1 Selecting Equilibria in Simultaneous Games

Recall that the two key questions that economic theorists ask are:

1. Does an equilibrium exist?
2. Is that equilibrium unique?

   Some theorems about existence were provided, but little was stated about uniqueness (unless all players have a *strictly* dominant strategy – then the Nash equilibrium is unique). A third question that arises is:

3. If the equilibrium is NOT unique, which one is selected?

These notes are an introduction to choosing among equilibria, and until we get to the topics sections, much of the time will be spent defining different solution concepts for general games so that we can eliminate Nash equilibria. Then we can use all the tools we have developed when analyzing topics such as auctions and voting.

1.1 A note on the expected value of Nash equilibria

In order to use these initial selection criteria we need to be able to compare payoffs for the players at different Nash equilibria. Note that we are concerned only with comparing payoffs of outcomes that are Nash equilibria, and we will not compare payoffs of equilibria to those of non-equilibria. The reason is because non-equilibria outcomes are not self-enforcing – players do not have the incentive to remain at these non-equilibrium outcomes.

The expected values for PSNE are easy to calculate – they are just the payoffs to the players. Refer back to the Boxing-Opera coordination game:

		Player 2	
		Boxing	Opera
Player 1	Boxing	$2,1$	$0,0$
	Opera	$0,0$	$1,2$

The three equilibria and their associated outcomes are listed in the table below (it is NOT a matrix – this is just a table).

P1 strategy	P2 strategy	P1 payoff	P2 payoff
Boxing	Boxing	2	1
Opera	Opera	1	2
$\frac{2}{3}$ Boxing, $\frac{1}{3}$ Opera	$\frac{1}{3}$ Boxing, $\frac{2}{3}$ Opera	???	???

The first thing that we have to do is determine the expected value of the MSNE. The matrix below is modified by including the MSNE probabilities of the players for each strategy.

			Player 2	
			$\frac{1}{3}$	$\frac{2}{3}$
			Boxing	Opera
Player 1	$\frac{2}{3}$	Boxing	$2,1$	$0,0$
	$\frac{1}{3}$	Opera	$0,0$	$1,2$

Now, just find the probability that each of the 4 outcomes occurs. For P1 choose Boxing, Player 2 choose Boxing this is $\frac{2}{9}$. For P1 choose Boxing, Player 2 choose Opera this is $\frac{4}{9}$. For P1 choose Opera, Player 2 choose Boxing this is $\frac{1}{9}$. For P1 choose Opera, Player 2 choose Opera this is $\frac{2}{9}$. These probabilities are found by multiplying the probability that each player chooses a particular strategy. So for the Boxing, Boxing outcome it is $\frac{2}{3} * \frac{1}{3} = \frac{2}{9}$. Now, instead of the payoffs I will put the probability that the players end up at each outcome when using the MSNE in each outcome cell:

			Player 2	
			$\frac{1}{3}$	$\frac{2}{3}$
			Boxing	Opera
Player 1	$\frac{2}{3}$	Boxing	$\frac{2}{9}$	$\frac{4}{9}$
	$\frac{1}{3}$	Opera	$\frac{1}{9}$	$\frac{2}{9}$

Note that these probabilities sum to 1 (some outcome occurs). Now, what is Player 1's expected value for the MSNE? Player 1 receives a payoff of 2 if Boxing, Boxing and he receives this payoff $\frac{2}{9}$ of the time. He receives 0 if Boxing, Opera and he receives this payoff $\frac{4}{9}$ of the time. He receives 0 if Opera, Boxing and he receives this payoff $\frac{1}{9}$ of the time. Finally, if Opera, Opera he receives 1 and he receives this $\frac{2}{9}$ of the time. So his expected value of the MSNE is:

$$\begin{aligned}
E_1[MSNE] &= 2*\frac{2}{9}+0*\frac{4}{9}+0*\frac{1}{9}+1*\frac{2}{9} \\
E_1[MSNE] &= \frac{4}{9}+\frac{2}{9} \\
E_1[MSNE] &= \frac{6}{9}=\frac{2}{3}
\end{aligned}$$

So if the players choose the MSNE then Player 1 would earn, on average $\frac{2}{3}$ from playing the MSNE. Player 2 would also earn $\frac{2}{3}$ and his calculation would be:

$$\begin{aligned}
E_2[MSNE] &= 1*\frac{2}{9}+0*\frac{4}{9}+0*\frac{1}{9}+2*\frac{2}{9} \\
E_2[MSNE] &= \frac{2}{9}+\frac{4}{9} \\
E_1[MSNE] &= \frac{6}{9}=\frac{2}{3}
\end{aligned}$$

Now the table can be filled in:

P1 strategy	P2 strategy	P1 payoff	P2 payoff
Boxing	Boxing	2	1
Opera	Opera	1	2
$\frac{2}{3}$ Boxing, $\frac{1}{3}$ Opera	$\frac{1}{3}$ Boxing, $\frac{2}{3}$ Opera	$\frac{2}{3}$	$\frac{2}{3}$

An alternative method of finding each player's expected value of the MSNE is to simply calculate the expected value for a particular player if that player chose one of his pure strategies all the time. Why would this work? This works because the goal of each player in an MSNE is to choose probabilities that make the other player indifferent over his pure strategies, which also makes the other player indifferent over any mixture of those pure strategies (including the mixed strategy that player uses in the MSNE). To see this, simply calculate the expected value if Player 1 chose Boxing when Player 2 uses his mixed strategy of $\frac{1}{3}$ Boxing, $\frac{2}{3}$ Opera – Player 1's expected value is $\frac{2}{3}$. The same is true if Player 1 chose Opera, or $\frac{2}{3}$ Boxing, $\frac{1}{3}$ Opera, or $\frac{1}{2}$ Boxing, $\frac{1}{2}$ Opera, or any other mixed strategy.

