1 Perfect Bayesian Equilibrium (PBE)

The final type of game that we will discuss is one that is dynamic (or sequential) and where players have imperfect information. Recall that a game of perfect information is a game like Chess or Checkers – all players know exactly where they are at every point in the game. When drawing out the game tree for games of perfect information, each information set contained a single decision node. Now, however, we allow for the possibility that some (or all) players do not know which node they are at, so that an information set may contain multiple decision nodes. This basic structure captures many types of card games, such as Bridge, Spades, and Poker, in which one player does not know what cards the other player(s) is holding. When playing games of this type people often use both the knowledge of the entire game as well as the actions that have previously occurred in the game to update their beliefs about which node in the information set they are at.

1.1 Definition and structure of a PBE

With a PBE we will still require that all players choose strategies that are best responses to the other player's strategies. However, when there is a player who has multiple decision nodes within an information set we now require that this player specifies a *belief* about which node in the information set he is at. The belief is simply a probability. Note that these probabilities (or beliefs) must follow the laws of probability – no probabilities greater than 1 or less than zero, and the probabilities for all decision nodes within an information set must sum to 1. Thus the first new requirement is that beliefs for the uninformed players must be specified – exactly how these beliefs (or probabilities) are specified will be discussed shortly.

The second requirement that we make is that given the players' beliefs, the strategy choices must be sequentially rational. Thus, each player must be acting optimally at each information set given his beliefs and the other players' subsequent strategies (the strategies that follow the information set). So the second requirement basically says that strategies must now also be best responses to beliefs, in addition to best responses to other players' strategies.

The third and fourth requirements for a PBE specify how the beliefs must be updated. At information sets along the equilibrium path (along the equilibrium path means that the information set is reached when the equilibrium is played) beliefs are determined by Bayes' rule and the players' equilibrium strategies. These first 3 requirements constitute what is known as a *weak perfect Bayesian equilibrium* (WPBE). A fourth requirement is that off the equilibrium path beliefs are also determined by Bayes' rule and the players' equilibrium strategies where possible. The 4 requirements together define a *strong perfect Bayesian equilibrium* (SPBE).

1.2 WPBE and SPBE

Now we will differentiate between a WPBE and a SPBE. Look at the following game:
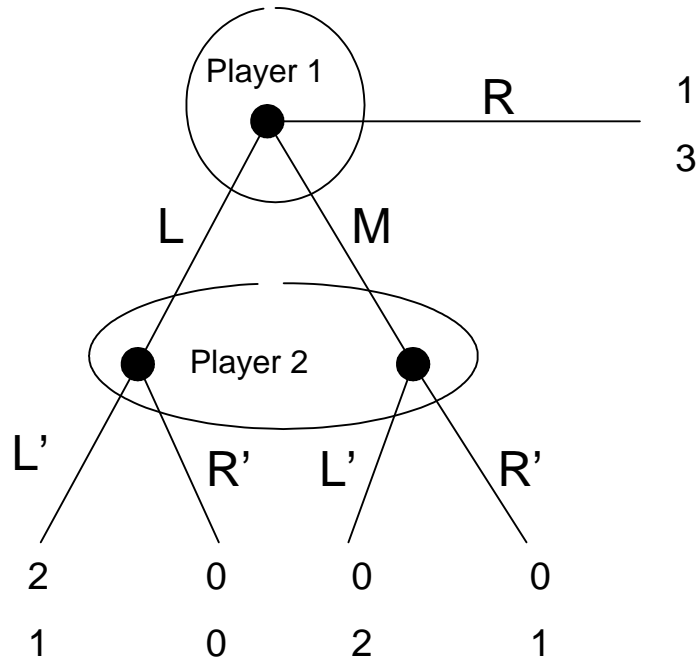
Player 2 cannot tell which node he is at if player 1 chooses L or M. There are 2 SPNE. One is that Player 1 chooses R and Player 2 chooses R', and the other is that Player 1 chooses L and Player 2 chooses L'. However, L' strictly dominates R', so Player 1 knows that if he chooses L he will get 2 (choosing M yields a payoff of 0 and R yields a payoff of 1). Player 2 knows this as well, and so his belief is that Player 1 chooses L with probability 1. Thus, Player 2 has updated his belief about which strategy Player 1 is using if Player 2 gets to make a decision. A weak perfect Bayesian equilibrium for this game is that Player 1 chooses L, Player 2 believes that Player 1 chooses L with probability 1, and Player 2 chooses L'. Note that this equilibrium also satisfies requirement 4 because there are no off-the-equilibrium path information sets, so it is also a SPBE.

Let's look at another game to illustrate the difference between the weak and strong Perfect Bayesian equilibrium concepts.
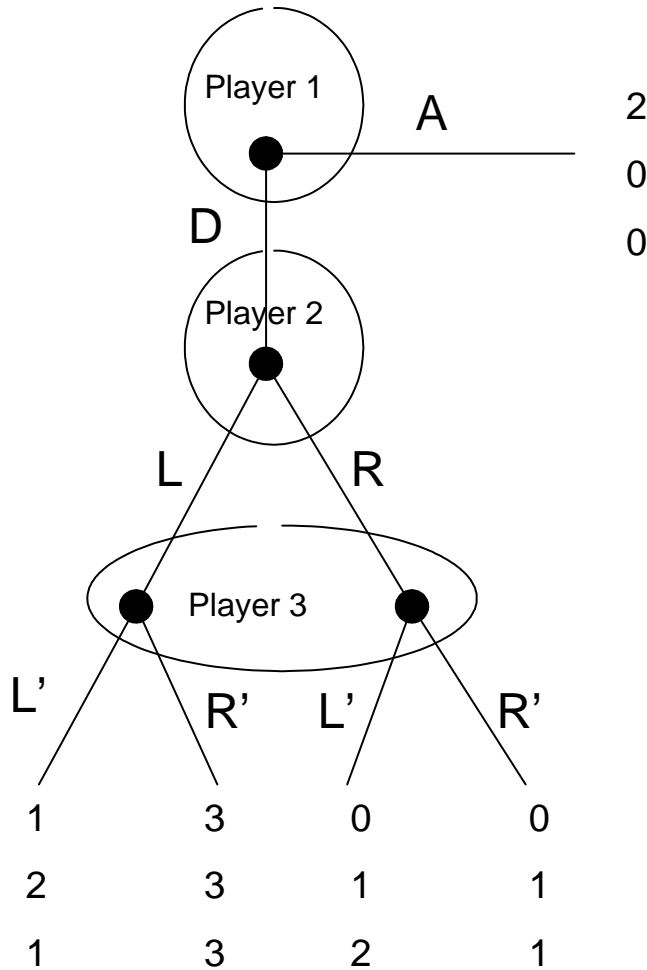
First begin by analyzing the subgame that begins at Player 2's decision node.

