Consumer Theory

These notes essentially correspond to chapter 1 of Jehle and Reny.

1 Consumption set

The consumption set, denoted X, is the set of all possible combinations of goods and services that a consumer could consume. Some of these combinations may seem impractical for many consumers, but we allow the possibility that a consumer could have a combination of 500 Ferraris and 1400 yachts. Assume there is a fixed number of goods, n, and that n is finite. Consumers may only consume nonnegative amounts of these goods, and we let $x_i \in \mathbb{R}_+$ be the amount consumed of good i. Note that this implies that the consumption of any particular good i is infinitely divisible. The n-vector x consists of an amount of each of the n goods and is called the consumption bundle. Note that $x \in X$ and typically $X = \mathbb{R}^n_+$. Thus the consumption set is usually the nonnegative n-dimensional space of real numbers.

Standard assumptions made about the consumption set are:

1. $\varnothing \neq X \subseteq \mathbb{R}^n_+$
2. X is closed
3. X is convex
4. The n-vector of zeros, $0 \in X$.

The feasible set, B (soon we will call it the budget set), is a subset of the consumption set so $B \subset X$. The feasible set represents the subset of alternatives which the consumer can possibly consume given his or her current economic situation. Generally consumers will be restricted by the amount of wealth (or income or money) which they have at their disposal.

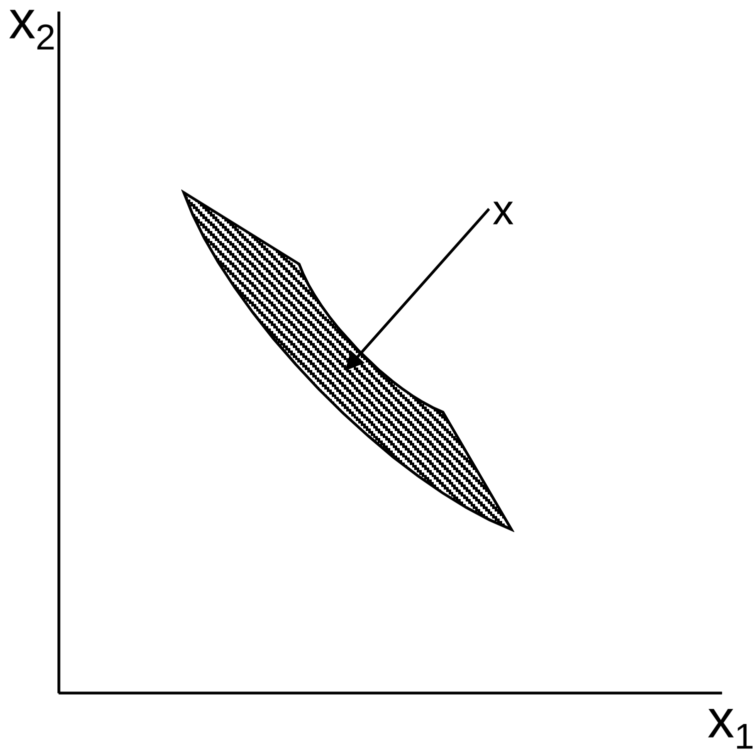
2 Preferences and utility

The basic building block of consumer theory is a binary relation on the consumption set X. The particular binary relation is the preference relation $\succsim$, which we call "at least as good as". We can use this preference relation to compare any two bundles $x_1, x_2 \in X$. If we have $x_1 \succsim x_2$ we say "x_1 is at least as good as x_2".

We will make some minimal restrictions about the preference relation $\succsim$. In general, our goal will be to make the most minimal assumptions possible. We make two assumptions about our preference relation $\succsim$:

1. Completeness: For any $x^1 \neq x^2$ in X, either $x^1 \succsim x^2$ or $x^2 \succsim x^1$ or both.
2. Transitivity: For any three elements x^1, x^2, and x^3 in X, if $x^1 \succsim x^2$ and $x^2 \succsim x^3$, then $x^1 \succsim x^3$.

Completeness means that the consumer can make choices or rank all the possible bundles in the consumption set. Transitivity imposes some minimal sense of consistency on those choices. The book lists a third assumption, reflexivity. However, if the preference relation $\succsim$ is complete and transitive, then we can also show that it is reflexive. Reflexivity simply means that an element of the consumption set is at least as good as itself, or $x^1 \succsim x^1$ for all $x \in X$. These very basic assumptions comprise the conditions of rationality in economic models. The notion of rationality in economics is one that is often misunderstood – all we assume for a "rational" economic agent is that preferences are complete and transitive (and reflexive).



A thick indifference set.

Whether or not the bulk of society considers a choice to be a good one (say a bright orange tuxedo at a formal event), if an individual agent possesses complete and transitive preferences then that consumer is considered to be rational.

Now that we have established the preference relation $\succsim$ we can define (1) the strict preference relation and (2) the indifference relation.

Definition 1 *The strict preference relation, $\succ$, on the consumption set X is defined as $x^1 \succ x^2$ if and only if $x^1 \succsim x^2$ but not $x^2 \succsim x^1$.*

Definition 2 *The indifference relation, $\sim$, on the consumption set X is defined as $x^1 \sim x^2$ if and only if $x^1 \succsim x^2$ and $x^2 \succsim x^1$.*

Note that neither $\succ$ nor $\sim$ is complete, both are transitive, and only $\sim$ is reflexive. Once we have $\sim$ and $\succ$ we can see that either $x^1 \succ x^2$, $x^2 \succ x^1$, or $x^1 \sim x^2$. Thus, the consumer is able to rank bundles of goods. However, these assumptions of completeness, transitivity, and reflexivity only impose some minimum order on the ranking of bundles. We will impose a little more structure on our consumer's preferences.

Assumption: *Continuity. For all $x \in \mathbb{R}^n_+$, the "at least as good as" set, $\succsim (x)$, and the "no better than set" $\precsim (x)$ are closed in $\mathbb{R}^n_+$.*

Continuity is primarily a mathematical assumption, but the intuitive reason behind imposing it is so that sudden preference reversals do not happen.

