1 Sequential Games

We call games where players take turns moving "sequential games". Sequential games consist of the same elements as normal form games – there are players, rules, outcomes, and payoffs. However, sequential games have the added element that history of play is now important as players can make decisions conditional on what other players have done. Thus, if two people are playing a game of Chess the second mover is able to observe the first mover's initial move prior to making his initial move. While it is possible to represent sequential games using the strategic (or matrix) form representation of the game it is more instructive at first to represent sequential games using a game tree. In addition to the players, actions, outcomes, and payoffs, the game tree will provide a history of play or a path of play.

A very basic example of a sequential game is the Entrant-Incumbent game. The game is described as follows:

> Consider a game where there is an entrant and an incumbent. The entrant moves first and the incumbent observes the entrant's decision. The entrant can choose to either enter the market or remain out of the market. If the entrant remains out of the market then the game ends and the entrant receives a payoff of 0 while the incumbent receives a payoff of 2. If the entrant chooses to enter the market then the incumbent gets to make a choice. The incumbent chooses between fighting entry or accommodating entry. If the incumbent fights the entrant receives a payoff of -3 while the incumbent receives a payoff of -1. If the incumbent accommodates the entrant receives a payoff of 2 while the incumbent receives a payoff of 1.

With this structure we can easily write down the strategic form of the game. The Entrant has 2 strategies, Enter or Stay Out, while the Incumbent also has 2 strategies, Fight or Accommodate. So this is a simple 2x2 matrix:

		Incumbent	
		Fight	Accommodate
Entrant	Enter	$-3, -1$	$2, 1$
	Stay Out	$0, 2$	$0, 2$

We can now find the pure strategy Nash equilibria (PSNE) to this game. The two PSNE to this game are that the Entrant chooses Enter and the Incumbent chooses Accommodate; and that the Entrant chooses Stay out and the Incumbent chooses Fight:

		Incumbent	
		Fight	Accommodate
Entrant	Enter	$-3, -1$	$\boxed{2},\boxed{1}$
	Stay Out	$\boxed{0},\boxed{2}$	$0,\boxed{2}$

Note that both of these outcomes are Nash equilibria since they both have the property that neither player would wish to unilaterally change his strategy given what the other is doing. For completeness, there is also a mixed strategy Nash equilibrium where the Entrant plays Stay Out with probability 1 (note that the Incumbent is indifferent between Fight and Accommodate if the Entrant always plays Stay Out) and the Incumbent plays Fight with probability $\frac{2}{5}$ and Accommodate with probability $\frac{3}{5}$. The actual *payoffs* with the MSNE will be the same as the payoffs where the Incumbent always chooses Fight and the Entrant always chooses Stay Out, but the *strategies* are different (because the Incumbent is using a mixed strategy).

Now, the new question is whether either of these PSNE (we will ignore the MSNE) are more reasonable than the other when looking at the game as it is played sequentially. Consider the Stay Out, Fight equilibrium from the perspective of the Entrant. The Entrant chooses Stay Out because the incumbent is threatening to Fight if the Entrant should choose to Enter. This seems somewhat logical, but would the Incumbent really choose Fight if the Entrant chose Enter? Of course not, because in this game that is only played once the Incumbent does better if it chooses to Accommodate when the Entrant chooses Enter.[1] We can see this by noting that Enter, Accommodate is an equilibrium to this game. The Entrant, moving first, knows that if it chooses Enter the Incumbent will choose Accommodate. Thus, the Entrant should always choose Enter in the *sequential game* because it knows that the Incumbent will always follow a play of Enter

[1] We will eventually discuss games that are repeated but for now our focus is on the game in question, which is played only once.

with a play of Accommodate. The process of eliminating one (or some) of the Nash equilibria is formalized soon.

1.1 A Game Tree

Game trees consist of the following pieces. There is an *initial node* to the game tree. This is the starting point of the game (think about the setup of the board when a game of Chess is begun – this is the initial node). From that initial node there are actions that the first mover can take. These actions are represented as *branches* to the game tree. At the end of each branch is a node. If the first mover makes a move and the game ends after that move, then we say that the game has reached a terminal node. A *terminal node* is a node at which no more actions can be taken. If the first mover makes a move and the second mover then gets to choose an action, we call this a *decision node*. The second mover's actions are then represented by branches extending from the decision node. The game tree extends until all the nodes are terminal nodes. At the terminal nodes, the payoffs to the players are listed. It is the convention to list the payoffs in the order that the players moved. One other important aspect of the game tree is the *information set*. For the games we will initially consider all decision nodes will also be information sets. However, it is possible that a game is being played and a player is uncertain as to which of a few decision nodes the player is at. In this case, the collection of decision nodes is that player's information set.