		Player 3	
		L'	R'
Player 2	L	2, 1	3, 3
	R	1, 2	1, 1

The Nash equilibrium to this subgame is Player 2 chooses L and Player 3 chooses R'. Player 1 knows this, and chooses D. So D, L, R' is a SPNE to the game, and if Player 3 has a belief that Player 2 chooses Left with probability 1 (which Player 3 should have since L is a strictly dominant strategy for Player 2), then requirements 1-3 are satisfied for this to be a weak perfect Bayesian equilibrium. Again, requirement 4 is satisfied because there are no off the equilibrium path information sets.

Now, consider the potential equilibrium where Player 1 chooses A, Player 2 chooses L, Player 3 believes that Player 2 chooses R with probability 1, and Player 3 chooses L'. Note that Player 1 is playing a best response to the strategies L and L' by Players 2 and 3 (Player 1 receives 1 if he plays D and 2 if he plays A). Player 3 is playing a best response given his beliefs about Player 2's actions (if he believes Player 2 is choosing R then Player 3 does better by choosing L'). Player 2 is choosing his strictly dominant strategy of L, and even if he switched his strategy to R he would still receive 0, so he is playing a best response to A, R'. Thus, this set of strategies and beliefs satisfies the first 3 requirements and is a weak perfect Bayesian equilibrium. However, the 4th requirement is not satisfied because Player 3's belief is inconsistent with the fact that Player 2 has a dominant strategy to play L. To implement this consistency requirement, Player 3 must believe Player 2 plays L with probability 1, but then L' is NOT an optimal response (R' is the optimal response) and we are now led back to D, L, R' with Player 3 believing that Player 2 chooses L with probability 1.

2 Separating and Pooling equilibria

There are many instances in which one player knows his own type and then takes an action and another player cannot observe the first player's type but only his action. These types of games generally fall under the category of signaling games, because the action taken by the first mover may (or may not) signal which type the first mover is. In a pooling equilibrium, all types choose the same action (or send the same signal). In a separating equilibrium, different types choose different actions. Thus, in a pooling equilibrium the player without the information on type is unable to update his belief about which type of player chose which action since all types are choosing the same action. We will consider the following game:
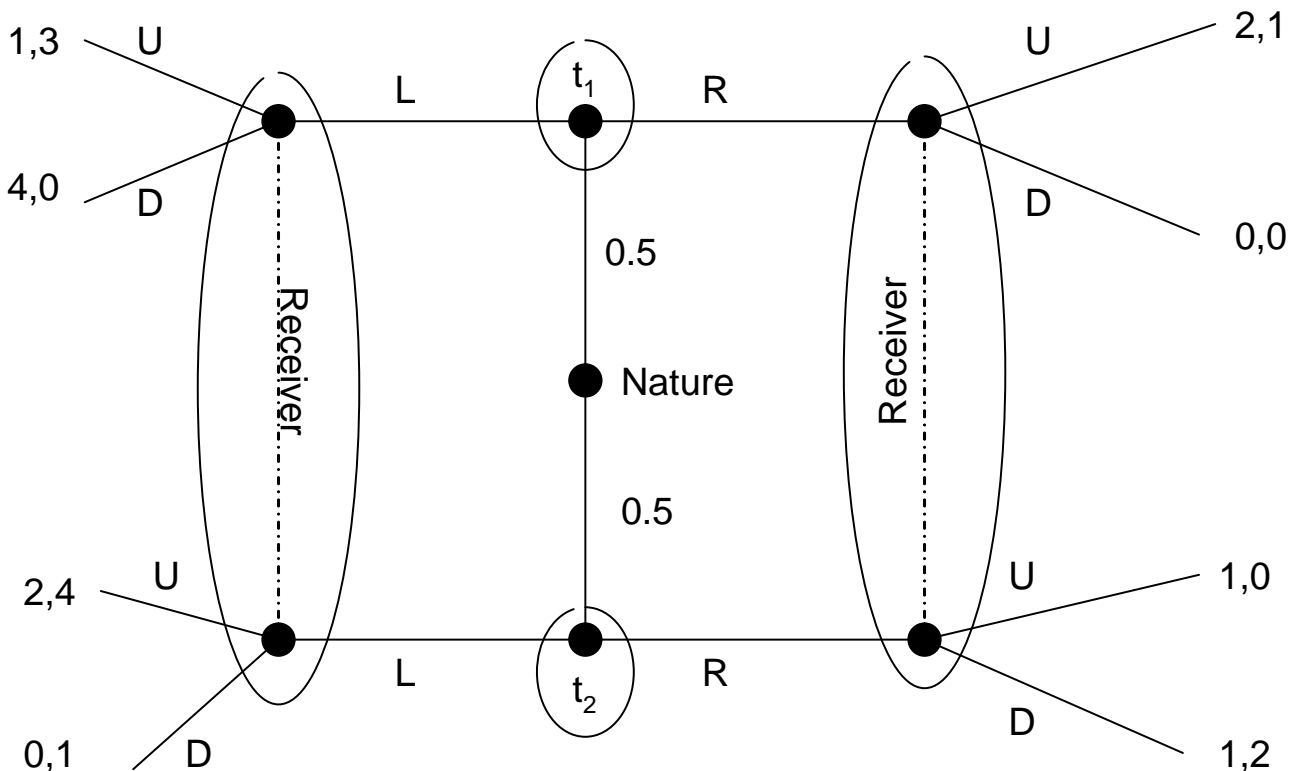
Note how the game plays out. Nature first determines the type of the first mover (the Sender). With probability of 0.5 the Sender is type t_1 and with probability 0.5 the Sender is type t_2. The Sender knows which type he is. Each Sender type can choose either L or R. The Receiver observes only the choice of L or R and not the Sender's actual type. Based upon the observation of L or R the Receiver can then choose U or D. Payoffs then follow, with the Sender's (first mover's) payoff listed first and the Receiver's (second mover's) payoff listed second. There are two potential pooling equilibria and two potential separating equilibria. The two potential pooling equilibria involve either (1) both types t_1 and t_2 choosing L or (2) both types t_1 and t_2 choosing R. The two potential separating equilibria involve either (1) type t_1 choosing R and type t_2 choosing L or (2) type t_1 choosing L or type t_2 choosing R. We will discuss these potential equilibria in detail.