Assumption: *Local Nonsatiation. For all $x^0 \in \mathbb{R}^n_+$, and for all $\varepsilon > 0$, there exists some $x \in B_\varepsilon \left(x^0\right) \cap \mathbb{R}^n_+$ such that $x \succ x^0$.*

When local nonsatiation is assumed, this means that there is some bundle close to a specific bundle which will be preferred to that specific bundle. There is nothing in local nonsatiation that specifies the direction of the preferred bundle. What local nonsatiation does is rule out "thick" preferences.

Assumption: *Strict monotonicity. For all $x^0, x^1 \in \mathbb{R}^n_+$, if $x^0 \geq x^1$ then $x^0 \succsim x^1$, while if $x^0 >> x^1$ then $x^0 \succ x^1$.*

In a principles or intermediate microeconomics class this is what we would call the "more is better" assumption. Note that when an individual compares bundles of goods, if the individual has bundle x^0 with more of at least one good (and the same level of all other goods) than is in x^1, then this individual deems x^0 at least as good as x^1. And if x^0 has more of all goods than x^1, then the individual strictly prefers x^0 to x^1.

Assumption: *Convexity. If $x^1 \succsim x^0$, then $tx^1 + (1-t)\,x^0 \succsim x^0$ for all $t \in [0,1]$.*

Assumption: *Strict convexity. If $x^1 \neq x^0$ and $x^1 \succsim x^0$, then $tx^1 + (1-t)\,x^0 \succ x^0$ for all $t \in (0,1)$.*

Either of these assumptions rules out concave to the origin preferences. The intuition behind these convexity assumptions is that consumers (generally) prefer balanced consumption bundles to unbalanced consumption bundles. Thus, since these convex combinations of consumption bundles provide a more balanced consumption plan, the consumer would prefer them.

Alternatively, think about any particular indifference set in $\mathbb{R}^2_+$. The slope of an indifference curve is called the marginal rate of substitution.[1] If we have strict monotonicity and either form of convexity then this means that the marginal rate of substitution should not increase as we move from bundles along the same indifference which have a lot of good 1 to those which have relatively less of good 1. Thus, when the consumer has a little of good 1 he should be willing to give up more of good 2 to get an extra unit of good 1 than when he has a lot of good 1.

As a summary, the assumptions of completeness, transitivity, and reflexivity are the basis for the rational consumer. The assumption of continuity is primarily a mathematical one to make the problem slightly more tractable. The remaining assumptions represent assumptions about a consumer's tastes.

2.1 Utility

The utility function is a nice way to summarize preferences, particularly if one wants to use calculus methods to solve problems (as we will want to do). We can establish results that show that with a minimal amount of structure that there will be a utility function which represents our preference relation $\succsim$.

Definition 3 *A real-valued function $u : \mathbb{R}^n_+ \to \mathbb{R}$ is called a utility function representing preference relation $\succsim$, if for all $x^0, x^1 \in \mathbb{R}^n_+$, $u\left(x^0\right) \geq u\left(x^1\right) \iff x^0 \succsim x^1$.*

The question is which assumptions that we made about our preference relation will be needed to establish that a utility function which represents $\succsim$ exists? There is a theorem which states that all we need is completeness, transitivity, reflexivity, and continuity. Note that monotonicity, convexity (of any type), and local nonsatiation are NOT needed to guarantee the existence of a utility function which represents $\succsim$.

Theorem 4 *If the binary relation $\succsim$ is copmlete, reflexive, transitive, and continuous then there exists a continuous real-valued function, $u : \mathbb{R}^n_+ \to \mathbb{R}$, which represents $\succsim$.*

We will not go through the proof of this theorem but we will use the result. Note that in the book the provide the proof of a slightly less general result in Theorem 3.1 as they assume strict monotonicity. Again, note that strict monotonicty is NOT required to ensure the existence of a utility function which represents $\succsim$.

When specifying utility functions economists are primarily concerned with preserving ordinal relationships, not cardinal ones. Thus, two utility functions which preserve the order of preferences over bundles will be viewed the same UNLESS the cardinality of the utility function is important for a particular application. So if there are two utility functions, $u\,(x_1, x_2) = x_1 + x_2$ and $v\,(x_1, x_2) = x_1 + x_2 + 5$ it should be clear that the resulting utility level from the same (x_1, x_2) bundle is higher in $v\,(\cdot)$ than in $u\,(\cdot)$. However, since the order of preferences over bundles is preserved between the two utility functions, they are generally viewed the same by economists.

Now consider the same function $u\,(x_1, x_2) = x_1 + x_2$ and another function $g\,(x_1, x_2) = x_1 x_2$. If we look at the table for three different bundles of x_1 and x_2 we see that:

[1] Note that this text refers to the marginal rate of substitution as a positive number even though the slope will be nonpositive for convex preferences.

x_1	x_2	$u(x_1, x_2)$	$g(x_1, x_2)$
8	0	8	0
2	2	4	4

Since $u(8,0) > u(2,2)$ but $g(2,2) > g(8,0)$, we can see that these utility functions do not preserve the order of preferences for the bundles so that they are not viewed as the same by economists. Given this notion of ordinality of utility functions, we have the following theorem.

Theorem 5 *Let $\succsim$ be a preference relation on $\mathbb{R}^n_+$ and suppose $u(x)$ is a utility function that represents it. Then $v(x)$ also represents $\succsim$ if and only if $v(x) = f(u(x))$ for every x, where $f : \mathbb{R} \to \mathbb{R}$ is strictly increasing on the set of values taken on by u.*

We have been developing a model of the consumer based upon the preference relation $\succsim$ and some assumptions (hopefully realistic) about the preference relation. We would like to represent our preference relation $\succsim$ with a utility function (so that we can use calculus to solve the problem). Based on a rational preference relation $\succsim$, we ensure that the utility function has certain properties when we impose monotonicity and convexity on our preference relation $\succsim$.

