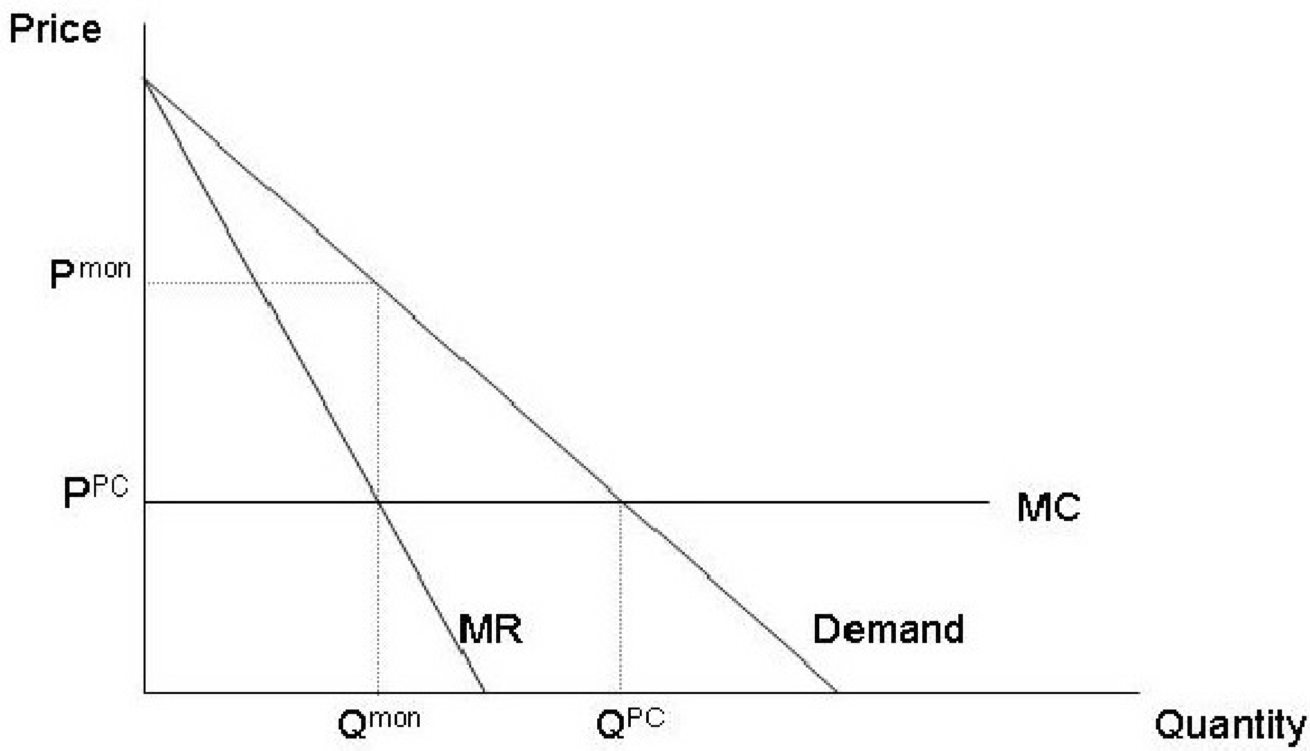
1 Market models

One area in economics in which game theory has made a large impact is that of oligopoly theory. Traditionally, an oligopoly is defined as a small group of sellers whose economic decisions are dependent upon what other firms do. Compared to the extremes of monopoly and perfect competition, one might expect that the price and total output in an oligopoly market lie somewhere between those two. The following figure:



shows a basic picture of a market with downward sloping demand where firms have constant marginal costs. We would expect that the total output in an oligopoly lie between Q^{mon} and Q^{PC}, where Q^{mon} is the monopoly quantity and Q^{PC} is the total output from perfect competition. We would also expect price in the market to lie between P^{mon} and P^{PC}. We will discuss a few standard game theoretic treatments of oligopoly – you should realize that there are many variants of these basic models and that changing the structure of the model can alter the Nash equilibrium and the outcome that occurs.

1.1 Benchmark models

We will use the monopoly model and the perfectly competitive model as benchmarks. For now we will assume that the inverse market demand function is linear and given by:

$$P(Q) = a - bQ$$

where a is the intercept of the line and $-b$ is the slope of the line, with $a > 0$ and $b > 0$. When discussing a market with more than one firm, we will assume that the firms are identical and if there are N firms that:

$$Q = q_1 + q_2 + ... + q_N$$

Thus, the total market quantity is just the sum of all the individual firm quantities. A final simplifying assumption is that firms have a total cost function of:

$$TC\left(q_i\right) = cq_i$$

so that a firms total cost is just the product of how much it produces and some constant $c > 0$. This means that firms will have constant marginal costs of c. One final assumption we will make is that $a > c$. If this is not the case then no market would exist as the cost of supplying a unit would be greater than the maximum willingness to pay of ANY consumer.

1.1.1 Perfect competition

In a perfectly competitive market we know that price equals marginal cost, or $P = MC$. Using the structure above, we have:

$$\begin{aligned} a - bQ &= c \\ \frac{a-c}{b} &= Q \end{aligned}$$

Thus, the total market output is $\frac{a-c}{b}$. We are assuming all firms are identical, so that we just assume that each firm produces $\frac{1}{N}$ of the total output. Price in this market is then:

$$\begin{aligned} P\left(Q\right) &= a - b\left(\frac{a-c}{b}\right) \\ P\left(Q\right) &= a - a + c \\ P\left(Q\right) &= c \end{aligned}$$

which we already knew but it is useful to check. Profits to the firms are given by:

$$\Pi_i = P\left(Q\right) * q_i - cq_i$$

We can plug in numbers for q_i but there is no need for this problem. Simply plug in $P\left(Q\right) = c$ to find:

$$\begin{aligned} \Pi_i &= P\left(Q\right) * q_i - cq_i \\ \Pi_i &= cq_i - cq_i \\ \Pi_i &= 0 \end{aligned}$$