2.1 Potential Separating Equilibria

As mentioned there are two potential separating equilibria. Note that all of these equilibria will consist of (1) a strategy for the Sender (which is an action if the Sender is type t_1 and an action if the Sender is type t_2), (2) a set of beliefs for the Receiver about which decision node in the information set the Receiver is at, and (3) a strategy for the Receiver (which is an action if L is observed and an action if R is observed). We begin by analyzing the one where type t_1 chooses R and type t_2 chooses L.

2.1.1 Type t_1 chooses R and type t_2 chooses L

We begin by specifying the *potential* strategy choice of the Sender. Suppose the Sender uses the separating strategy: R if t_1 and L if t_2. If this is the case, what does the Receiver believe? Note that when forming the Receiver's beliefs it is as if the Receiver knows precisely which equilibrium is being played. Thus, what is the probability that the Sender is type t_1 if the Receiver observes R? We can abbreviate "the probability that the Sender is type t_1 if the Receiver observes R" as $\Pr(t_1|R)$. In this particular potential equilibrium, only the t_1 type chooses R. Thus, if the Receiver observes a choice of R it should believe with 100% probability that it was type t_1 who chose R. Thus, we have $\Pr(t_1|R) = 1$. Now, what is $\Pr(t_2|R)$? Since type t_2 is NEVER choosing R in this potential separating equilibrium, the Receiver should believe that the probability a type t_2 chose R is equal to 0, or $\Pr(t_2|R) = 0$. We are not yet done with beliefs – we still need to specify the Receiver's beliefs about which node he is at when a choice of L is observed. What is $\Pr(t_1|L)$? If the Receiver observes L it knows with certainty that, in this potential equilibrium, it was type t_2 who chose L. Thus, $\Pr(t_1|L) = 0$ and $\Pr(t_2|L) = 1$. We are now done specifying the Receiver's beliefs.

The Receiver must now specify a strategy, so an action if he observes L and an action if he observes R. If L is observed the Receiver knows it is type t_2, and also knows that if he chooses U he will receive 4 and if he chooses D he will receive 1. Since $4 > 1$ the Receiver chooses U. If R is observed the Receiver knows it is type t_1, and also knows that if he chooses U he will receive 1 and if he chooses D he will receive 0. Since $1 > 0$ the Receiver chooses U. Thus, a *potential* separating PBE of the game is:

$$
\begin{array}{l}
t_1 \text{ choose } R \\
t_2 \text{ choose } L
\end{array} \quad \{\text{Sender's strategy}
$$

$$
\begin{array}{l}
\Pr(t_1|L) = 0 \\
\Pr(t_2|L) = 1 \\
\Pr(t_1|R) = 1 \\
\Pr(t_2|R) = 0
\end{array} \quad \{\text{Receiver's beliefs}
$$

$$
\begin{array}{l}
\text{choose } U \text{ if } L \text{ observed} \\
\text{choose } U \text{ if } R \text{ observed}
\end{array} \quad \{\text{Receiver's strategy}
$$

Again, as of now this is a *potential* separating PBE of the game. We need to make sure that (1) the Receiver is playing a best response to the Sender's strategy and his (the Receiver's) beliefs and (2) the Sender is playing a best response to the Receiver's strategy. We have already done part (1) in constructing the Receiver's strategy. However, we still need to check part (2). Under the proposed equilibrium, type t_1 receives 2; if type t_1 were to switch to L he would receive 1 (because the Receiver is choosing U if L is observed) and so type t_1 does not wish to deviate. Under the proposed equilibrium type t_2 receives 2; if type t_2 were to switch to R he would receive 1 (because the Receiver is choosing U if R is observed) and so type t_2 does not wish to deviate. Thus, since no player wishes to deviate, the proposed PBE is a separating PBE of the game.

2.1.2 Type t_1 chooses L and type t_2 chooses R

Again, begin with the *potential* separating PBE of the game. Type t_1 chooses L and type t_2 chooses R. The Receiver's beliefs if L is observed are that $\Pr(t_1|L) = 1$ and $\Pr(t_2|L) = 0$ because in this equilibrium only type t_1 chooses L. The Receiver's beliefs if R is observed are that $\Pr(t_1|R) = 0$ and $\Pr(t_2|R) = 1$ because in this equilibrium only type t_2 chooses R. If L is observed the Receiver gets 3 if U is chosen and 0 if D is chosen. Thus, the Receiver would choose U if L is observed. If R is observed the Receiver gets 0

if U is chosen and 2 if D is chosen and so chooses D. A *potential* separating PBE is:

$$
\begin{aligned}
&t_1 \text{ choose } L \\
&t_2 \text{ choose } R
\end{aligned} \quad \{\text{Sender's strategy}
$$

$$
\begin{aligned}
&\Pr(t_1|L) = 1 \\
&\Pr(t_2|L) = 0 \\
&\Pr(t_1|R) = 0 \\
&\Pr(t_2|R) = 1
\end{aligned} \quad \{\text{Receiver's beliefs}
$$

$$
\begin{aligned}
&\text{choose } U \text{ if } L \text{ observed} \\
&\text{choose } D \text{ if } R \text{ observed}
\end{aligned} \quad \{\text{Receiver's strategy}
$$

Again, we need to check to see if either type t_1 or type t_2 would deviate. In the proposed equilibrium type t_1 receives 1 (because the Receiver chooses U if L is chosen); if type t_1 were to switch to R he would receive 0 (because the Receiver chooses D if R is observed). Thus, type t_1 would not wish to deviate. For type t_2, in the proposed equilibrium he receives 1; if type t_2 were to switch to L he would receive 2 (because the Receiver chooses U if L is chosen). Thus, type t_2 WOULD deviate from the proposed equilibrium, so the proposed equilibrium is NOT a separating PBE to this game. Thus, there is no separating equilibrium where type t_1 chooses L and type t_2 chooses R.