Theorem 6 *Let $\succsim$ be represented by $u : \mathbb{R}^n_+ \to \mathbb{R}$. Then:*

1. *$u(x)$ is strictly increasing if and only if $\succsim$ is strictly monotonic*
2. *$u(x)$ is quasiconcave if and only if $\succsim$ is convex*
3. *$u(x)$ is strictly quasiconcave if and only if $\succsim$ is strictly convex*

In order to facilitate finding a solution to the consumer's problem we impose differentiability of the consumer's utility function $u(\cdot)$. Like continuity, differentiability is a mathematical assumption. When $u(\cdot)$ is differentiable, we can find the first-order partial derivatives. The first-order partial derivative of $u(x)$ with respect to x_i, $\frac{\partial u(x)}{\partial x_i}$, is called the marginal utility of good i. We can now define the marginal rate of substitution (MRS) between two goods as the ratio of the marginal utilities of the two goods. So the marginal rate of substitution of good i for good j is:

$$MRS_{ij}(x) \equiv \frac{\partial u(x)/\partial x_i}{\partial u(x)/\partial x_j} \tag{1}$$

What the MRS tells us is the rate at which we can substitute one good for the other, keeping utility constant.

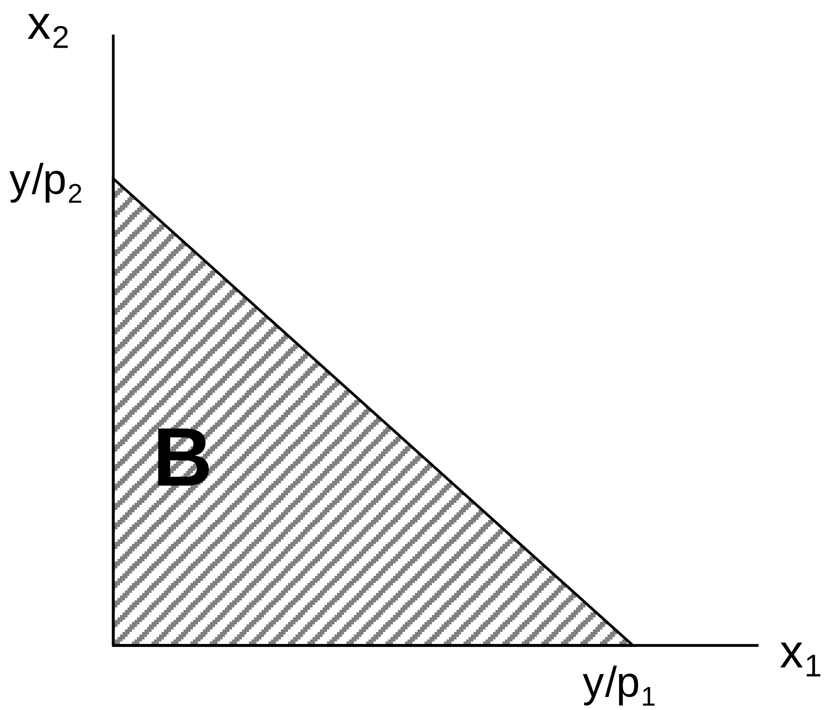
3 Consumer's problem

The consumer's general problem is to choose $x^* \in X$ such that $x^* \succsim x$ for all $x \in X$. However, when $X = \mathbb{R}^n_+$ this simply means that the consumer chooses an infinite amount of all goods. Thus, we restrict the consumption set to a feasible set $B \subset X = \mathbb{R}^n_+$. The consumer's problem then is to choose $x^* \in B$ such that $x^* \succsim x$ for all $x \in B$. Since it is easier to work with utility functions than preference relations, we make the following assumptions about our preference relation $\succsim$. Assume the preference relation $\succsim$ is complete, reflexive, transitive, continuous, strictly monotonic, and strictly convex on $\mathbb{R}^n_+$. This means that $\succsim$ can be represented by a real-valued utility function that is continuous, strictly increasing, and strictly quasiconcave on $\mathbb{R}^n_+$.

3.1 Market economy

In a market economy the consumer will face a price vector p, where there is one price for each of the n goods. We assume that the price vector p is strictly positive, or $p >> 0$ so that each $p_i > 0$. Also, the price vector is fixed and exogenous to the consumer's decisions – therefore, an individual consumer has no impact on the price of ANY good.[2] The consumer also has a fixed amount of money $y > 0$. This is an endowment (for

[2] This is just an assumption that can be changed.



now), meaning that the consumer simply receives this sum of money y. The consumer CANNOT spend more than this particular amount of income. The combination of positive prices, finite income, and the assumption that the consumer cannot spend more than his income restrict the consumption set, X, to the feasible set B. Thus, the consumer's budget constraint is given by:

$$\sum_{i=1}^{n} p_i x_i \leq y \tag{2}$$

With the budget constraint we can now create the budget set B, where:

$$B = \left\{ x | x \in \mathbb{R}_+^n, px \leq y \right\} \tag{3}$$

When there are 2 goods, the budget set B is:

Because of our assumptions about $\succsim$ and its relationship to $u(\cdot)$, we can formulate the consumer's problem as:

$$\max_{x \in \mathbb{R}_+^n} \; u(x) \text{ subject to } px \leq y \tag{4}$$

Thus, our consumer's problem is an inequality constrained maximization problem. The solution to the problem, x^*, is the x such that $u(x^*) \geq u(x)$ for all $x \in B$. Given the relationship between $\succsim$ and $u(\cdot)$, this means that $x^* \succsim x$ for all $x \in B$ and that x^* solves our original consumer's problem with the preference relation. Note that the particular solution x^* will depend upon the parameters of the problem, or the prices and income that the consumer faces. Thus we will write x^* as $x(p, y)$, with $x_i(p, y)$ representing the particular quantity of good i.