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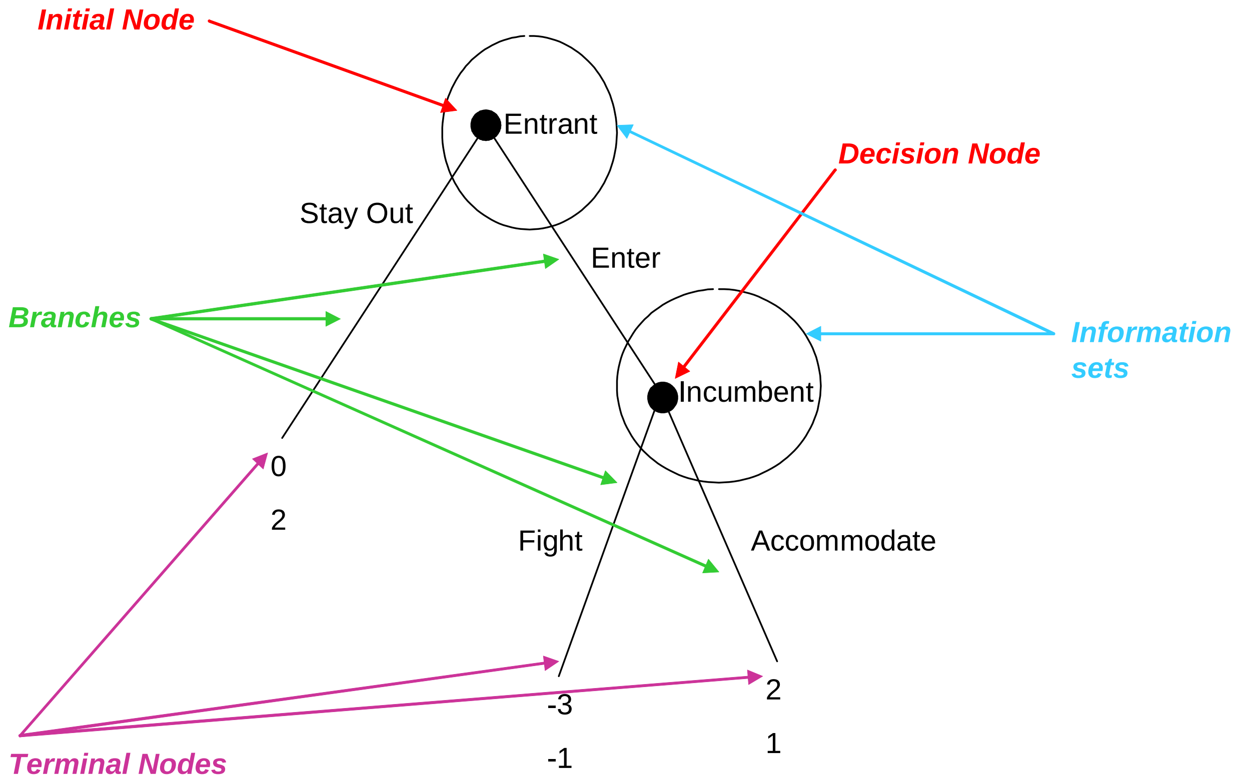
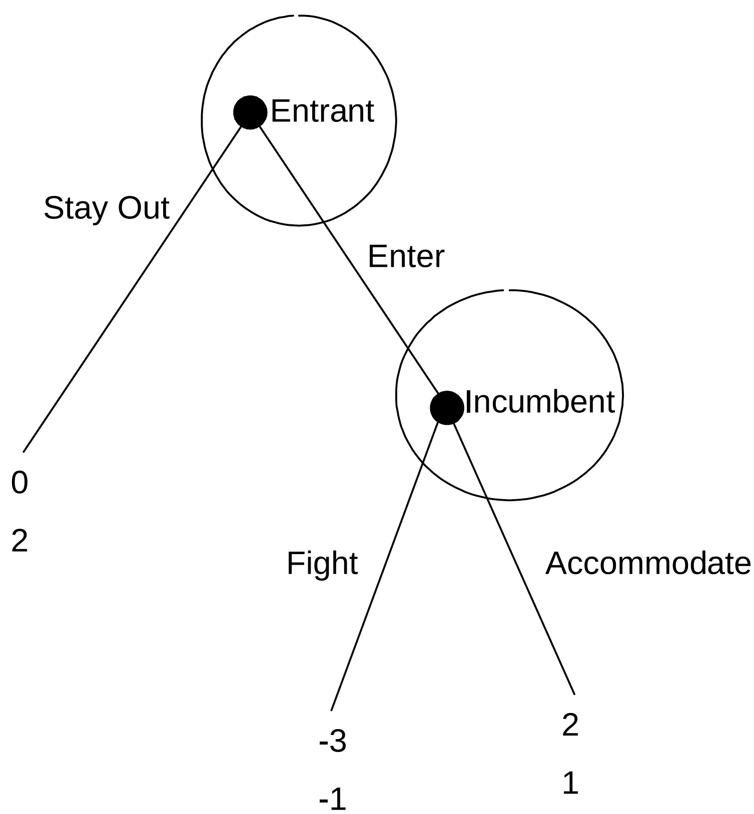
The ultimate goal is to solve this game, but first we can display it as a game tree (or in *extensive form*, which is the technical name for a game tree). I have two "versions" of the extensive form of the Entry Game. The first has labels for each of the components of the game (node, branch, information set) and the second is the actual game without all the components labeled.