2.2 Potential Pooling Equilibria

We now shift our focus to pooling equilibria. With pooling equilibria all types choose the same action so that the uninformed party (the Receiver in our game) cannot condition his belief upon the action chosen. There are two potential pooling equilibria in our game – one where both types t_1 and t_2 choose L and another where both types t_1 and t_2 choose R.

2.2.1 Both types choose L

Suppose that both Sender types choose L. If this is the case then what is the probability that the sender is type t_1 if the Receiver observes L? It is just the starting (or initial or prior) probability of type t_1 being drawn by nature, which in this example is 0.5. Thus, $\Pr(t_1|L) = 0.5$. Now, what is the probability that the Sender is type t_2 if the Receiver observes L? Again, it is just the initial probability of 0.5. Thus, $\Pr(t_2|L) = 0.5$. That is the easy part – since it is a pooling equilibrium there is no updating to be done on the action upon which the Senders pool. However, we still need to specify $\Pr(t_1|R)$ and $\Pr(t_2|R)$. But there really is no good reason for any particular probability at this point, so we just let $\Pr(t_1|R) = q$ and $\Pr(t_2|R) = (1-q)$ for now.

Now, what is the Receiver's best response if L is observed? If the Receiver chooses U then he gets 3 half of the time and 4 the other half of the time, so his expected value is $3 * \frac{1}{2} + 4 * \frac{1}{2} = \frac{7}{2}$ if he chooses U. If he chooses D he gets 0 half of the time and 1 the other half of the time, so his expected value is $0 * \frac{1}{2} + 1 * \frac{1}{2} = \frac{1}{2}$. So if the Receiver observes L the Receiver will choose U (note that technically U is a strictly dominant strategy for the Receiver if L is observed).

What is the Receiver's best response if R is observed? If the Receiver chooses U then he gets 1 with probability q and he gets 0 with probability $1-q$, so his expected value is $1 * q + 0 * (1-q) = q$. If the Receiver chooses D then he gets 0 with probability q and 2 with probability $1-q$, so his expected value from choosing D is $0 * q + 2 * (1-q) = 2 - 2q$. When will his payoff from choosing U be greater than his payoff from choosing D? When $q \geq 2 - 2q$, or when $q \geq \frac{2}{3}$. Thus, if the Receiver believes (for whatever reason – remember, this is off the equilibrium path) that $q \geq \frac{2}{3}$ then the Receiver will choose U, while if $q < \frac{2}{3}$ the

Receiver will choose D. So, our proposed pooling PBE is:

$$
\begin{array}{ll}
\begin{array}{l} t_1 \text{ choose } L \\ t_2 \text{ choose } L \end{array} & \{\text{Sender's strategy} \\[2ex]
\begin{array}{l} \Pr\left(t_1|L\right) = \frac{1}{2} \\ \Pr\left(t_2|L\right) = \frac{1}{2} \\ \Pr\left(t_1|R\right) \geq \frac{2}{3} \\ \Pr\left(t_2|R\right) < \frac{2}{3} \end{array} & \{\text{Receiver's beliefs} \\[3ex]
\begin{array}{l} \text{choose } U \text{ if } L \text{ observed} \\ \text{choose } U \text{ if } R \text{ observed} \end{array} & \{\text{Receiver's strategy}
\end{array}
$$

or alternatively:

$$
\begin{array}{ll}
\begin{array}{l} t_1 \text{ choose } L \\ t_2 \text{ choose } L \end{array} & \{\text{Sender's strategy} \\[2ex]
\begin{array}{l} \Pr\left(t_1|L\right) = \frac{1}{2} \\ \Pr\left(t_2|L\right) = \frac{1}{2} \\ \Pr\left(t_1|R\right) \leq \frac{2}{3} \\ \Pr\left(t_2|R\right) > \frac{2}{3} \end{array} & \{\text{Receiver's beliefs} \\[3ex]
\begin{array}{l} \text{choose } U \text{ if } L \text{ observed} \\ \text{choose } D \text{ if } R \text{ observed} \end{array} & \{\text{Receiver's strategy}
\end{array}
$$

We can check to see if either or neither or both of these are pooling PBE (note the difference in the two potential equilibria is in the inequality sign for the Receiver's beliefs). Checking the first one, would type t_1 deviate from L to R? In the proposed equilibrium, where he chooses L, he receives 1. If he deviates to R, he receives 2 (because he plays R and the Receiver is playing U). Thus, we can rule out the first proposed pooling PBE already.

What about the second proposed pooling PBE where both play senders play L? In the proposed equilibrium Sender type t_1 chooses L and receives 1. If he switches to R, he receives 0 (because the Receiver is choosing D). So type t_1 does not wish to deviate. For Sender type t_2, he receives 2 when he chooses L in the proposed equilibrium. If he switches to R, he receives 1 (because the Receiver is choosing D – actually, in this case he receives 1 if he chooses R regardless of what the Receiver chooses).

So this second proposed equilibrium is a pooling equilibrium. The key is that the Receiver must believe that $\Pr\left(t_2|R\right) > \frac{2}{3}$.

