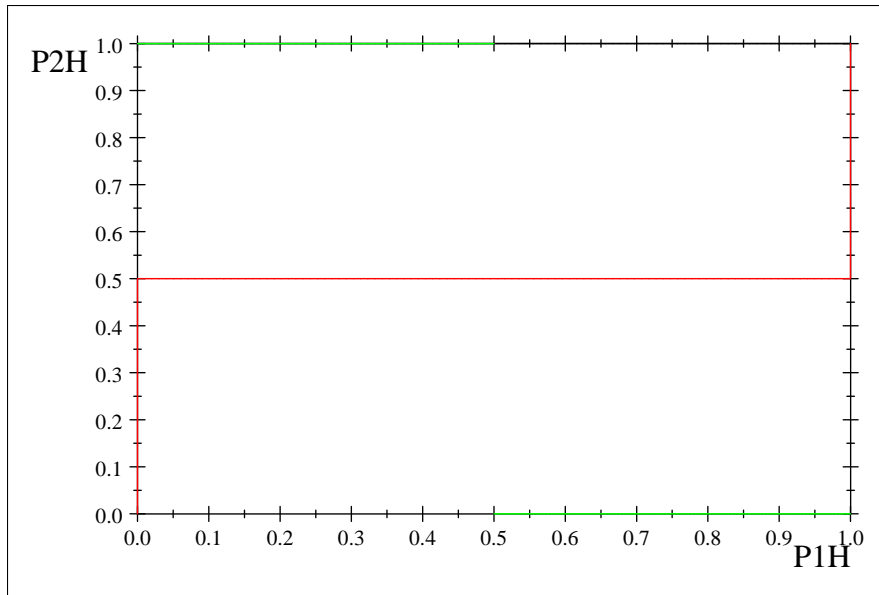
The last question to ask is what strategy would maximize Player 1's expected utility if Player 2 uses a strategy of 50% Heads and 50% Tails. The answer is ... ANY STRATEGY, pure or mixed. We found that if Player 2 uses a 50/50 split that Player 1 is indifferent among all his strategies, so it does not matter which strategy Player 1 uses. We can graph this as well:



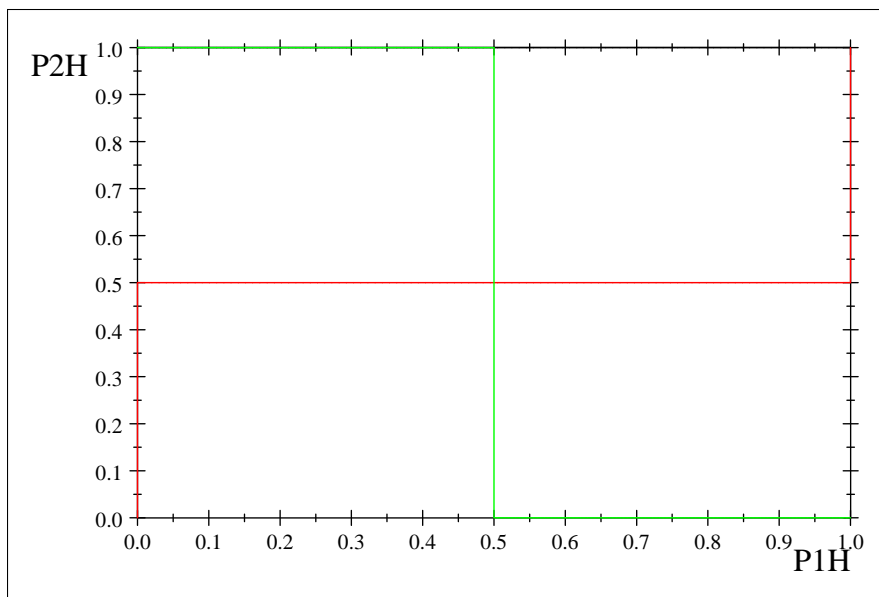
Now we have Player 1's best response correspondence. Now we need to find Player 2's best responses. If Player 1 were to choose a strategy more heavily weighted towards Tails, Player 2 would always choose Heads (recall that Player 2 "wins" when the coins do not match). This part of Player 2's best response can be graphed:



If Player 1 chooses a strategy more heavily weighted towards Heads, then Player 2 would always choose Tails. This part of Player 2's best response can be graphed:



If Player 1 chooses 50% Heads and 50% Tails, then Player 2 is indifferent among all of his strategies so we have:



This game shows the best responses for both players. Notice that they intersect when both players choose to play Heads 50% of the time and Tails 50% of the time. This intersection point is the Nash equilibrium to the game. When there are only 2 strategies (or a continuum of strategies, as we will see later in quantity choice games) for each player we can draw graphs of best responses. It is possible to draw a graph for the Prisoner's Dilemma game – you should find that the only intersection point is where both players are choosing Confess with 100% probability (there should just be a straight line along the confess with probability 100% axis for both players and they should intersect at one of the corners of the square).