Game tree with its components labeled.



Game tree without the components labeled.

One method of solving sequential games is to use the process of backward induction. This simply means to start at the end of the game and work towards the beginning. In this case, we would start with the Incumbent's decision and then move to the Entrant's decision after determining what the Incumbent will do.

2 Subgame Perfect Nash Equilibrium (SPNE)

While there have been criteria used to eliminate Nash equilibria, this is the first major refinement of the basic concept of Nash equilibrium that we have seen. While all subgame perfect Nash equilibria of a game are Nash equilibria, not all Nash equilibria are subgame perfect. This means that if you set up the matrix and find all the pure strategy Nash equilibria to the game, if there is a subgame perfect Nash equilibrium it will be one of those you found, but not all of those equilibria will be subgame perfect.

Before discussing a subgame perfect Nash equilibrium the term *subgame* must be defined. A subgame is part of a game, but a subgame has some particular features. A subgame must:

1. Begin at an information set that contains a single decision node.
2. Contain all the decision nodes and terminal nodes that follow that decision node, but no decision or terminal nodes that do not follow.
3. A subgame cannot contain part of an information set (if an information set contains 2 or more nodes the subgame must contain all of the nodes in the information set).

Technically, the entire game is a subgame (consider whether all of the conditions needed for a subgame are met using the initial node as the starting point) but sometimes we want to focus on those subgames that are not the entire game. Thus, a subgame that is not the entire game is called a *proper subgame*. If there are x proper subgames in a particular game, then there will be $x+1$ subgames (including the entire game). I point out the terminology so that you know the difference in the terms – it is not a major point, so do not get bogged down in it.

In the Incumbent-Entrant game there is one proper subgame, that being the subgame that starts with the Incumbent's decision node. The idea behind the subgame perfection refinement is to look at the smallest (or last) subgame in the game that contains terminal nodes. The smallest subgame should contain no other subgames. In the Entry game there is only the one proper subgame so that is where we begin. We know that if the decision node where the Incumbent chooses to Fight or to Accommodate is ever reached the Incumbent will choose to Accommodate because the payoff to Accommodate is greater than the payoff to Fight (i.e. $1 > -3$). In effect, this "removes" Fight as a viable strategy for the Incumbent. Since the game is one of complete and perfect information, the Entrant knows that the Incumbent will never choose Fight if the Incumbent is forced to make a decision. Thus, the Entrant can also "remove" the Fight branch from the game. Now we move to the next subgame, which happens to be the initial node. If the Entrant chooses Stay Out it will earn a payoff of 0. If the Entrant chooses Enter it will earn a payoff of 2 because the Entrant knows the Incumbent will never choose Fight. Thus, the Entrant chooses Enter at the initial node since $2 > 0$. Thus, the Enter, Accommodate equilibrium is the one that "survives" this refinement process. Note that this process of finding the subgame perfect Nash equilibrium is *identical to the process of backward induction* that we have discussed. The key is to find the action which yields the highest payoff at each subgame. The picture below shows how I keep track of which action is chosen by each player for each information set. I simply mark an arrow over those actions.