Again, we should already have known this – in perfectly competitive markets firms make zero ECONOMIC profit. To summarize:

$$\begin{aligned} Q &= \frac{a-c}{b} \\ q_i &= \frac{1}{N} * \frac{a-c}{b} \\ P\left(Q\right) &= c \\ \Pi_i &= 0 \end{aligned}$$

1.1.2 Monopoly

The monopoly problem is closer to the problems we will be solving. The monopolist's problem is non-game theoretic (well, as it is proposed here) so it is a simple decision. Given the market demand function and the total cost function how much should the monopolist produce to maximize profit. The monopolist's profit is:

$$\Pi_M = P\left(Q\right) * Q - cQ$$

Notice that I have removed the q_i from the profit function – there is no need to distinguish between firms since there is only one firm. Now, to maximize profit simply take the derivative with respect to Q. We will first do this for the general profit function and then for the profit function with linear inverse demand. Differentiating Π_M with respect to Q we have:

$$\frac{d\Pi_M}{dQ} = P'(Q)\,Q + P(Q) - c$$

Setting this equal to zero we have:

$$\begin{aligned} P'(Q)\,Q + P(Q) - c &= 0 \\ P'(Q)\,Q + P(Q) &= c \end{aligned}$$

This is just $MR = MC$. Now, let's solve for $P(Q)$. We have:

$$P(Q) = c - P'(Q)\,Q$$

If you are told that $P'(Q) < 0$ if $Q > 0$ (this just means that demand curves slope downward) then we can see that the monopolist charges a price greater than marginal cost. This is because we have c, which is positive, minus $P'(Q)\,Q$, which is negative since Q is positive and $P'(Q)$ is negative, and a positive minus a negative is a positive number. We will use a result like this when solving one of our models.

For our specific inverse linear demand function of $a - bQ$ we have:

$$\begin{aligned} \Pi_M &= (a - bQ)\,Q - cQ \\ \frac{d\Pi_M}{dQ} &= a - 2bQ - c \\ a - 2bQ - c &= 0 \\ \frac{a-c}{2b} &= Q \\ \frac{1}{2} * \left(\frac{a-c}{b}\right) &= Q \end{aligned}$$

Thus, the monopolist's quantity is $\frac{1}{2}$ of the total output in the perfectly competitive market. The price in the market is:

$$\begin{aligned} P(Q) &= a - b\left(\frac{a-c}{2b}\right) \\ P(Q) &= a - \frac{a}{2} + \frac{c}{2} \\ P(Q) &= \frac{a}{2} + \frac{c}{2} \\ P(Q) &= \frac{a+c}{2} \end{aligned}$$

So the price is $\frac{a+c}{2}$ which is greater than c because $a > c$.[1] The monopolist's profit is:

$$\begin{aligned} \Pi_M &= \left(\frac{a+c}{2}\right) * \left(\frac{a-c}{2b}\right) - c * \left(\frac{a-c}{2b}\right) \\ \Pi_M &= \left(\frac{a+c}{2} - c\right) * \left(\frac{a-c}{2b}\right) \\ \Pi_M &= \left(\frac{a+c}{2} - \frac{2c}{2}\right) * \left(\frac{a-c}{2b}\right) \\ \Pi_M &= \left(\frac{a-c}{2}\right) * \left(\frac{a-c}{2b}\right) \\ \Pi_M &= \frac{(a-c)^2}{4b} \end{aligned}$$

[1] If $a = c$ we would have $P(Q) = \frac{c+c}{2} = \frac{2c}{2} = c$. Since $a > c$ we have some price greater than c.

Note that this is positive because $(a-c)^2 > 0$ and $4b > 0$. To summarize:

$$\begin{aligned} Q &= \frac{1}{2} * \frac{a-c}{b} \\ P(Q) &= \frac{a+c}{2} \\ \Pi_M &= \frac{(a-c)^2}{4b} \end{aligned}$$

Now we will consider oligopoly models.

2 Simultaneous oligopoly models

There are two standard models used in an oligopoly setting. One is the quantity choice model, proposed by Cournot in 1838. In Cournot's model all firms are identical and their choice variable is quantity. The other standard model is a pricing choice model, proposed by Bertrand in response to Cournot's model. Bertrand proposed that firms chose prices, not quantities. We will examine these two models and compare their resulting equilibria and outcomes. We begin with simultaneous games and then move to sequential games.

One goal of showing the Bertrand and Cournot games is that different model structures call for different techniques to find the equilibrium. We will see that in the Cournot game we can use differential calculus to find the PSNE while in the Bertrand game we will use the "method of exhaustion". The method of exhaustion simply means that you look at all possible cases and determine which one or ones are PSNE.

2.1 Quantity choice (Cournot) game

Suppose there are 2 firms who simultaneously choose what quantity level to produce. The firms face a downward sloping inverse market demand function, $P(Q)$, where Q is the market quantity and is simply the sum of the two individual firms quantities. The firms are identical in the products that they produce and have identical total cost functions $TC(q_i) = cq_i$ and marginal costs of c. Each firm wishes to maximize profit, which is:

$$\Pi_i = P(Q) * q_i - cq_i$$

Note that this is different from the monopolist problem because now Firm 2's quantity enters into Firm 1's profit function and vice versa. Again we could show that, in this model, price will be greater than marginal cost at the equilibrium. Also note that the Nash equilibrium in this game will be a pair of quantities, one for firm 1 (q_1) and one for firm 2 (q_2). It may be helpful when starting these problems to determine what exactly a Nash equilibrium is – it is a set of quantities, a set of prices, a set of best response functions, a best response function for one firm and a single quantity choice for the other firm, etc.