A few general results. We "know" that the optimal bundle $x^*(p, y)$ will lie on the budget constraint, or where $px = y$. This is because preferences are strictly monotonic. If the consumer chooses a bundle of goods on the interior of the budget set (not along the budget constraint), then that consumer will always be able to find another bundle that is preferred to the chosen bundle (because there is some feasible bundle with more of both goods). Also, because $y > 0$ and $x^* \neq 0$, we know that the consumer consumes a positive amount of at least one good. Since $\succsim$ is assumed to be strictly convex, the solution $x^*(p, y)$ will be unique, so that $x^*(p,y)$ is a demand function that specifies the amount of each good a consumer will choose given price vector p and income level y. We call these $x(p,y)$ the Marshallian demand functions. To find them, simply solve the inequality constrained maximization problem by setting up the Lagrangian, differentiating, and solving for each of the x_i.

$$\mathcal{L}(x,\lambda) = u(x) + \lambda[y - px] \tag{5}$$

Assuming that $x^* >> 0$, we know there is a $\lambda^* \geq 0$ such that (x^*, λ^*) satisfy:

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial x_i} &= \frac{\partial u(x)}{\partial x_i} - \lambda p_i = 0 \quad i = 1, ..., n & (6) \\ y - px^* &\geq 0 & (7) \\ \lambda^*[y - px^*] &= 0 & (8) \end{aligned}$$

Since we are assuming $\succsim$ is strictly monotone we have $y - px^* = 0$, which leaves us with $n+1$ equations and $n+1$ unknown. While it's possible that $\nabla u(x^*) = 0$ it is unlikely that this is so we assume $\nabla u(x^*) \neq 0$. So we will have $\frac{\partial u(x^*)}{\partial x_i} > 0$ for at least one $i = 1, ..., n$. Since $p_i > 0$, we have that $\lambda^* > 0$ because from:

$$\begin{aligned} \frac{\partial u(x^*)}{\partial x_i} &= \lambda^* p_i & (9) \\ \frac{\partial u(x^*)}{\partial x_i}/p_i &= \lambda^* > 0 & (10) \end{aligned}$$

For any two goods we can rewrite this as:

$$\begin{aligned} \frac{\partial u(x^*)}{\partial x_i}/p_i &= \frac{\partial u(x^*)}{\partial x_j}/p_j & (11) \\ \frac{\frac{\partial u(x^*)}{\partial x_i}}{\frac{\partial u(x^*)}{\partial x_j}} &= \frac{p_i}{p_j} & (12) \end{aligned}$$

Recall that $\frac{\partial u(x^*)}{\partial x_i}/\frac{\partial u(x^*)}{\partial x_j}$ is the marginal rate of substitution between goods i and j. Thus, at the optimum, the MRS between goods i and j will be equal to the slope of the budget constraint. This is simply the mathematical result that one would see in an intermediate microeconomics class. The figure illustrates that the consumer optimum is where the indifference curve is tangent to the budget constraint (point E), while also showing why other points cannot be optimal. If the consumer were at point G, there are many bundles that are strictly preferred to G (including point E). While point F is on the budget constraint it is not optimal as it is indifferent to point G (since it lies on the same indifference curve) and we have already seen that G is not optimal.

3.1.1 Example

Consider the utility function $u(x_1, x_2) = x_1^\alpha x_2^\beta$. The prices of good 1 is $p_1 > 0$ and the price of good 2 is $p_2 > 0$. The consumer has income $y > 0$. The consumer's problem is:

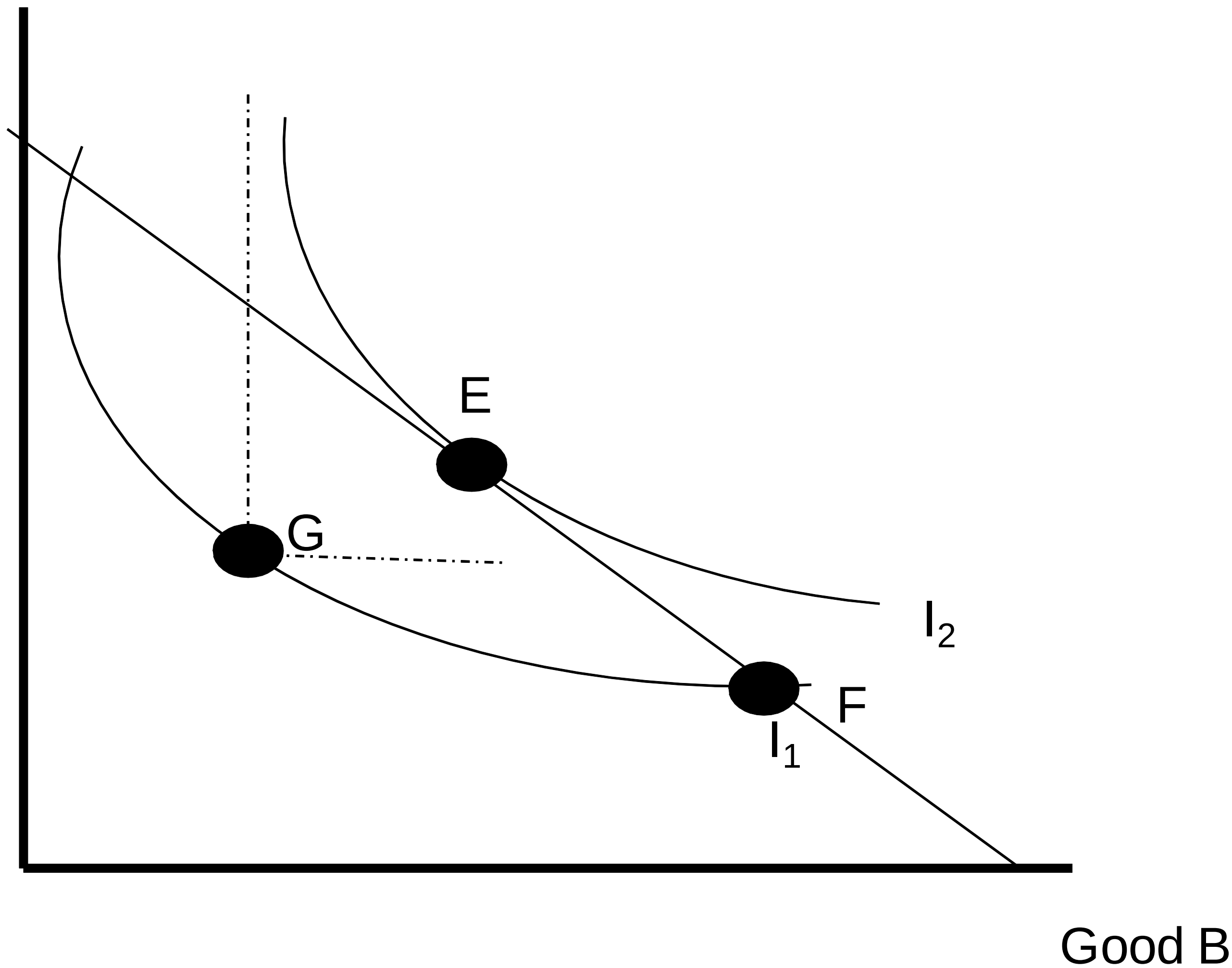
$$\max_{x_1 \geq 0, x_2 \geq 0} x_1^\alpha x_2^\beta \text{ s.t. } p_1 x_1 + p_2 x_2 \leq y \tag{13}$$