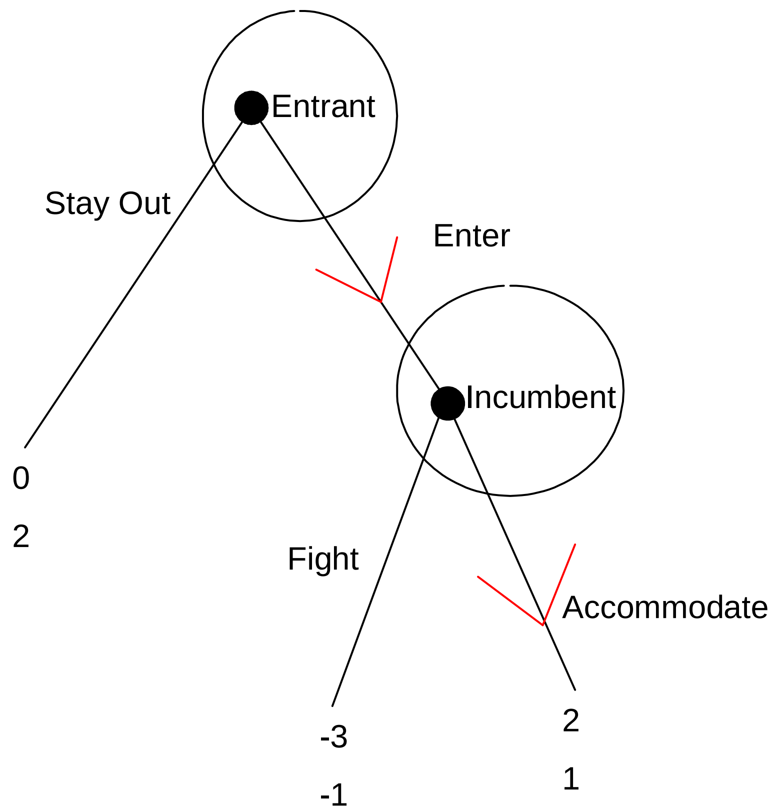


Illustration of SPNE of the Entry game.

Note that the definition of subgame perfection requires that every player acts in an optimal manner at each subgame. Thus, the Stay Out, Fight equilibrium is NOT a subgame perfect Nash equilibrium because for the Incumbent choosing Fight is NOT a best response if it actually gets to make a decision. Subgame perfection rules out non-credible threats, which is exactly what choosing Fight is for the Incumbent. The only way that the Incumbent can get the Entrant to Stay Out is by threatening to Fight, but if the Entrant does Enter then the Incumbent no longer has any incentive to Fight, so the threat is non-credible.

Definition 1 *A Nash equilibrium is subgame perfect if the players' strategies constitute a Nash equilibrium at every subgame.*

As for existence of NE and SPNE in sequential games, we have a proposition:

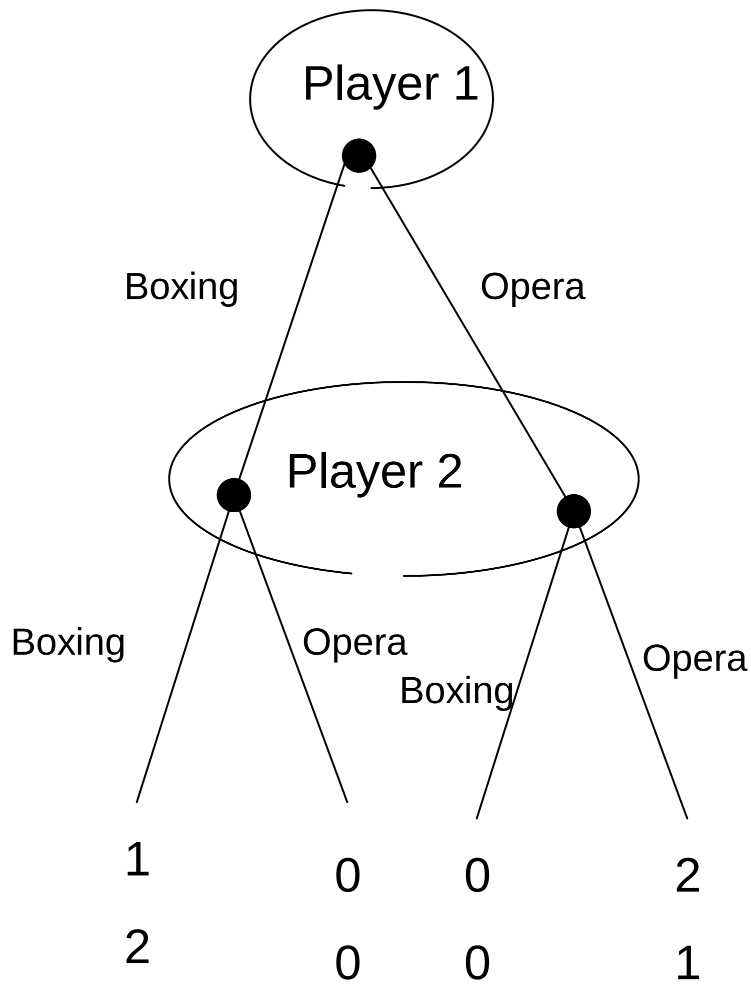
Theorem 2 *Every finite game of perfect information has a pure strategy subgame perfect Nash Equilibrium that can be derived through backward induction. Moreover, if no player has the same payoffs at any two terminal nodes, then there is a unique subgame perfect Nash Equilibrium that can be derived in this manner.*