2.1.1 Linear inverse demand

We will now work through an example where the inverse market demand function is linear, so that $P(Q) = a - bQ$. In this case, we now have $Q = q_1 + q_2$, so:

$$P(Q) = a - bq_1 - bq_2$$

Since there are two firms there are two maximization problems that we will need to solve, one for Firm 1 and one for Firm 2. Firm 1 maximizes its profit, which is:

$$\Pi_1 = (a - bq_1 - bq_2) * q_1 - cq_1$$

Technically, when we differentiate this profit function we will be treating q_2 as a constant (even though it is a variable) so that we will be taking the partial derivative. So we have:

$$\frac{\partial \Pi_1}{\partial q_1} = a - 2bq_1 - bq_2 - c$$

Now setting this equal to zero[2] and solving for our choice variable, q_1, we have:

$$\begin{aligned} a - 2bq_1 - bq_2 - c &= 0 \\ a - bq_2 - c &= 2bq_1 \\ \frac{a - bq_2 - c}{2b} &= q_1 \end{aligned}$$

Now, think about what this equation means – for any value of q_2, this equation tells us the quantity choice q_1 should produce to maximize its profit.[3] Thus, $\frac{a-bq_2-c}{2b}$ is firm 1's best response function as it tells us what firm 1's optimal quantity choice should be. Now, we are not done. We need to find firm 2's best response function (we still will not be done, but we will be closer). Firm 2 maximizes its profit which is:

$$\Pi_1 = (a - bq_1 - bq_2) * q_2 - cq_2$$

We will take the partial derivative with respect to q_2, treating q_1 as a constant, to see:

$$\frac{\partial \Pi_1}{\partial q_1} = a - bq_1 - 2bq_2 - c$$

Setting this equal to zero and solving for q_2 we have:

$$\begin{aligned} a - bq_1 - 2bq_2 - c &= 0 \\ a - bq_1 - c &= 2bq_2 \\ \frac{a - bq_1 - c}{2b} &= q_2 \end{aligned}$$

Again, this is firm 2's best response function. Now we know that $q_1 = \frac{a-bq_2-c}{2b}$ and $q_2 = \frac{a-bq_1-c}{2b}$. However, we are still not done as all we have found are the best response functions. We need to solve this system of equations to find a precise q_1 and q_2. So we substitute one in for the other to find:

$$\begin{aligned} q_1 &= \frac{a - b\left(\frac{a-bq_1-c}{2b}\right) - c}{2b} \\ 2bq_1 &= a - \left(\frac{a - bq_1 - c}{2}\right) - c \\ 4bq_1 &= 2a - a + bq_1 + c - 2c \\ 3bq_1 &= a - c \\ q_1 &= \frac{a - c}{3b} \end{aligned}$$

Now we substitute back in to find q_2:

$$\begin{aligned} q_2 &= \frac{a - b\left(\frac{a-c}{3b}\right) - c}{2b} \\ 2bq_2 &= a - \left(\frac{a - c}{3}\right) - c \\ 6b_2 &= 3a - a + c - 3c \\ 6bq_2 &= 2a - 2c \\ q_2 &= \frac{2a - 2c}{6b} \\ q_2 &= \frac{a - c}{3b} \end{aligned}$$

Thus, the PSNE to this game is firm 1 choose $q_1 = \frac{a-c}{3b}$ and firm 2 choose $q_2 = \frac{a-c}{3b}$.

[2] Just to ensure that we are finding a maximum, if we differentiate $a - 2bq_1 - bq_2 - c$ with respect to q_1 we get: $-2b$. This is negative as long as $b > 0$, which it is by assumption, so we have found a maximum.

[3] There is a slight technical detail here – we need $q_1 \geq 0$ as $q_1 < 0$ makes no sense. Thus, as long as q_2 does not produce more than $\frac{a-c}{b}$ (the entire market quantity for perfect competition) this function is the best response function. Technically, the true best response function is $q_1 = Max\left[\frac{a-bq_2-c}{2b}, 0\right]$, which simply means that firm 1 chooses the maximum of 0 and the best response based upon q_2's quantity choice.

Small technicality in each firm's best response functions Firm 2's best response function is $q_2 = \frac{a-c}{2b} - \frac{1}{2}q_1$. Note that if Firm 1 produces $\frac{a-c}{b}$ then Firm 2's best response is to produce 0 – this is fine. However, if Firm 1 produces MORE THAN $\frac{a-c}{b}$, then Firm 2's best response is to produce a negative amount. This cannot happen, so technically Firm 2's best response function should be:

$$q_2 = Max\left\{\frac{a-c-bq_1}{2b}, 0\right\}$$

and Firm 1's should be:

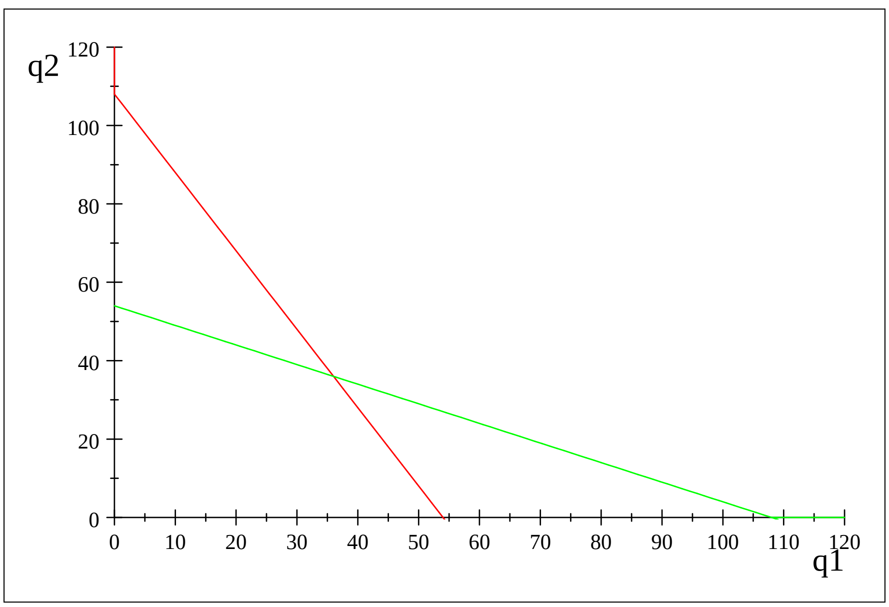
$$q_1 = Max\left\{\frac{a-c-bq_2}{2b}, 0\right\}$$