2.2.2 Both types choose R

Suppose that both Sender types choose R. Again, there is no chance for the Receiver to update his beliefs about which node he is at if he observes R, so we have $\Pr\left(t_1|R\right) = 0.5$ and $\Pr\left(t_2|R\right) = 0.5$. Also, we do not know what his beliefs are if he observes L (since he should never observe L in this equilibrium), so for now specify $\Pr\left(t_1|L\right) = p$ and $\Pr\left(t_2|L\right) = (1-p)$. If R is chosen and the Receiver chooses U he gets 1 half of the time and 0 half of the time, so his expected value is $0 * \frac{1}{2} + 1 * \frac{1}{2} = \frac{1}{2}$. If R is chosen and the Receiver chooses D he gets 0 half of the time and 2 the other half, so his expected value is $0 * \frac{1}{2} + 2 * \frac{1}{2} = 1$. Thus, the Receiver would choose D if R is observed. If L is observed and the Receiver chooses U he gets 3 with probability p and he gets 4 with probability $(1-p)$, so his expected value is $3p + 4 * (1-p) = 4 - p$. If L is observed and the Receiver chooses D he gets 0 with probability p and 1 with probability $(1-p)$, so his expected value is $0 * p + 1 * (1-p) = 1 - p$. His expected payoff from choosing U will be greater than his expected payoff from choosing D if $4 - p > 1 - p$, or $4 > 1$. All this means is that if L is observed the Receiver will choose U – we should have already known this because earlier we noted that U was a strictly dominant strategy if the Receiver observed L. Thus, it does not matter what the Receiver's beliefs are if L

is observed – the Receiver will always choose U. So a *potential* pooling PBE is:

$$
\begin{array}{ll}
t_1 \text{ choose } R & \\
t_2 \text{ choose } R & \{\text{Sender's strategy}
\end{array}
$$

$$
\begin{array}{ll}
\Pr(t_1|L) \geq 0 & \\
\Pr(t_2|L) \leq 1 & \\
\Pr(t_1|R) = \frac{1}{2} & \{\text{Receiver's beliefs} \\
\Pr(t_2|R) = \frac{1}{2} &
\end{array}
$$

$$
\begin{array}{ll}
\text{choose } U \text{ if } L \text{ observed} & \\
\text{choose } D \text{ if } R \text{ observed} & \{\text{Receiver's strategy}
\end{array}
$$

Note that the Receiver's beliefs state that no matter what the relative probabilities are between $\Pr(t_1|L)$ and $\Pr(t_2|L)$ the Receiver will always choose U (again, because it is strictly dominant).

Finally, is this potential pooling PBE actually an equilibrium? It is easy to see that it is not – we know that type t_2 receives 1 if he chooses R and the Receiver chooses D. However, since the Receiver is choosing U if L is observed, then type t_2 could switch to L and receive 2, so type t_2 would deviate from the proposed strategy. There is no need to check if type t_1 would deviate because we know at least one player type will deviate so the proposed equilibrium cannot be an equilibrium.

2.3 Summing up Sender-Receiver

We found that there was one separating PBE:

$$
\begin{array}{ll}
t_1 \text{ choose } R & \\
t_2 \text{ choose } L & \{\text{Sender's strategy}
\end{array}
$$

$$
\begin{array}{ll}
\Pr(t_1|L) = 0 & \\
\Pr(t_2|L) = 1 & \\
\Pr(t_1|R) = 1 & \{\text{Receiver's beliefs} \\
\Pr(t_2|R) = 0 &
\end{array}
$$

$$
\begin{array}{ll}
\text{choose } U \text{ if } L \text{ observed} & \\
\text{choose } U \text{ if } R \text{ observed} & \{\text{Receiver's strategy}
\end{array}
$$

and we found that there were a class of pooling PBE:

$$
\begin{array}{ll}
t_1 \text{ choose } L & \\
t_2 \text{ choose } L & \{\text{Sender's strategy}
\end{array}
$$

$$
\begin{array}{ll}
\Pr(t_1|L) = \frac{1}{2} & \\
\Pr(t_2|L) = \frac{1}{2} & \\
\Pr(t_1|R) \leq \frac{2}{3} & \{\text{Receiver's beliefs} \\
\Pr(t_2|R) > \frac{2}{3} &
\end{array}
$$

$$
\begin{array}{ll}
\text{choose } U \text{ if } L \text{ observed} & \\
\text{choose } D \text{ if } R \text{ observed} & \{\text{Receiver's strategy}
\end{array}
$$

The reason I write "class" is because there are many different sets of beliefs that will lead to this pooling equilibrium. If $q = \frac{1}{8}$ and $(1-q) = \frac{7}{8}$ then the above equilibrium is a pooling equilibrium. If $q = \frac{1}{9}$ and $(1-q) = \frac{8}{9}$ then this is also a pooling equilibrium. So there are a lot of pooling equilibria, as long as $q \leq \frac{2}{3}$.

3 More signaling

Consider the following game: Suppose there are two types of men in the world, Dapper Dans and Average Joes. Nature determines the type of man: Dans occur with probability $\Pr(Dan)$ and Joes occur with probability $\Pr(Joe)$. Women prefer to go on dates with the Dapper Dan types. For both types of men it costs them to look like a Dapper Dan; Dans must incur a cost of $c > 0$ while Joes, who need more work,