Note the key points of the theorem. First, the games have to be *finite* (eventually we will discuss infinitely repeated games). Second, the games have to be games of *perfect information* – this means that there are no information sets which contain multiple decision nodes. If these two conditions are met, then we know that there is at least one subgame perfect Nash equilibrium. For uniqueness we need a little more. Again, consider the popular games of Chess and Checkers and Connect Four and even Tic-Tac-Toe. These are all finite games of perfect information, so they all have a subgame perfect Nash equilibrium. Tic-Tac-Toe is by far the easiest one for which we can find the subgame perfect Nash equilibrium because it has the smallest game tree (plus many of the board layouts are identical), with Connect Four the next easiest, then Checkers, and finally Chess.[2]

[2] There are solutions for games like Checkers and Connect Four but they may not be the subgame perfect Nash equilibria to the game, which is what we are calling a "solution".

2.1 Comparing games with perfect and imperfect information

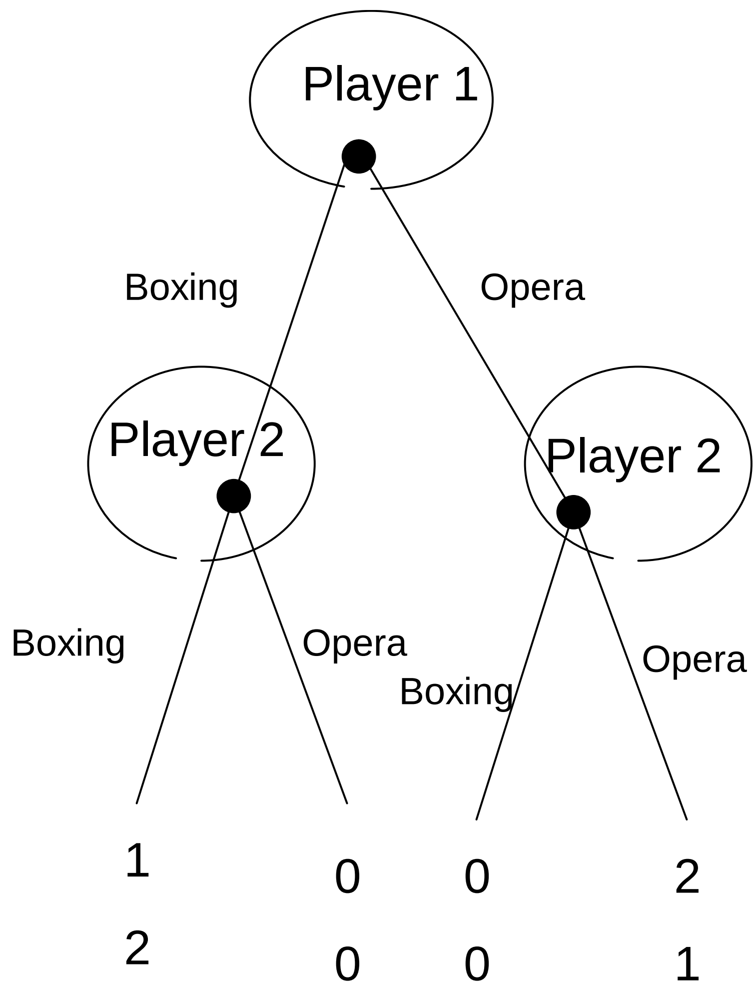
Consider the Boxing-Opera game. When the game was simultaneous there were 3 Nash equilibria – both play Boxing; both play Opera; and a MSNE. The MSNE was eliminated by the payoff dominance criterion, but the other two equilibria still remained. If the game tree for the *simultaneous* Boxing-Opera game is drawn, it will look like:



Simultaneous Boxing-Opera game

Note that both of Player 2's decision nodes are included in one information set. That is because, in the simultaneous game, Player 2 does NOT KNOW whether Player 1 has chosen Boxing or Opera. Thus, Player 2 does not know which decision node he is at. In this game there is only 1 subgame – the entire game.