2.1.2 Graphing best response functions

Another way to find the Cournot-Nash solution is to plot the best response functions. We can rewrite $q_1 = \frac{a-c-bq_2}{2b}$ and $q_2 = \frac{a-c-bq_1}{2b}$ as $q_1 = \frac{a-c}{2b} - \frac{1}{2}q_2$ and $q_2 = \frac{a-c}{2b} - \frac{1}{2}q_1$. Note that these are just lines. For firm 1's best response function it will make more sense (when plotting the function) if we rewrite it as q_2 as a function of q_1 (since this is how we normally think of functions). So we now have these 2 equations:

$$\begin{aligned} q_2 &= \frac{a-c}{b} - 2q_1\text{: Firm 1's best response function} \\ q_2 &= \frac{a-c}{2b} - \frac{1}{2}q_1\text{: Firm 2's best response function} \end{aligned}$$

Using $a = 120$, $b = 1$, and $c = 12$, if we plot these functions we will get:



Note that Firm 1's best response is in red while Firm 2's best response is in green. The intersection of these two best response curves is simply at the point $(36, 36)$, which is the equilibrium quantities of $\frac{a-c}{3b}$ when $a = 120$, $b = 1$, and $c = 12$.

2.1.3 N-firm Cournot game

We can derive general results for the N-firm Cournot game, where $N = 1, 2, 3, ..., \infty$. Suppose the exact same structure, with $P(Q) = a - bQ$, but now $Q = q_1 + q_2 + ... + q_N$. In this game we will focus on a *symmetric equilibrium*. A symmetric equilibrium is an equilibrium in which all firms use the same strategy – given that the firms are all identical this (hopefully) seems like a realistic assumption on our part. The mathematical reason for this is that it will make finding the firm's best response function easier. Firm 1 maximizes its profit which is:

$$\begin{aligned}
\Pi_1 &= (a - bq_1 - bq_2 - ... - bq_N) * q_1 - c * q_1 \\
\frac{\partial \Pi_1}{\partial q_1} &= a - 2bq_1 - bq_2 - ... - bq_N - c \\
0 &= a - 2bq_1 - bq_2 - ... - bq_N - c \\
2bq_1 &= a - bq_2 - ... - bq_N - c \\
q_1 &= \frac{a - bq_2 - ... - bq_N - c}{2b}
\end{aligned}$$

Note that this looks very similar to Firm 1's best response function when there were 2 firms, only now there are more bq_i terms in the numerator because there are more firms. Since we are looking for a symmetric equilibrium, in equilibrium we will have $q_1 = q_2 = q_3 = ... = q_N$. So we can substitute q_1 in for all the other terms to get:

$$\begin{aligned}
q_1 &= \frac{a - bq_1 - ... - bq_1 - c}{2b} \\
2bq_1 &= a - bq_1 - ... - bq_1 - c
\end{aligned}$$

The key step here is to remember how many other bq_i terms there are in the numerator. They range from 2 to N, so there are $N-1$ bq_i terms on the right hand side. So we have:

$$\begin{aligned}
2bq_1 &= a - (N-1)\,bq_1 - c \\
2bq_1 + (N-1)\,bq_1 &= a - c \\
(N+1)\,bq_1 &= a - c \\
q_1 &= \frac{a-c}{(N+1)\,b}
\end{aligned}$$

Thus, in our symmetric equilibrium we have that $q_1 = \frac{a-c}{(N+1)b}$. We also know, since $q_1 = q_2 = q_3 = ... = q_N$, that every firm will produce using $q_i = \frac{a-c}{(N+1)b}$. So the symmetric PSNE in this game is that $q_1 = q_2 = ... = q_N = \frac{a-c}{(N+1)b}$. Notice that when $N = 2$ we have $q_1 = q_2 = \frac{a-c}{3b}$, which is what we found for the duopoly example.

We can derive the price and profit results as well for the general case of N firms. We know that:

$$P(Q) = a - bQ$$

and that:

$$Q = N * \frac{a-c}{(N+1)\,b}$$

because there are N firms in the market and they each produce $\frac{a-c}{(N+1)b}$. So price is:

$$\begin{aligned}
P(Q) &= a - b * \left(N * \frac{a-c}{(N+1)\,b}\right) \\
P(Q) &= a - \left(\frac{Na - Nc}{(N+1)}\right) \\
P(Q) &= \frac{aN + a}{N+1} - \left(\frac{Na - Nc}{N+1}\right) \\
P(Q) &= \frac{aN + a - Na + Nc}{N+1} \\
P(Q) &= \frac{a + Nc}{N+1}
\end{aligned}$$

Finally, we can find profit which is:

$$\begin{aligned}
\Pi &= P*q-c*q \\
\Pi &= \left(\frac{a+Nc}{N+1}\right)*\frac{a-c}{(N+1)\,b}-c*\frac{a-c}{(N+1)\,b} \\
\Pi &= \left(\frac{a+Nc}{N+1}-c\right)*\left(\frac{a-c}{(N+1)\,b}\right) \\
\Pi &= \left(\frac{a+Nc}{N+1}-\left(\frac{cN+c}{N+1}\right)\right)*\left(\frac{a-c}{(N+1)\,b}\right) \\
\Pi &= \left(\frac{a+Nc-cN-c}{N+1}\right)*\left(\frac{a-c}{(N+1)\,b}\right) \\
\Pi &= \left(\frac{a-c}{N+1}\right)*\left(\frac{a-c}{(N+1)\,b}\right) \\
\Pi &= \frac{(a-c)^2}{(N+1)^2\,b}
\end{aligned}$$