We can form the Lagrangian, differentiate with respect to x_1, x_2, and λ, and find the solution as:

$$\mathcal{L}(x_1, x_2, \lambda) = x_1^\alpha x_2^\beta + \lambda[y - p_1 x_1 - p_2 x_2] \tag{14}$$

Differentiating we have:

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial x_1} &= \alpha x_1^{\alpha-1} x_2^\beta - \lambda p_1 = 0 & (15) \\ \frac{\partial \mathcal{L}}{\partial x_2} &= \beta x_1^\alpha x_2^{\beta-1} - \lambda p_1 = 0 & (16) \\ y - p_1 x_1 - p_2 x_2 &\leq 0 & (17) \\ \lambda[y - p_1 x_1 - p_2 x_2] &= 0 & (18) \end{aligned}$$


E
G
I2
F
I1
Good B

Again, the budget constraint will hold with equality so:

$$\begin{array}{c} \alpha x_1^{\alpha-1}x_2^{\beta} - \lambda p_1 = 0 \\ \beta x_1^{\alpha}x_2^{\beta-1} - \lambda p_1 = 0 \\ y - p_1x_1 - p_2x_2 = 0 \end{array} \tag{19}$$

Simplifying we have:

$$\begin{array}{c} \frac{\alpha x_1^{\alpha-1}x_2^{\beta}}{p_1} = \frac{\beta x_1^{\alpha}x_2^{\beta-1}}{p_2} \\ \frac{x_2^{\beta}}{x_2^{\beta-1}} = \frac{\beta x_1^{\alpha}p_1}{\alpha x_1^{\alpha-1}p_2} \\ x_2 = \frac{x_1p_1\beta}{\alpha p_2} \end{array} \tag{20}$$

Now, substituting into the budget constraint we have:

$$\begin{array}{c} y - p_1x_1 - p_2\left(\frac{x_1p_1\beta}{\alpha p_2}\right) = 0 \\ y - p_1x_1 - \frac{\beta}{\alpha}p_1x_1 = 0 \\ p_1x_1 + \frac{\beta}{\alpha}p_1x_1 = y \\ x_1\left(p_1 + \frac{\beta}{\alpha}p_1\right) = y \\ x_1\left(\frac{\alpha+\beta}{\alpha}p_1\right) = y \\ x_1 = \frac{\alpha y}{(\alpha+\beta)p_1} \end{array} \tag{21}$$

To find x_2 we simply plug x_1 back into $x_2 = \frac{x_1p_1\beta}{\alpha p_2}$:

$$\begin{array}{c} x_2 = \frac{x_1p_1\beta}{\alpha p_2} \\ x_2 = \frac{\alpha y}{(\alpha+\beta)p_1}\frac{p_1\beta}{\alpha p_2} \\ x_2 = \frac{\beta y}{(\alpha+\beta)p_2} \end{array} \tag{22}$$

Thus, if we have done the calculus and algebra correctly, we have:

$$x_1^*(p, y) = \frac{\alpha y}{(\alpha+\beta)p_1} \tag{23}$$

$$x_2^*(p, y) = \frac{\beta y}{(\alpha+\beta)p_2} \tag{24}$$

We can check that at the optimum we have:

$$\begin{array}{c} MRS = \frac{p_i}{p_j} \\ or \\ \frac{MU_1}{p_1} = \frac{MU_2}{p_2} \end{array} \tag{25}$$

Recall that $u(x_1, x_2) = x_1^{\alpha}x_2^{\beta}$, so that:

$$MU_1 = \alpha x_1^{\alpha-1}x_2^{\beta} \tag{26}$$

$$MU_2 = \beta x_1^{\alpha}x_2^{\beta-1} \tag{27}$$

Substituting x_1^* and x_2^* and dividing by the respective prices we have:

$$\frac{MU_1}{p_1} = \frac{MU_2}{p_2} \tag{28}$$

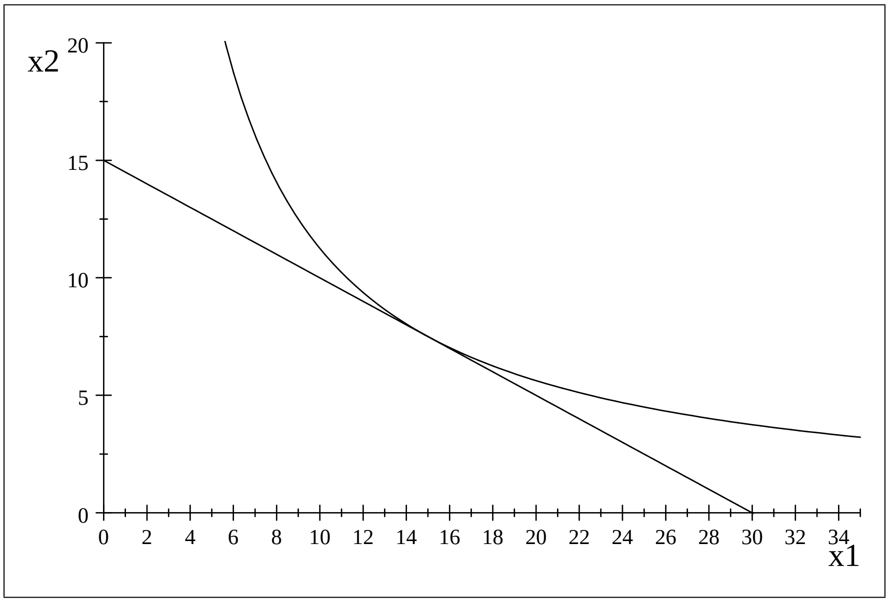
$$\frac{\alpha x_1^{\alpha-1}x_2^{\beta}}{p_1} = \frac{\beta x_1^{\alpha}x_2^{\beta-1}}{p_2} \tag{29}$$

$$\frac{\alpha x_2}{p_1} = \frac{\beta x_1}{p_2} \tag{30}$$

$$\frac{\alpha\frac{\beta y}{(\alpha+\beta)p_2}}{p_1} = \frac{\beta\frac{\alpha y}{(\alpha+\beta)p_1}}{p_2} \tag{31}$$

$$\frac{\alpha\beta y}{(\alpha+\beta)p_2p_1} = \frac{\beta\alpha y}{(\alpha+\beta)p_1p_2} \tag{32}$$

Technically we should check to make sure that the interior solution IS the optimal solution, and that there is not a better solution at a corner (when either $x_1^* = 0$ and $x_2^* = \frac{y}{p_2}$ or $x_1^* = \frac{y}{p_1}$ and $x_2^* = 0$). However, in this problem $u(x_1, x_2) = x_1^\alpha x_2^\beta$, so if either x_1 or x_2 equals 0 then the utility function is undefined. So the consumer should buy at least some small positive amount of each good with this utility function. Using $\alpha = \frac{1}{2}$, $\beta = \frac{1}{2}$, $p_1 = 5$, $p_2 = 10$, and $y = 150$, the following picture is a two-dimensional representation of the consumer's problem:



3.1.2 Second example

Now suppose that $u(x_1, x_2) = \alpha x_1 + \beta x_2$, with prices $p_1 > 0$ and $p_2 > 0$ respectively, and $y > 0$. We can start by setting up the Lagrangian and following our steps:

$$\mathcal{L}(x_1, x_2, \lambda) = \alpha x_1 + \beta x_2 + \lambda [y - p_1 x_1 - p_2 x_2] \tag{33}$$

Differentiating we have:

$$\frac{\partial \mathcal{L}}{\partial x_1} = \alpha - \lambda p_1 = 0 \tag{34}$$

$$\frac{\partial \mathcal{L}}{\partial x_2} = \beta - \lambda p_2 = 0 \tag{35}$$

$$y - p_1 x_1 - p_2 x_2 \leq 0 \tag{36}$$

$$\lambda [y - p_1 x_1 - p_2 x_2] = 0 \tag{37}$$

While the budget constraint will still hold with equality, combining the first two equations we get:

$$\frac{\alpha}{p_1} = \frac{\beta}{p_2} = \lambda \tag{38}$$

Since this condition does not depend on x_1 or x_2 it will only be true for certain parameters. If the parameters are such that $\frac{\alpha}{p_1} = \frac{\beta}{p_2}$, then any combination of x_1 and x_2 such that the budget constraint holds with equality will be a solution to the problem (in this case we have a Marshallian demand correspondence, not a Marshallian demand function). However, if $\frac{\alpha}{p_1} \neq \frac{\beta}{p_2}$, then the consumer would like to spend all of his income on either x_1 or x_2. Thus we can check the "corners" to see which gives higher utility. The next section discusses checking corner solutions more generally.

3.1.3 A more general example

While we may not have a guarantee of an interior solution, we still want to restrict $x_1 \geq 0$ and $x_2 \geq 0$. So, our consumer's problem is still to maximize utility subject to his budget constraint, but now we have the additional constraints that $x_1 \geq 0$ and $x_2 \geq 0$. Writing this out for a two good problem we have:

$$\max_{x_1, x_2} u(x_1, x_2) \text{ s.t. } p_1 x_1 + p_2 x_2 \leq y,\ x_1 \geq 0,\ x_2 \geq 0.$$

We have already seen this general example for an inequality constrained optimization problem. For our specific problem, we need all of the inequality constraints as $\geq$ constraints. Since $x_1 \geq 0$ and $x_2 \geq 0$ are already written in this manner, that just leaves rewriting the budget constraint as $y - p_1 x_1 - p_2 x_2 \geq 0$. Now we can form the Lagrangian:

$$\mathcal{L}(x_1, x_2, \lambda_1, \lambda_2, \lambda_3) = u(x_1, x_2) + \lambda_1 [y - p_1 x_1 - p_2 x_2] + \lambda_2 [x_1] + \lambda_3 [x_2]$$