Now consider the *sequential* Boxing-Opera game, in which Player 2 knows which decision is made by Player 1. We will assume that Player 1 is the player who prefers meeting at the Opera. The game tree for the *sequential* Boxing-Opera game now looks like:



Sequential Boxing-Opera game

Note that it is very similar to the simultaneous game tree, only that now since Player 2 knows which choice is made by Player 1 he has two information sets. Basically, he knows which decision node he is at. If we use backward induction to find the subgame perfect Nash equilibrium we find that the *OUTCOME* of the SPNE is that the players end up at the Opera. Note that this is NOT the SPNE, but the outcome. The actual SPNE is: Player 1 chooses Opera; Player 2 chooses Boxing if Player 1 chooses Boxing; Player 2 chooses Opera if Player 1 chooses Opera. It seems rather lengthy, but recall that a strategy for a player is a complete, contingent plan of action. Thus, each player must specify an action for every information set that he or she has, whether or not that information set is reached in the course of playing the game.

There is an important point that is glossed over in the previous paragraph, which is the difference between an *action* and a *strategy*. In simultaneous games these terms are identical. However, in sequential games they have a different meaning. An *action* is simply an available choice a player may take at an information set. A *strategy* is essentially a collection of specific actions that the player can take at all information sets. Thus, when specifying a strategy one must specify an action for every information set that a player has. Yes, it's repeated but it's important.

Why must actions be specified at decision nodes that are not reached? In order for a player who makes takes an action earlier in the game to know which action to take, that player needs to know which actions will be taken by the players who follow. For example, Player 1 needs to know whether Player 2 is choosing Boxing or Opera when he (Player 1) chooses Boxing so that he can determine whether or not to choose Boxing or Opera.[3] If this is still unclear (it may be) follow this simple rule for now:

[3] I realize this is not the best example because if Player 1 knows Player 2 will choose Opera when he (Player 1) chooses Opera, then Player 1 knows he will choose Opera because that yields his highest payoff. I will provide another example shortly

Remark 3 *Whenever specifying a Nash equilibrium in a sequential game, subgame perfect or just a regular one, be sure to specify an action for each information set a player has.*

2.2 Strategic (or matrix) form of a sequential game

The strategic form for the Entrant-Incumbent game was simple to draw because each player only had two strategies (as well as two actions). The strategic form for the Boxing-Opera game is a little more complicated to draw, but can still be drawn. Recall that all that is needed to draw the strategic form of the Boxing-Opera game is the strategies available to each player. Player 1 has 2 strategies, choose Boxing or choose Opera. Player 2, however, now has *FOUR* strategies. These strategies are:

1. Choose Boxing if Player 1 chooses Boxing, choose Boxing if Player 1 chooses Opera (shorthand: Always choose Boxing)

2. Choose Boxing if Player 1 chooses Boxing, choose Opera if Player 1 chooses Opera (shorthand: Copy Player 1)

3. Choose Opera if Player 1 chooses Boxing, choose Boxing if Player 1 chooses Opera (shorthand: Opposite of Player 1)

4. Choose Opera if Player 1 chooses Boxing, choose Opera if Player 1 chooses Opera (shorthand: Always choose Opera)

These are the four possible strategies that Player 2 has. It is no longer simply a choice of Boxing or Opera, but a choice *contingent* upon what action Player 1 takes. To draw the strategic form of the game we need a 4x2 (or a 2x4) matrix. I prefer the 4x2 matrix because it is easier to display – plus I get worried someone will become angry and take the 2x4 and whack me on the head. Note that Player 2 is the row player and Player 1 is the column player (this might explain what could be confusion about the payoffs).

		Player 1	
		Boxing	Opera
	Always Boxing	$2,1$	$0,0$
Player 2	Copy Player 1	$2,1$	$1,2$
	Opposite of Player 1	$0,0$	$0,0$
	Always Opera	$0,0$	$1,2$

If we find all the pure strategy Nash equilibria for this game we have:

		Player 1	
		Boxing	Opera
	Always Boxing	$\boxed{2},\boxed{1}$	$0,0$
Player 2	Copy Player 1	$\boxed{2},1$	$\boxed{1},\boxed{2}$
	Opposite of Player 1	$0,\boxed{0}$	$0,\boxed{0}$
	Always Opera	$0,0$	$\boxed{1},\boxed{2}$