Now, what happens as N gets really large, or as $N\to\infty$? Profit is easy to see – it goes to zero because the denominator increases rapidly while the numerator stays constant. We should have that $P(Q)\to c$ as $N\to\infty$. To see this we take:

$$\begin{aligned}
&\lim_{N\to\infty}\frac{a+Nc}{N+1} \\
&\lim_{N\to\infty}\frac{a}{N+1}+\lim_{N\to\infty}\frac{Nc}{N+1} \\
&0+\frac{\infty}{\infty}
\end{aligned}$$

Now, the 0 is fine but the $\frac{\infty}{\infty}$ is not. We can do one of two things – use L'Hopital's Rule, which states that if:

$$\lim_{N\to K}\frac{f(N)}{g(N)}=\frac{\infty}{\infty}$$

for some constant K then

$$\lim_{N\to K}\frac{f'(N)}{g'(N)}=\lim_{N\to K}\frac{f(N)}{g(N)}.$$

Thus, the limit of the ratio of the derivatives of the functions is equal to the limit of the ratio of the functions. Taking the derivative of Nc and $N+1$ with respect to N we have:

$$\lim_{N\to\infty}\frac{c}{1}=c$$

So that price approaches marginal cost as the number of firms goes to infinity. Again, this is consistent with our model of perfect competition. The alternative method of finding that price goes to marginal cost is to multiply $\frac{Nc}{N+1}$ by $\frac{1/N}{1/N}$ and take that limit. We would have:

$$\lim_{N\to\infty}\frac{\frac{Nc}{N}}{\frac{N+1}{N}}=\lim_{N\to\infty}\frac{c}{1+\frac{1}{N}}=\frac{c}{1}=c.$$

2.2 Pricing (Bertrand) game

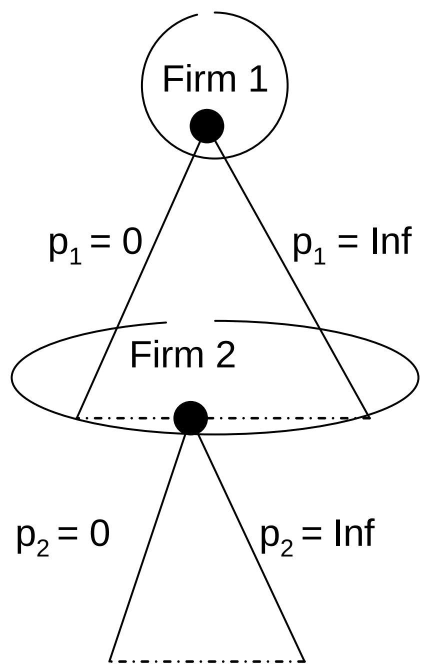
About 50-60 years after Cournot, another economist (Bertrand) found fault with Cournot's work. Bertrand believed that firms competed by choosing prices, and then letting the market determine the quantity sold. Recall that if a monopolist wishes to maximize profit it can choose either price or quantity while allowing the market to determine the variable that the monopolist did not choose. The resulting price and quantity in the market is unaffected by the monopolist's decision of which variable to use as its strategic variable. We will see that this is not the case for a duopoly market.

The general structure of the game is as follows. There are identical 2 firms competing in the market – the firms produce identical products, have the same cost structure ($TC = c * q$ and $MC = c$), and face the same downward sloping inverse demand function, $P(Q) = a - bQ$. However, in this game it is more useful to structure the inverse demand function as an actual demand function (because the firms are choosing prices and allowing the market to determine the quantity sold), so we can rewrite the inverse demand function as a demand function, $Q(P) = \frac{a}{b} - \frac{1}{b}P$. Consumers have no brand or firm loyalty, and it is assumed that all consumers know the prices of both firms in the market. Consumers will purchase from the lowest priced producer according to the demand function. This last assumption means that each firm's quantity is determined by the table below (p_1 is Firm 1's price choice and p_2 is Firm 2's price choice):

	q_1	q_2
if $p_1 > p_2$	0	$\frac{a-p_2}{b}$
if $p_1 = p_2$	$\frac{1}{2} * \frac{a-p_1}{b}$	$\frac{1}{2} * \frac{a-p_2}{b}$
if $p_1 < p_2$	$\frac{a-p_1}{b}$	0

Thus, the firm with the lowest price will sell the entire market quantity at that price. If the firms have equal prices then they will each sell $\frac{1}{2}$ the total market quantity at that price. A key piece of information is that the price spaces is continuous, not discrete. Thus, firms can charge prices such as \$12.0004 or \$$\sqrt{2}$.

Even though the game is simultaneous it is difficult to represent in a matrix because both firms have an infinite amount of possible strategies. Thus, we use a game tree as follows:



Game tree for simultaneous Bertrand game

to represent the simultaneous Bertrand game. Note that a PSNE to this game involves Firm 1 choosing a single price p_1 and Firm 2 choosing a single price p_2. Since the demand function for the firms are not differentiable we will have to skip the calculus and rely on intuition to find the PSNE to this game. We should first note that neither firm will choose a price below marginal cost of c because then one firm (or both) will earn a negative profit and could do better by charging a higher price (say a price equal to c). Now, there are 4 potential cases which could occur:

1. $p_i > p_j > c$

   In this case one firm has a higher price than the other firm. This is NOT a NE. The reason it is not is because Firm i has profit $\Pi_i = 0$ since it is selling zero. Firm i could increase profit by charging slightly less than Firm j – now it would capture the entire market and make a positive profit rather than selling 0 and making a zero profit.