We will now have a full set of Kuhn-Tucker conditions for both our choice variables and our Lagrange multipliers:

$$\begin{array}{c} \frac{\partial \mathcal{L}}{\partial x_1} = \frac{\partial u}{\partial x_1} - \lambda_1 p_1 + \lambda_2 \leq 0, \quad x_1 \geq 0, \quad x_1 * \frac{\partial \mathcal{L}}{\partial x_1} = 0 \\ \frac{\partial \mathcal{L}}{\partial x_2} = \frac{\partial u}{\partial x_2} - \lambda_1 p_2 + \lambda_3 \leq 0, \quad x_2 \geq 0, \quad x_2 * \frac{\partial \mathcal{L}}{\partial x_2} = 0 \\ \frac{\partial \mathcal{L}}{\partial \lambda_1} = y - p_1 x_1 - p_2 x_2 \geq 0, \quad \lambda_1 \geq 0, \quad \lambda_1 * y - p_1 x_1 - p_2 x_2 = 0 \\ \frac{\partial \mathcal{L}}{\partial \lambda_2} = x_1 \geq 0, \quad \lambda_2 \geq 0, \quad \lambda_2 * x_1 = 0 \\ \frac{\partial \mathcal{L}}{\partial \lambda_3} = x_2 \geq 0, \quad \lambda_3 \geq 0, \quad \lambda_3 * x_2 = 0 \end{array}.$$

Note that in this case we have complementary slackness conditions for the choice variables because we are uncertain as to whether or not the constraints are binding. Technically we would have 32 cases to check, one for each possible combination of x_1, x_2, λ_1, λ_2, and λ_3 being either strictly positive or zero. However, we know that the budget constraint will bind, and we know that either $x_1 \geq 0$ or $x_2 \geq 0$ so we really only have to check if $x_1 = 0$ or $x_2 = 0$ (with more goods, for example three goods, we would still have to check whether $x_1 = x_2 = 0$ and $x_3 > 0$, $x_1 = x_3 = 0$ and $x_2 > 0$, or $x_2 = x_3 = 0$ and $x_1 > 0$).

In general, the process I would use to find the optimal value would be to set up the Lagrangian function and assume an interior solution (or argue that the solution must be interior) and then check the potential corner solutions. If you cannot find a unique optimal interior solution (which would be the case if with our linear utility function example we had $\frac{\alpha}{p_1} = \frac{\beta}{p_2}$), then I would suggest checking the various "corners". Continuing with the linear function example, if $u\left(\frac{y}{p_1}, 0\right) > u\left(0, \frac{y}{p_2}\right)$, then the consumer would choose to consume only x_1. This would be true if:

$$\frac{\alpha y}{p_1} > \frac{\beta y}{p_2} \tag{39}$$

To make it easier to see that this is optimal, assume $\alpha = \beta$. Then the consumer would simply choose to consume only the good that is less expensive.

4 Additional formulations of the consumer's problem

We will look at two additional formulations of the consumer's problem. In the first we create the consumer's indirect utility function. The indirect utility function possesses a few useful properties that we can take advantage of. In the second we formulate the consumer's problem as an expenditure minimization problem. In this problem, the consumer's goal is to set a target level of utility and then find the bundle that minimizes expenditure.

4.1 Indirect utility function

When we set up the consumer's utility function we have the consumer maximizing $u(x)$ by choosing a bundle of goods. For any set of prices p and income y the consumer chooses $x(p, y)$ that maximizes $u(x)$. The value of the utility function at $x(p, y)$ is the maximum utility for a consumer given prices p and income y.

We can define a function that relates the maximum value of utility to the different price vectors and income levels a consumer may face. Define a real-valued function $v : \mathbb{R}^{n+1}_{++} \to \mathbb{R}$ as:

$$v(p, y) = \max_{x \in \mathbb{R}^n_+} u(x) \text{ s.t. } px \leq y \tag{40}$$

The function $v(p, y)$ is called the *indirect utility function* because the consumer is not directly maximizing v but indirectly maximizing v by maximizing u. If $u(x)$ is continuous and strictly quasiconcave, then there is a unique solution to this optimization problem, and that is the consumer's demand function $x(p, y)$. There is a relationship between $v(p, y)$ and $u(x)$:

$$v(p, y) = u(x(p, y))$$

for some price vector p and income level y. There are a number of properties the indirect utility function possesses and they are summarized in the theorem below:

Theorem 7 *If $u(x)$ is continuous and strictly increasing on $\mathbb{R}^n_+$, then $v(p, y)$ is:*

1. *Continuous on $\mathbb{R}^n_{++} \times \mathbb{R}_+$*
2. *Homogeneous of degree zero in (p, y)*
3. *Strictly increasing in y*
4. *Decreasing in p*
5. *Quasiconvex in (p, y)*
6. *Roy's Identity: If $v(p, y)$ is differentiable at (p^0, y^0) and $\partial v(p^0, y^0)/\partial y \neq 0$, then:*

$$x_i(p^0, y^0) = -\frac{\partial v(p^0, y^0)/\partial p_i}{\partial v(p^0, y^0)/\partial y}, \quad i = 1, ..., n$$

The first five points are simply restrictions on $v(p, y)$ given restrictions on $u(x)$. The sixth point greatly simplifies finding the consumer's Marshallian demand function if the indirect utility function is known. While we will not prove these results (proofs and sketches of proofs are in the text), we will work through an example using the following indirect utility function:

$$v(p, y) = \frac{\alpha_1^{\alpha_1} \alpha_2^{\alpha_2} \alpha_3^{\alpha_3} y}{p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3}}$$

with $\alpha_i > 0$ and $\alpha_1 + \alpha_2 + \alpha_3 = 1$.