must incur a cost of $c + x$, where $x > 0$. Thus, the men must decide if they wish to incur the cost and look like a Dapper Dan. The woman must then decide if she wishes to go on a date with a man – however, she cannot tell his type, she can only see his outward appearance. If the woman goes on a date with a Dapper Dan, she gets a payoff of V. If she goes on a date with an Average Joe, she gets a payoff of W, where $V > W$. If she decides to stay home, she gets a payoff of M. For the men, if they get asked out on a date they go and receive a payoff of K. If they are not asked out on a date they receive a payoff of zero. The game tree is below.
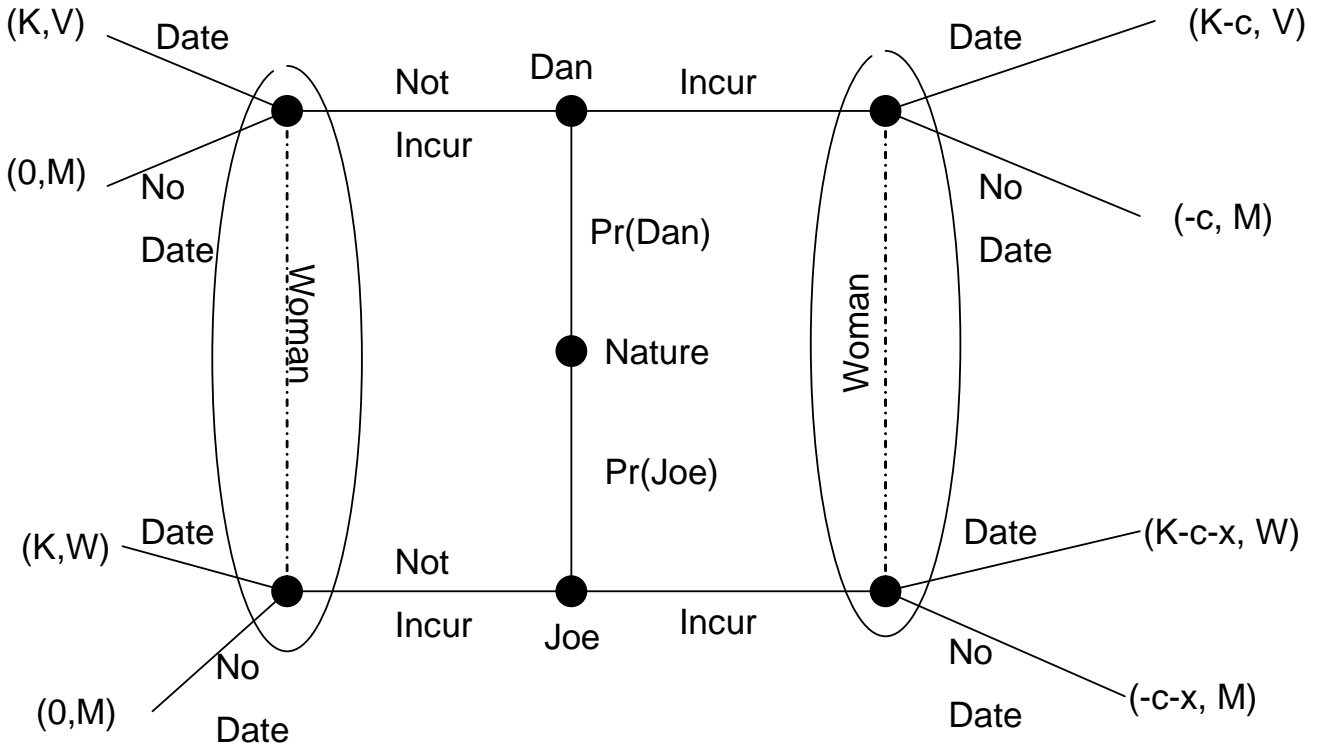
The question now is, without actual numbers but with only variables how do we determine if there is a pooling or separating equilibrium? We know that there are two potential pooling equilibria in this game – (1) both Dans and Joes choose to Incur the cost, or (2) both Dans and Joes choose to Not Incur the cost. There are also two potential separating equilibria in this game (1) Dans Incur and Joes choose Not Incur or (2) Dans choose Not Incur and Joes choose Incur.

Separating equilibrium (2) Focus on separating equilibrium (2) where Dans choose Not Incur and Joes choose Incur. The woman believes that $\Pr(Joe|Incur) = 1$ and $\Pr(Dan|Incur) = 0$ while $\Pr(Joe|Not\ Incur) = 0$ and $\Pr(Dan|Not\ Incur) = 1$. The woman will choose to date a man who incurs if:

$$W > M$$

because only Joes are incurring (note that we have made no assumptions about the relationship of W to M). The woman will choose to date a man who chooses Not Incur if:

$$V > M$$

However, this cannot be an equilibrium because the Joes could switch to Not Incur and save the $c + x$ cost because the woman is choosing Date if a man chooses Not Incur.

   What if the woman chooses to Not Date a man who chooses Not Incur but Date a man who chooses Incur? This would be true if:

$$
\begin{aligned}
W &> M \\
M &> V
\end{aligned}
$$

but this cannot be true because $V > M$.

What if a woman chooses No Date regardless of the Incur/Not Incur decision? Then we would need:

$$
\begin{aligned}
M &> W \\
M &> V
\end{aligned}
$$

but then the Joes would choose Not Incur because if the woman is choosing No Date regardless then there is no point in choosing Incur.

Finally, if the woman chooses Date if Not Incur and No Date if Incur then we would need:

$$
\begin{aligned}
M &> W \\
V &> M
\end{aligned}
$$

This is certainly possible (and plausible), but if this were the case then again the Joes would switch to Not Incur. Thus, there can be no separating equilibrium in which the Joes choose Incur and the Dans choose Not Incur.

Separating equilibrium (1) Now focus on separating equilibrium (1), where Dans Incur and Joes choose Not Incur. The woman will believe that $\Pr(Dan|Incur) = 1$ and $\Pr(Joe|Incur) = 0$, and that $\Pr(Dan|Not\ Incur) = 0$ and $\Pr(Joe|Not\ Incur) = 1$. Suppose a woman chooses Date if Incur and Date if Not Incur, then we need:

$$
\begin{aligned}
V &> M \\
W &> M
\end{aligned}
$$

Again, this is plausible, but if the woman is choosing Date regardless of the Incur/Not Incur choice then the Dans should save the cost from Incur and choose Not Incur. A similar argument holds if the woman is choosing No Date if Incur and No Date if Not Incur, in which case we would have:

$$
\begin{aligned}
M &> V \\
M &> W
\end{aligned}
$$

Once again the Dans should simply save the cost from Incur and choose Not Incur because the woman has chosen No Date regardless.

What if the woman chose to Date if Not Incur and No Date if Incur? This would mean:

$$
\begin{aligned}
M &> V \\
W &> M
\end{aligned}
$$

which leads to $W > V$ which contradicts our assumption so this potential separating equilibrium is ruled out.