2. $p_i = p_j > c$

   In this case both firms are charging the same price and that price is greater than marginal cost. This is NOT a NE. In this case both firms are only selling to $\frac{1}{2}$ of the market. However, if either firm slightly lowered its price it would capture the entire market, and because the price space is continuous, the loss in revenue from slightly lowering the price would be more than offset by a gain in revenue from selling to more people.[4]

3. $p_i > p_j = c$

   In this case one firm is charging marginal cost and the other firm is charging some price above marginal cost. This is NOT a NE. So far we have focused on the firm with the higher price undercutting the firm with the lower price – in this case that will not work because the firm with the higher price does not want to undercut the firm with the lower price because then it would have to choose $p_i < c$ which will lead to a negative profit. So now focus on the firm chargining marginal cost. Since the price space is continuous no matter what price is chosen by the other firm there is always some price between p_j such that $p_i > p_j > c$ and the firm which was charging marginal cost can raise its price and earn a positive profit. Note that we already know that $p_i > p_j > c$ is NOT a NE because that is case 1.

4. $p_i = p_j = c$

   In this case both firms are charging price equal to marginal cost and earning a zero profit. Does either firm wish to lower price? No because then it would make itself worse off because it would be earning a negative profit. Does either firm have the incentive to raise price? No, because if either firm unilaterally raises price it will still earn a zero profit. Thus, this is the NE of the game, and it is the unique PSNE of this game (there may be MSNE to this game – we will not worry about those).

In equilibrium to the Bertrand game all it takes is 2 firms to drive the outcome to the perfectly competitive outcome with $p = c$ and $\Pi = 0$. Note that there are very special features of this game that are driving this equilibrium. One is that the demand function is not continuous in that if one firm changes its price a small amount it can go from selling 0 quantity to selling the entire market quantity. So if we had a demand function such that one firm did not lose all of its sales when it priced above another firm then this perfectly competitive outcome need not hold. Also, we could use calculus to solve the problem rather than working through it one a case-by-case basis. Another feature of the model that drives the perfectly competitive outcome is the continuous price space. As was already mentioned if the price space is discrete it is possible to have an equilibrium in which both firms charge above marginal cost. Although there are many other changes that could be made, these are two of the most basic changes that could be made.

2.2.1 Weakly dominated strategy as part of NE

Earlier in the semester I mentioned that there was a classic game in which the unique PSNE involves both firms playing a weakly dominated strategy. This is true in the Bertrand game. Consider firm 1 charging $p_1 = c$ or alternatively charging $p_1 = p^M$, where p^M is the monopoly price. The following table shows the profit to Firm 1 for these 2 strategies when Firm 2 uses a certain set of strategies:

	$0 \leq p_2 < c$	$p_2 = c$	$c < p_2 < p^M$	$p_2 \geq p^M$
$p_1 = c$	$\Pi_1 = 0$	$\Pi_1 = 0$	$\Pi_1 = 0$	$\Pi_1 = 0$
$p_1 = p^M$	$\Pi_1 = 0$	$\Pi_1 = 0$	$\Pi_1 = 0$	$\Pi_1 =$ positive

If we compare the profitability of the two strategies, $p_1 = p^M$ weakly dominates $p_1 = c$ because both strategies earn a payoff of zero for many choices by firm 2, but $p_1 = p^M$ earns a positive payoff for *some* price choices by firm 2 (specifically those in which $p_2 \geq p^M$), whereas $p_1 = c$ never does.

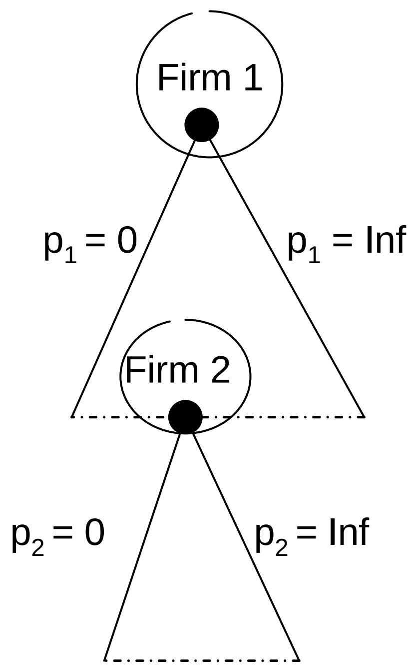
[4] If the price space is discrete then it is possible for an outcome like this to be a PSNE. The derivation of this equilibrium is left for the enterprising student.

3 Sequential games

In this section we consider sequential pricing and quantity choice games. The goal is to examine how (1) the equilibria are different in the two types of games and (2) to determine whether the outcomes differ if the game is played sequentially. Note that we will now be focusing on SPNE to these games.

3.1 Pricing game (Bertrand)

Consider the exact same setup as the simultaneous Bertrand game, only now one firm sets its price first and the other observes this price and sets it. Again, both firms are identical and both have constant marginal cost of c. If one firm prices above the other then the firm with the higher price sells zero units and makes zero profit while the firm with the lower price sells the entire market quantity at its chosen price. If the firms charge the same price then they will both sell $\frac{1}{2}$ the market quantity at that price. Suppose that Firm 1 makes its pricing decision first and then Firm 2 makes its pricing decision. The game tree would look like:



Game tree for sequential Bertrand game

Note that there is little difference between the game tree for the simultaneous Bertrand game and the sequential one. The difference is that we circle the entire continuum of strategies for the first player in the simultaneous game while we only circle part of the continuum for the sequential game.