Now that the expected values of all three Nash equilibria are known, is there a Nash equilibrium on which you might suppose the players would NOT coordinate? This is the type of question we aim to answer – among the Nash equilibria, which one or ones are likely to be played by the players.

1.2 Selection criteria

Consider two additional games. The first is a 2x2 game:

		Player 2	
		Right	Left
Player 1	Right	$5,5$	$1,1$
	Left	$1,1$	$2,2$

There are 3 Nash equilibria to this game.[1] They are: both players choose Right; both players choose Left; and a MSNE where both players choose Right with probability $\frac{1}{5}$ and Left with probability $\frac{4}{5}$. The expected value of these Nash equilibria are:

P1 strategy	P2 strategy	P1 payoff	P2 payoff
Right	Right	5	5
Left	Left	2	2
$\frac{1}{5}$ Right, $\frac{4}{5}$ Left	$\frac{1}{5}$ Right, $\frac{4}{5}$ Left	$\frac{9}{5}$	$\frac{9}{5}$

[1] You should convince yourself that there are in fact 3 Nash equilibria to this game.

Among these 3 Nash equilibria, both players are strictly better off when playing the Right, Right equilibrium than when playing either of the other two equilibria. When considering the Boxing-Opera game, both players are strictly better off when comparing either the Boxing, Boxing equilibrium or the Opera, Opera equilibrium to the mixed strategy Nash equilibrium. Comparisons of this type lead to our first criterion: *equilibrium payoff dominance*. Equilibrium payoff dominance is based on the concept of Pareto superiority (or dominance). In neoclassical economics (this is the material you learned in intermediate microeconomics), an equilibrium is Pareto optimal if there are no other equilibria at which at least one individual can be made strictly better off while no other individuals are made worse off. Consider the Right, Right equilibrium and the Left, Left equilibrium. It is clear that both players are strictly better off in the Right, Right equilibrium (since they each receive 5 when playing Right, Right but only receive 2 when playing Left, Left), so that it is Pareto superior to the Left, Left equilibrium (as well as the MSNE). Using our criterion, we would say that the Right, Right equilibrium is payoff dominant over the Left, Left equilibrium.

When considering the Boxing-Opera game, note that both the Boxing, Boxing equilibrium AND the Opera, Opera equilibrium are payoff dominant over the MSNE. However, neither the Boxing, Boxing equilibrium nor the Opera, Opera equilibrium is payoff dominant over the other (since one player would have to be made worse off when moving from one of these equilibria to the other), so the only equilibrium in the Boxing-Opera game which can be removed using the payoff dominance criterion is the MSNE.

Now consider the following 3x3 game:

		Player 2		
		Low	Medium	High
	Low	$2, 2$	$1, 1$	$0, 2$
Player 1	Medium	$1, 1$	$3, 3$	$1, 4$
	High	$2, 0$	$4, 1$	$0, 0$

There are 3 PSNE to this game. One is Low, Low. A second is Player 1 chooses High, Player 2 chooses Medium. A third is Player 1 chooses Medium, Player 2 chooses High. Note that none of these equilibria are payoff dominant over any of the other equilibria.[2] However, what is true is that High weakly dominates Low for both players ($2 = 2$, $4 > 1$, $0 = 0$). Thus, why would either player ever choose Low when there is another strategy that does just as well against two of the other player's three strategies and strictly better against the other strategy? A second criterion for selecting among Nash equilibria is that of *undominated Nash equilibrium*. For an undominated Nash equilibrium, no player should use a weakly dominated strategy. This would remove the Low, Low equilibrium from play. There are now two pure strategy equilibria remaining, but we have no criterion for removing one or the other. Recall that the goal is to reduce the set of equilibria to as few as we can – if the number can be reduced to 1 using our criteria, great; if not, just eliminating some is a step towards uniqueness.

1.2.1 When the criteria are at odds

It is possible that these criteria are at odds. Consider the modified Prisoner's Dilemma game from class:

		Player 2	
		$Q = 10$	$Q = 20$
Player 1	$Q = 10$	$11, 11$	$5, 16$
	$Q = 20$	$16, 3$	$5, 5$

Again, there are 2 PSNE. One occurs when both players choose $Q = 20$, the other occurs when Player 1 chooses $Q = 10$ and Player 2 chooses $Q = 20$. Note that the equilibrium when Player 1 chooses $Q = 10$ and Player 2 chooses $Q = 20$ is payoff dominant. Player 1 can earn no more than 5 by switching to the other equilibrium and Player 2 is strictly better off. However, when Player 1 chooses $Q = 10$ and Player 2 chooses $Q = 20$ is NOT undominated. This is because Player 1 is using a weakly dominated strategy in $Q = 10$. Thus, it is possible for the criteria to be at odds.

[2] We will not consider the MSNE in this game.