Finally, what if we try the following (logical one in my mind): Dans Incur, Joes choose Not Incur, the woman has beliefs that $\Pr(Dan|Incur) = 1$ and $\Pr(Joe|Incur) = 0$, and that $\Pr(Dan|Not\ Incur) = 0$ and $\Pr(Joe|Not\ Incur) = 1$, and the woman chooses Date if Incur and No Date if Not Incur. Then we have:

$$
\begin{aligned}
V &> M \\
M &> W
\end{aligned}
$$

The question now is, would Dans or Joes switch? Dans would switch if $0 > K - c$, meaning if the cost of incurring (c) was greater than the value of the date they would switch to Not Incur. Joes would switch to

Incur if $K - c - x > 0$, or if $K > c + x$. Thus, in order for the following potential PBE:

$$\begin{array}{l} Dan \text{ choose } Incur \\ Joe \text{ choose } Not\ Incur \end{array} \quad \{\text{Man's strategy}$$

$$\begin{array}{l} \Pr\left(Dan|Incur\right) = 1 \\ \Pr\left(Joe|Incur\right) = 0 \\ \Pr\left(Dan|Not\ Incur\right) = 0 \\ \Pr\left(Joe|Not\ Incur\right) = 1 \end{array} \quad \{\text{Woman's beliefs}$$

$$\begin{array}{l} \text{choose } Date \text{ if } Incur \text{ observed} \\ \text{choose } No\ Date \text{ if } Not\ Incur \text{ observed} \end{array} \quad \{\text{Woman's strategy}$$

to actually be a PBE we need:

$$\begin{aligned} V &> M \\ M &> W \\ K &> c \\ c + x &> K \end{aligned}$$

If all of those conditions hold then we have a separating PBE where the Dans Incur and the Joes choose Not Incur.

Pooling equilibrium (1) If both Dans and Joes decide to Incur the cost, the only reasonable potential equilibrium is:

$$\begin{array}{l} Dan \text{ choose } Incur \\ Joe \text{ choose } Incur \end{array} \quad \{\text{Man's strategy}$$

$$\begin{array}{l} \Pr\left(Dan|Incur\right) = \Pr\left(Dan\right) \\ \Pr\left(Joe|Incur\right) = \Pr\left(Joe\right) \\ \Pr\left(Dan|Not\ Incur\right) = p \\ \Pr\left(Joe|Not\ Incur\right) = 1 - p \end{array} \quad \{\text{Woman's beliefs}$$

$$\begin{array}{l} \text{choose } Date \text{ if } Incur \text{ observed} \\ \text{choose } No\ Date \text{ if } Not\ Incur \text{ observed} \end{array} \quad \{\text{Woman's strategy}$$

Why is this the only reasonable potential equilibrium? Consider what would happen if the woman chose Date if Not Incur (regardless of whether she chose Date or No Date if Incur)? Then both men would switch from Incur to Not Incur because they go on the date without incurring the cost.

Now consider what would happen if the woman used No Date if Incur and No Date if Not Incur as her strategy. Then the men would choose Not Incur because they are not getting a date. Thus the only reasonable pooling equilibrium where both Dans and Joes choose Incur is the one proposed above. What conditions need to hold in order for this to be true? For the woman we need:

$$\Pr\left(Dan\right) * V + \Pr\left(Joe\right) * W \quad \geq \quad M$$
$$\text{and}$$
$$M \quad \geq \quad p * V + (1 - p) * W$$

Thus, suppose the probabilities by nature are such that $\Pr\left(Dan\right) = 0.3$ and $\Pr\left(Joe\right) = 0.7$. These two conditions suggest that the woman must believe that $\Pr\left(Joe|Not\ Incur\right) \geq 0.7$, or that $(1 - p) \geq 0.7$.

For the men, we need that:

$$K - c - x \geq 0 \ \{\text{for Joe's}$$
$$K - c \geq 0 \ \{\text{for Dan's}$$

The reason that we need this condition is because the men would receive 0 if they chose not incur, and we just need to check to see that the men prefer incurring to not incurring.

Pooling equilibrium (2)   If both Dans and Joes decide to Not Incur the cost, there are a few reasonable pooling equilibria.   They are (2a):

$$Dan \text{ choose } Not \text{ } Incur$$
$$Joe \text{ choose } Not \text{ } Incur \quad \{\text{Man's strategy}$$

$$\Pr\left(Dan|Incur\right) = q$$
$$\Pr\left(Joe|Incur\right) = 1 - q$$
$$\Pr\left(Dan|Not \text{ } Incur\right) = \Pr\left(Dan\right) \quad \{\text{Woman's beliefs}$$
$$\Pr\left(Joe|Not \text{ } Incur\right) = \Pr\left(Joe\right)$$

$$\text{choose } No \text{ } Date \text{ if } Incur \text{ observed}$$
$$\text{choose } No \text{ } Date \text{ if } Not \text{ } Incur \text{ observed} \quad \{\text{Woman's strategy}$$

or (2b):

$$Dan \text{ choose } Not \text{ } Incur$$
$$Joe \text{ choose } Not \text{ } Incur \quad \{\text{Man's strategy}$$

$$\Pr\left(Dan|Incur\right) = q$$
$$\Pr\left(Joe|Incur\right) = 1 - q$$
$$\Pr\left(Dan|Not \text{ } Incur\right) = \Pr\left(Dan\right) \quad \{\text{Woman's beliefs}$$
$$\Pr\left(Joe|Not \text{ } Incur\right) = \Pr\left(Joe\right)$$

$$\text{choose } Date \text{ if } Incur \text{ observed}$$
$$\text{choose } Date \text{ if } Not \text{ } Incur \text{ observed} \quad \{\text{Woman's strategy}$$

or (2c):

$$Dan \text{ choose } Not \text{ } Incur$$
$$Joe \text{ choose } Not \text{ } Incur \quad \{\text{Man's strategy}$$

$$\Pr\left(Dan|Incur\right) = q$$
$$\Pr\left(Joe|Incur\right) = 1 - q$$
$$\Pr\left(Dan|Not \text{ } Incur\right) = \Pr\left(Dan\right) \quad \{\text{Woman's beliefs}$$
$$\Pr\left(Joe|Not \text{ } Incur\right) = \Pr\left(Joe\right)$$

$$\text{choose } Date \text{ if } Incur \text{ observed}$$
$$\text{choose } No \text{ } Date \text{ if } Not \text{ } Incur \text{ observed} \quad \{\text{Woman's strategy}$$

or (2d):

$$Dan \text{ choose } Not \text{ } Incur$$
$$Joe \text{ choose } Not \text{ } Incur \quad \{\text{Man's strategy}$$

$$\Pr\left(Dan|Incur\right) = q$$
$$\Pr\left(Joe|Incur\right) = 1 - q$$
$$\Pr\left(Dan|Not \text{ } Incur\right) = \Pr\left(Dan\right) \quad \{\text{Woman's beliefs}$$
$$\Pr\left(Joe|Not \text{ } Incur\right) = \Pr\left(Joe\right)$$

$$\text{choose } No \text{ } Date \text{ if } Incur \text{ observed}$$
$$\text{choose } Date \text{ if } Not \text{ } Incur \text{ observed} \quad \{\text{Woman's strategy}$$