The first question to ask is: What constitutes a strategy for each firm? A strategy for the first mover, Firm 1, is a single price. Firm 1 has only one information set, and thus need only specify one action for its strategy. Firm 2, however, must specify an action FOR EACH POSSIBLE price choice by Firm 1. Now, it will take a long time to write that down, but we can summarize what Firm 2 would do for broad classes (or ranges or intervals) of price choices by Firm 1. As always, the game is sequential and finite so we can start from the end and work towards the beginning. Let's suppose that Firm 1 chooses some price $p_1 \in [0, c)$. This simply means that Firm 1's price is somewhere below cost but not equal to it. What would Firm 2's best response be? Well, it could be many things, but anything that leads to Firm 2 not selling any items is the important part. There are a lot of actions that Firm 2 could take here, but we just need to specify one of them. Let's say that if Firm 1 chooses some price $p_1 \in [0, c)$ that Firm 2 chooses $p_2 = c$. Again, there are many, many different prices Firm 2 could specify to ensure it sells no items, and this is just one of them.[5]

[5] Firm 2 could specify that $p_2 = p_1 + \varepsilon$, where $\varepsilon > 0$ is a small amount above Firm 1's price. Again this ensures that Firm 2 will not sell any units, and it's profit will be zero, so it is also a best response.

Now, what if Firm 1 chooses $p_1 = c$? We know that Firm 2 will NOT chooses $p_2 < c$ because then it will earn a negative profit. Any other price choice by Firm 2 will lead to zero profit, so we just need to specify one of them. We can choose $p_2 = c$. Then both Firm 1 and Firm 2 split the market and receive 0 profit. This is the same outcome as in the simultaneous game.

We are not done yet. Let p^M be the monopoly price. Suppose that Firm 1 chooses a price $p_1 \in \left(c, p^M\right]$. What is Firm 2's best response now? Firm 2's best response is to price slightly under Firm 1, so $p_2 = p_1 - \varepsilon$, where $\varepsilon > 0$ is some small amount. Thus, Firm 2 just undercuts Firm 1 and captures the whole market, leaving Firm 1 with zero customers and zero profit. If Firm 1 were to charge \$17, Firm 2 would then charge \$16.999999999999. If Firm 1 charged \$16.999999999999, Firm 2 would charge \$16.999999999998. And so on.

Finally, suppose that Firm 1 chooses $p_1 > p^M$. Now, Firm 2 COULD just undercut Firm 1 and capture the whole market, but that is NOT a best response. Technically, the best response is to simply choose $p_2 = p^M$ if Firm 1 chooses $p_1 > p^M$. Why is this the best response? Because if Firm 2 is going to be the only firm in the market AND it can charge the monopoly price and still be the only firm in the market, then that is Firm 2's best response because it can do no better than being the only firm in the market and charging the monopoly price. Thus, if we make a table of Firm 2's best responses then we have Firm 2's strategy is:

	$p_1 \in [0, c)$	$p_1 = c$	$p_1 \in \left(c, p^M\right]$	$p_1 > p^M$
$p_2 =$	c	c	$p_1 - \varepsilon$	p^M

This is Firm 2's strategy as it specifies an action for every possible price choice by Firm 1. To be complete, when Firm 1 chooses some price $p_1 \in [0, c]$ (note that we are including c in this interval) then there are MANY other best responses we could specify. So this particular game does NOT have a unique solution. But the goal is to find one of them, not all of them, and this is one set of best responses for Firm 2.

Now we need to find Firm 1's choice. We know that Firm 1 will not choose $p_1 < c$ because then Firm 1's profit will be negative so we can rule those choices out.[6] What does Firm 1 earn if it chooses $p_1 = c$? It earns 0. What does Firm 1 earn if it chooses $p_1 \in \left(c, p^M\right]$? It earns 0 because Firm 2 slightly undercuts it and captures the entire market. What does Firm 1 earn if it chooses $p_1 > p^M$? It earns 0 because Firm 2 undercuts it and captures the entire market. Basically, Firm 1 has 2 options – choose some $p_1 < c$ and earn a $\Pi_1 < 0$, or choose some $p_1 \geq c$ and earn a $\Pi_1 = 0$. So you can choose ANY price (but just one) for Firm 1 as long as that price is above c and it is a best response to Firm 2's strategy. So a SPNE to the game could be: that Firm 1 choose $p_1 = c$ and Firm 2 uses the strategy as specified in the table above. The OUTCOME of that game would be $p_1 = c$ and $p_2 = c$ and both firms split the market but earn zero profit. A second SPNE would be: Firm 1 choose $p_1 = p^M + \varepsilon$ and Firm 2 uses the strategy as specified in the table above. The outcome in this case would be $p_1 = p^M + \varepsilon$ and $p_2 = p^M$ with Firm 1 earning $\Pi_1 = 0$ and Firm 2 earning the monopoly profit. Thus, in this game the second mover has an advantage over the first. While the first mover can never earn a positive profit, there are many equilibria in which the second mover may.

3.2 Quantity choice game (Stackelburg)

Now we consider a sequential quantity choice game known as the Stackelburg game. Again, the structure is identical to the simultaneous quantity choice game except that now one firm makes its quantity choice and then then the other observes this choice and makes its quantity choice. Both firms produce identical products, both have constant marginal cost of c, and the inverse demand function is $P(Q) = a - bQ$, where $Q = q_1 + q_2$.

As always, the first step is to figure out exactly what constitutes a SPNE for this game. Assume that Firm 1 moves first and Firm 2 observes this quantity choice by Firm 1 and then makes its quantity choice. Now, a strategy for Firm 1 is a single quantity choice – Firm 1 has one information set, so one action is all that is needed to specify a complete strategy for Firm 1. But Firm 2 must specify a quantity choice for EVERY POSSIBLE quantity choice made by Firm 1. Thus, it will be a similar problem as the pricing problem, although we will use calculus rather than intuition because $a - bQ$ is continuous and differentiable.