Note that there are three pure strategy Nash equilibria to this game. One is where Player 1 chooses Boxing and Player 2 always chooses Boxing. A second is Player 1 chooses Opera and Player 2 chooses to Copy Player 1. A third is Player 1 chooses Opera and Player 2 chooses Always Opera. While these are all viable Nash equilibria (no player can unilaterally change his strategy and make himself strictly better off), the only one of these three strategies which is a subgame perfect Nash equilibrium is the second one, where Player 1 chooses Opera and Player 2 chooses Copy Player 1. The other two Nash equilibria rely on noncredible threats. In the first one, Player 2 is threatening to choose Boxing if Player 1 chooses Opera so that the players end up going to see the Boxing. In the third one, Player 2 is threatening to choose Opera if Player 1 chooses Boxing so that the players end up going to see the Opera (not that this threat really matters to Player 1 since Player 1's preferred outcome is to see the Opera). Once again, the subgame perfect Nash equilibrium criterion is similar to the undominated Nash equilibrium criterion in that the Nash equilibrium outcomes in which Player 2 uses a weakly dominated strategy do not survive the subgame perfection refinement.

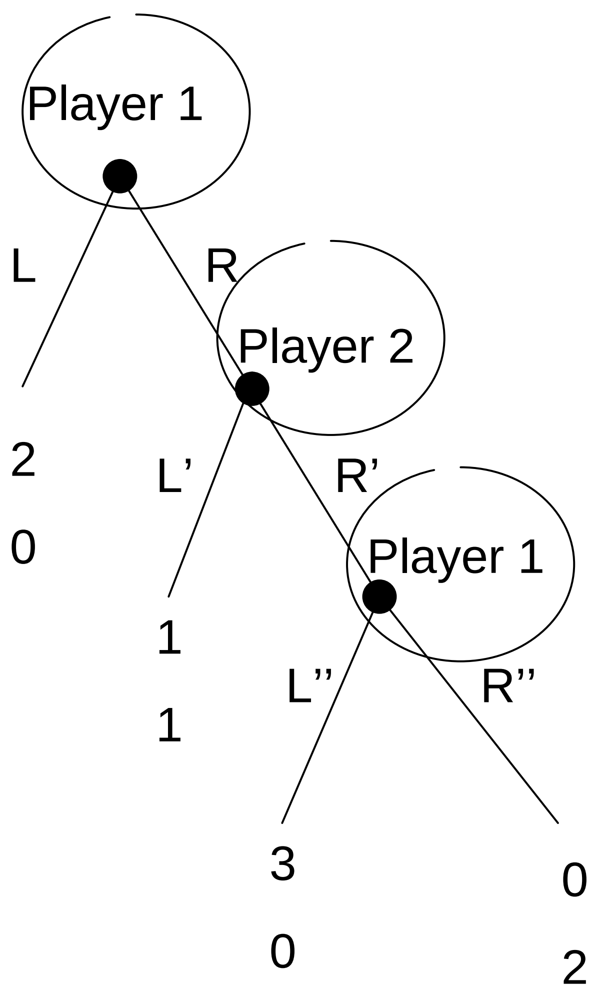
where this is not the case.

3 More Games

It is possible that there are (1) games in which a player makes two decisions, one before the second player and one after the second player as well as (2) sequential games with more than 2 players. We will consider an example of each.

3.1 When a player has more than one decision

Consider the following game with 2 players. Player 1 begins the game and can choose L or R. If Player 1 chooses L then the game ends. Player 1 receives 2 and Player 2 receives 0. If Player 1 chooses R then Player 2 observes this and makes a decision. Player 2 can choose L' or R'. If Player 2 chooses L' the game ends. Player 1 receives 1 and Player 2 receives 1. If Player 2 chooses R' then Player 1 observes this decision and makes a decision. Player 1 can now choose L'' or R''. If Player 1 chooses L'' then the game ends. Player 1 receives 3 and Player 2 receives 0. If Player 1 chooses R'' then the game ends. Player 1 receives 0 and Player 2 receives 2. The game tree, or extensive form, of this game is:



The strategic (matrix) form of the game is:

Player 1	Player 2	L'	R'
	L, L''	$2, 0$	$2, 0$
	L, R''	$2, 0$	$2, 0$
	R, L''	$1, 1$	$3, 0$
	R, R''	$1, 1$	$0, 2$

Again, note that Player 1 has 4 strategies while Player 2 only has 2 strategies. Player 1 has 2 information sets with 2 actions each at the information sets and thus has $2x2 = 4$ strategies (the twos are for the number of actions available at each information set – use the number of information sets as the amount of numbers you multiply together). Player 2 has 1 information set with 2 actions at it and thus has just 2 strategies (there is no multiplication here because there is only 1 information set). More on this in a minute.