The first equilibrium (2a) basically states that the woman will not date a man regardless of whether he incurs the cost or not.   Thus, there is no reason for the man to incur the cost because the men would get a negative amount if the cost is incurred (either $-c$ or $-c-x$) and 0 if the cost is not incurred.   The necessary conditions for the woman are:

$$M \geq q * V + (1 - q) * W$$
$$M \geq \Pr\left(Dan\right) * V + \Pr\left(Joe\right) * W$$

For the second potential equilibrium (2b), where the woman is choosing Date regardless, once again it makes little sense for the men to incur because if they choose Incur they receive either $K - c$ (if Dan) or $K - c - x$ (if

Joe) versus K if they choose Not Incur. The necessary conditions for the woman to choose Date regardless of the man's choice are:

$$
\begin{aligned}
q * V + (1 - q) * W &\geq M \\
\Pr(Dan) * V + \Pr(Joe) * W &\geq M
\end{aligned}
$$

Note that for this potential equilibrium the inequalities are simply reversed for the woman's conditions. For the third potential equilibrium (2c), where the woman is choosing Date if Incur and No Date if Not Incur, the woman would need:

$$
\begin{aligned}
q * V + (1 - q) * W &\geq M \\
M &\geq \Pr(Dan) * V + \Pr(Joe) * W
\end{aligned}
$$

The key here is the condition for the men – both men would like to go on the date and receive K, but in order for this to be an equilibrium it must be too costly for them to switch, so we must have $K - c < 0$ (also $K - c - x < 0$ but that must be true given that $K - c < 0$).

Finally, consider the potential equilibrium where the woman chooses No Date if Incur and Date if Not Incur (2d). If this is the case we must have:

$$
\begin{aligned}
M &\geq q * V + (1 - q) * W \\
\Pr(Dan) * V + \Pr(Joe) * W &\geq M
\end{aligned}
$$

For the men it really does not matter what the relationship between K and c is – we could have $K < c$, we could have $K = c$, or $K > c$ – regardless, they are not switching.

3.1 An example

Suppose that $K = 20$, $c = 2$, $x = 8$, $V = 20$, $W = 5$, $M = 15$ and that $\Pr(Dan) = 0.25$ and $\Pr(Joe) = 0.75$. Are there pooling equilibria? What about separating equilibria?

Pooling equilibria To check for pooling equilibria we know we need to check whether or not the woman's expected value of a date is greater than or equal to that of staying home. If she stays home she receives 15. If she goes on a date (remember, this is for a pooling equilibrium) she gets:

$$
\Pr(Dan) * V + \Pr(Joe) * W = 0.25 * 20 + 0.75 * 5 = 8.75
$$

Thus, if both Dans and Joes choose the same action (either Incur or Not Incur), the woman will NOT go on a date because:

$$
\begin{aligned}
M &\geq \Pr(Dan) * V + \Pr(Joe) * W \\
15 &\geq 8.75
\end{aligned}
$$

Thus, we cannot have pooling equilibrium (1) where both types of men choose Incur – this is because the only possible way this potential equilibrium will be an equilibrium is if the woman chooses Date when both men choose Incur and she will not. We could have a pooling equilibrium (2), where both men choose Not Incur, but which one? It will not involve the woman choosing Date if Not Incur, because we have just shown that $15 \geq 8.75$. This rules out 2b and 2d. Pooling equilibrium 2c requires that both types of men find it too costly to switch to Date, and this is not true because $K - c = 20 - 2 = 18 > 0$ and $K - c - x = 20 - 2 - 8 = 10 > 0$. This rules out pooling equilibrium 2c. The only remaining pooling equilibrium is 2a, where the woman chooses No Date regardless of the decision by the man. The condition for the Dans and Joes is satisfied, the woman's condition if Not Incur is satisfied, and so now we must check the woman's condition to choose No Date if Incur (we know that she will choose No Date if Not Incur

because $15 \geq 8.75$):

$$
\begin{aligned}
M &\geq q * V + (1 - q) * W \\
15 &\geq q20 + (1 - q)5 \\
15 &\geq 20q + 5 - 5q \\
10 &\geq 15q \\
\frac{2}{3} &\geq q
\end{aligned}
$$

As long as $q \leq \frac{2}{3}$ (meaning she believes $\Pr(Dan|Incur) \leq \frac{2}{3}$), then the woman will choose Not Date if Incur.

Separating equilibria

Note that we cannot have a separating equilibrium where the Dans choose Not Incur and the Joes choose Incur – we already ruled out this possibility. The only other possibility is:

$$
\begin{array}{l}
Dan \text{ choose } Incur \\
Joe \text{ choose } Not \ Incur
\end{array} \quad \{\text{Man's strategy}
$$

$$
\begin{array}{l}
\Pr(Dan|Incur) = 1 \\
\Pr(Joe|Incur) = 0 \\
\Pr(Dan|Not \ Incur) = 0 \\
\Pr(Joe|Not \ Incur) = 1
\end{array} \quad \{\text{Woman's beliefs}
$$

$$
\begin{array}{l}
\text{choose } Date \text{ if } Incur \text{ observed} \\
\text{choose } No \ Date \text{ if } Not \ Incur \text{ observed}
\end{array} \quad \{\text{Woman's strategy}
$$

But in order for this to be an equilibrium we need:

$$
\begin{aligned}
V &> M \\
M &> W \\
K &> c \\
c + x &> K
\end{aligned}
$$

or:

$$
\begin{aligned}
20 &> 15 \\
15 &> 5 \\
20 &> 2 \\
10 &> 20
\end{aligned}
$$

Three out of four is not too bad, but in this case it shows that we CANNOT have a separating equilibrium in which the Dans Incur and the Joes choose Not Incur. Thus, the only equilibrium with these parameters is the pooling equilibrium where both Dans and Joes choose Not Incur and where the woman chooses to Not Date regardless of whether Dans or Joes Incur. And this only occurs if she believes $\Pr(Dan|Incur) \leq \frac{2}{3}$.