As always, start from the end of the game and work to the beginning. Firm 2 wishes to maximize its

[6] Even though we can rule these choices for p_1 out we still must specify what Firm 2 would do in that situation.

profit, which is:

$$\begin{aligned}
\Pi_2 &= (a - bq_1 - bq_2) * q_2 - c * q_2 \\
\frac{\partial \Pi_2}{\partial q_2} &= a - bq_1 - 2bq_2 - c \\
&\quad \text{set this equal to zero} \\
&\quad \text{and solve for } q_2 \\
0 &= a - bq_1 - 2bq_2 - c \\
q_2 &= \frac{a - bq_1 - c}{2b}
\end{aligned}$$

This is the exact same best response function we found before. Note that as far as finding Firm 2's equilibrium strategy we are done (almost – we have one little technical detail to think about). Firm 2's best response function specifies the optimal quantity choice given any quantity choice by Firm 1, which is exactly what we need to find. The only problem is that Firm 2 must produce a positive quantity, and for some q_1 choices the optimal choice for q_2 is negative. We fix this problem by using:

$$q_2 = Max\left\{\frac{a - bq_1 - c}{2b}, 0\right\}$$

which simply means that Firm 2 will choose whichever is larger, 0 or $\frac{a-bq_1-c}{2b}$. We use the 0 portion of this best response function when Firm 1 chooses $q_1 > \frac{a-c}{b}$, which is the perfectly competitive quantity.

That is Firm 2's strategy. What is Firm 1's optimal quantity choice then? Firm 1 maximizes its profit:

$$\Pi_1 = (a - bq_1 - bq_2) * q_1 - c * q_1$$

However, Firm 1 now knows that Firm 2 will best respond with $q_2 = \frac{a-bq_1-c}{2b}$. We can ignore the 0 portion of $Max\left\{\frac{a-bq_1-c}{2b}, 0\right\}$ because Firm 1 will not choose a quantity that large as it would earn negative profit. So we now substitute $q_2 = \frac{a-bq_1-c}{2b}$ directly into Firm 1's profit function to find:

$$\begin{aligned}
\Pi_1 &= \left(a - bq_1 - b\left(\frac{a - bq_1 - c}{2b}\right)\right) * q_1 - c * q_1 \\
\Pi_1 &= \left(a - bq_1 - \left(\frac{a - bq_1 - c}{2}\right)\right) * q_1 - c * q_1 \\
\Pi_1 &= \left(a - bq_1 - \frac{a}{2} + \frac{bq_1}{2} + \frac{c}{2}\right) * q_1 - c * q_1 \\
\Pi_1 &= \left(\frac{a}{2} - \frac{bq_1}{2} + \frac{c}{2}\right) * q_1 - c * q_1 \\
\frac{d\Pi_1}{dq_1} &= \frac{a}{2} - \frac{2bq_1}{2} + \frac{c}{2} - c \\
\frac{d\Pi_1}{dq_1} &= \frac{a}{2} - \frac{2bq_1}{2} - \frac{c}{2} \\
0 &= \frac{a}{2} - \frac{2bq_1}{2} - \frac{c}{2} \\
0 &= a - 2bq_1 - c \\
2bq_1 &= a - c \\
q_1 &= \frac{a - c}{2b}
\end{aligned}$$

Thus, Firm 1's best response is to produce the monopoly quantity in this game, $q_1 = \frac{a-c}{2b}$. So the SPNE to the game is:

$$\begin{aligned}
q_1 &= \frac{a - c}{2b} \\
q_2 &= Max\left\{\frac{a - bq_1 - c}{2b}, 0\right\}
\end{aligned}$$

As a result of playing this equilibrium strategy, the OUTCOME is as follows. Firm 2 will produce:

$$
\begin{aligned}
q_2 &= \frac{a - b\left(\frac{a-c}{2b}\right) - c}{2b} \\
2bq_2 &= a - \left(\frac{a-c}{2}\right) - c \\
4bq_2 &= 2a - a + c - 2c \\
4bq_2 &= a - c \\
q_2 &= \frac{a-c}{4b}
\end{aligned}
$$

Thus, the total market quantity will be:

$$
\begin{aligned}
Q &= q_1 + q_2 \\
Q &= \frac{a-c}{2b} + \frac{a-c}{4b} \\
Q &= \frac{3(a-c)}{4b}
\end{aligned}
$$

The price will be:

$$
\begin{aligned}
P(Q) &= a - bQ \\
P(Q) &= a - b\left(\frac{3(a-c)}{4b}\right) \\
P(Q) &= a - \left(\frac{3a-3c}{4}\right) \\
P(Q) &= \frac{4a - 3a + 3c}{4} \\
P(Q) &= \frac{a+3c}{4}
\end{aligned}
$$

Firm 1's profit will be:

$$
\begin{aligned}
\Pi_1 &= \left(\frac{a+3c}{4}\right) * \left(\frac{a-c}{2b}\right) - c * \left(\frac{a-c}{2b}\right) \\
\Pi_1 &= \left(\frac{a+3c}{4} - c\right) * \left(\frac{a-c}{2b}\right) \\
\Pi_1 &= \left(\frac{a-c}{4}\right) * \left(\frac{a-c}{2b}\right) \\
\Pi_1 &= \frac{(a-c)^2}{8b}
\end{aligned}
$$

Firm 2's profit will be:

$$
\begin{aligned}
\Pi_2 &= \left(\frac{a+3c}{4}\right) * \left(\frac{a-c}{4b}\right) - c * \left(\frac{a-c}{4b}\right) \\
\Pi_2 &= \left(\frac{a+3c}{4} - c\right) * \left(\frac{a-c}{4b}\right) \\
\Pi_2 &= \left(\frac{a-c}{4}\right) * \left(\frac{a-c}{4b}\right) \\
\Pi_2 &= \frac{(a-c)^2}{16b}
\end{aligned}
$$

Thus, the first mover has the advantage in this game as it can commit to its quantity choice. The second firm then must best respond to that quantity choice, and chooses a smaller quantity ($\frac{1}{2}$ of the first mover's quantity to be exact). Thus, the first mover ends up earning a higher profit than the second firm. In general, if one firm moves first and then is followed by mulitple firms who play a simultaneous game (so if Firm 1 chooses first and then Firms 2 and 3 play a simultaneous quantity choice game after observing Firm 1's decision) then the first mover will choose the monopoly quantity and let the other firms best respond to that.