The SPNE to the game, found by backward induction, is the set of strategies: Player 1 choose L, L'', Player 2 choose L'. There is one other pure strategy Nash equilibrium to this game as we can see below:

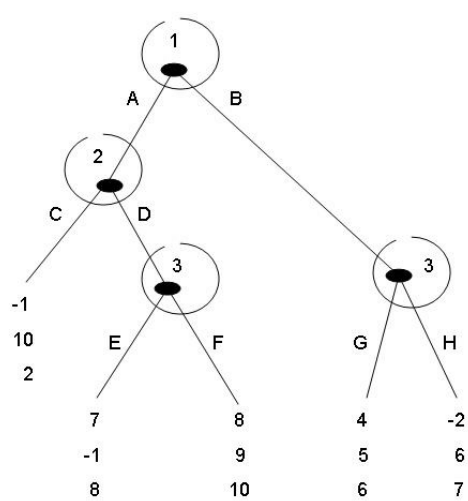
		Player 2	
		L'	R'
	L, L''	$\boxed{2}, \boxed{0}$	$2, \boxed{0}$
Player 1	L, R''	$\boxed{2}, \boxed{0}$	$2, \boxed{0}$
	R, L''	$1, \boxed{1}$	$\boxed{3}, 0$
	R, R''	$1, 1$	$0, \boxed{2}$

Notice that the other PSNE is Player 1 choose L, R'', Player 2 choose L'. Also notice that Player 1's choice of strategy (L, L'')in the SPNE does NOT weakly dominate the strategy choice in the non-SPNE PSNE (L, R'') since both strategies yield exactly the same payoffs. So it is not necessarily the case that the SPNE can be found by ruling out the "dominated" Nash equilibrium using the criterion from chapter 5 (no player uses a weakly dominated strategy in a PSNE). As this example shows, you cannot tell from the strategic form which of these PSNE are the SPNE, which is why we need the game tree.

Counting strategies in an extensive form You can use the method above to calculate how many strategies each player has. As an example, if a player had 3 information sets with two actions at each information set then the Player would have: $2x2x2 = 8$ strategies. If a player had 2 information sets with 3 actions at each information set then the player would have $3x3 = 9$ strategies. If a player had 3 information sets with 3 actions at one information set, and 2 actions at the other 2 information sets, the player would have $3x2x2 = 12$ strategies. If a player had 4 information sets with 7 actions at one, 6 at another, 5 at another, and 3 at the other, the player would have: $7x6x5x3 = 630$ strategies. Now think about Chess, and about (1) how many information sets each player has and (2) how many actions each player can take at each information set. It is a very, very large number. Now back to games.

3.2 More than 2 players

Consider the following game with 3 players. Note that Player 1 has 2 strategies (A or B), Player 2 has 2 strategies (C or D), and Player 3 has FOUR strategies (2 decision nodes, 2 actions at each information set, so $2x2 = 4$). Player 3's strategies are (E, G); (E, H); (F, G); and (F, H).



Now, what is the SPNE to this game? Again, start from the end and work towards the beginning. Player 3 will choose H over G because $7 > 5$. Player 2 will choose F over E because $10 > 8$. Now that we know that, can we determine what Player 1 will do? Not yet, we need to determine what Player 2 will do. Player 2 will choose C over D because $10 > 9$. Note that Player 2 is not concerned with what happens when Player 3 chooses E since Player 3 will not choose E. Now we can determine what Player 1 will do. If Player 1 chooses A, Player 2 chooses C and the game ends with Player 1 receiving (-1). If Player 1 chooses B, then Player 3 chooses H, and Player 1 receives (-2). So Player 1 would choose A. Thus, the SPNE to this game is: Player 1 choose A, Player 2 choose C, Player 3 choose F and Player 3 choose H. You could write it out shorthand as:

P1: A

P2: C

P3: F,H

The strategic form of this game, for expositional purposes, is a set of two 4x2 matrices. Or it could be four 2x2 matrices. It would look like (the two 4x2 matrices):

Player 1: A

Player 3 \ Player 2	C	D
E,G	$2, 10, -1$	$8, -1, 7$
E,H	$2, 10, -1$	$8, -1, 7$
F,G	$2, 10, -1$	$10, 9, 8$
F,H	$2, 10, -1$	$10, 9, 8$

Player 1: B

Player 3 \ Player 2	C	D
E,G	$6, 5, 4$	$6, 5, 4$
E,H	$7, 6, -2$	$7, 6, -2$
F,G	$6, 5, 4$	$6, 5, 4$
F,H	$7, 6, -2$	$7, 6, -2$