This indirect utility function can be found by solving the consumer's maximization problem and substituting the Marshallian demands into the utility function. Consider:

$$u(x_1, x_2, x_3) = x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3} \tag{41}$$

where $\alpha_i > 0$ and $\alpha_1 + \alpha_2 + \alpha_3 = 1$. We know that:

$$x_i(p, y) = \frac{\alpha_i y}{p_i} \tag{42}$$

so that:

$$v(p, y) = \left(\frac{\alpha_1 y}{p_1}\right)^{\alpha_1} \left(\frac{\alpha_2 y}{p_2}\right)^{\alpha_2} \left(\frac{\alpha_3 y}{p_3}\right)^{\alpha_3} \tag{43}$$

$$v(p, y) = \frac{\alpha_1^{\alpha_1} \alpha_2^{\alpha_2} \alpha_3^{\alpha_3} y^{\alpha_1 + \alpha_2 + \alpha_3}}{p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3}} \tag{44}$$

$$v(p, y) = \frac{\alpha_1^{\alpha_1} \alpha_2^{\alpha_2} \alpha_3^{\alpha_3} y}{p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3}} \tag{45}$$

To show that $v(p,y)$ is homogeneous of degree zero, we have:

$$\begin{aligned} v(tp,ty) &= \frac{\alpha_1^{\alpha_1}\alpha_2^{\alpha_2}\alpha_3^{\alpha_3}ty}{(tp_1)^{\alpha_1}(tp_2)^{\alpha_2}(tp_3)^{\alpha_3}} \\ v(tp,ty) &= \frac{\alpha_1^{\alpha_1}\alpha_2^{\alpha_2}\alpha_3^{\alpha_3}ty}{t^{\alpha_1+\alpha_2+\alpha_3}p_1^{\alpha_1}p_2^{\alpha_2}p_3^{\alpha_3}} \\ v(tp,ty) &= \frac{\alpha_1^{\alpha_1}\alpha_2^{\alpha_2}\alpha_3^{\alpha_3}y}{p_1^{\alpha_1}p_2^{\alpha_2}p_3^{\alpha_3}} \\ v(tp,ty) &= v(p,y) \end{aligned}$$

To show that $v(p,y)$ is increasing in y we simply find the partial derivative with respect to y and show that this derivative is strictly positive:

$$\frac{\partial v(p,y)}{\partial y} = \frac{\alpha_1^{\alpha_1}\alpha_2^{\alpha_2}\alpha_3^{\alpha_3}}{p_1^{\alpha_1}p_2^{\alpha_2}p_3^{\alpha_3}} > 0$$

This partial derivative is strictly greater than 0 since $p_i > 0$ and $\alpha_i > 0$.

To show that $v(p,y)$ is decreasing in p, we can find the partial derivative with respect to any price we rewrite as:

$$\begin{aligned} v(p,y) &= \frac{y\alpha_i^{\alpha_i}\alpha_j^{\alpha_j}\alpha_k^{\alpha_k}p_i^{-\alpha_i}}{p_j^{\alpha_j}p_k^{\alpha_k}} \\ \frac{\partial v(p,y)}{\partial p_i} &= \frac{-\alpha_i\alpha_i^{\alpha_i}\alpha_j^{\alpha_j}\alpha_k^{\alpha_k}yp_i^{-\alpha_i-1}}{p_j^{\alpha_j}p_k^{\alpha_k}} \\ \frac{\partial v(p,y)}{\partial p_i} &= \frac{-\alpha_i\alpha_i^{\alpha_i}\alpha_j^{\alpha_j}\alpha_k^{\alpha_k}y}{p_i^{\alpha_i+1}p_j^{\alpha_j}p_k^{\alpha_k}} < 0 \end{aligned}$$

To provide an example of Roy's identity we have:

$$\begin{aligned} \frac{\partial v(p,y)}{\partial p_i} &= \frac{-\alpha_i\alpha_i^{\alpha_i}\alpha_j^{\alpha_j}\alpha_k^{\alpha_k}y}{p_i^{\alpha_i+1}p_j^{\alpha_j}p_k^{\alpha_k}} \\ \frac{\partial v(p,y)}{\partial y} &= \frac{\alpha_i^{\alpha_i}\alpha_j^{\alpha_j}\alpha_k^{\alpha_k}}{p_i^{\alpha_i}p_j^{\alpha_j}p_k^{\alpha_k}} \end{aligned}$$

Taking the ratio:

$$\begin{aligned} x_i(p,y) &= -\frac{\frac{-\alpha_i\alpha_i^{\alpha_i}\alpha_j^{\alpha_j}\alpha_k^{\alpha_k}y}{p_i^{\alpha_i+1}p_j^{\alpha_j}p_k^{\alpha_k}}}{\frac{\alpha_i^{\alpha_i}\alpha_j^{\alpha_j}\alpha_k^{\alpha_k}}{p_i^{\alpha_i}p_j^{\alpha_j}p_k^{\alpha_k}}} \\ x_i(p,y) &= \frac{\alpha_i y}{p_i^{\alpha_i+1}p_j^{\alpha_j}p_k^{\alpha_k}} * p_i^{\alpha_i}p_j^{\alpha_j}p_k^{\alpha_k} \\ x_i(p,y) &= \frac{\alpha_i y}{p_i} \end{aligned}$$

Note that these demand functions are similar to those that we found when working through the standard maximization problem when $u(x_1,x_2) = x_1^{\alpha}x_2^{\beta}$. Recall that the demand functions in that example were $x_1(p,y) = \frac{\alpha y}{(\alpha+\beta)p_1}$ and $x_2(p,y) = \frac{\beta y}{(\alpha+\beta)p_2}$. If we impose $\alpha+\beta = 1$ (as we have done in the indirect utility function example), we have the same form for the demand functions. So the indirect utility function for the Cobb-Douglas utility function (with 3 goods) is $v(p,y) = \frac{\alpha_i^{\alpha_i}\alpha_j^{\alpha_j}\alpha_k^{\alpha_k}y}{p_1^{\alpha_1}p_2^{\alpha_2}p_3^{\alpha_3}}$.

4.2 Expenditure function

With the standard utility maximization problem we assume that the consumer has a fixed budget constraint and then determines what the maximum level of utility can be achieved given p and y. However, we can formulate a similar problem where the consumer fixes a target level of utility and then chooses the income level which minimizes expenditure to attain that target level of utility. In essence, the indifference curve is fixed and the consumer is shifting the budget constraint back and forth trying to find the lowest possible cost to achieve his target utility level.

Formally we can define the expenditure function as:

$$e(p,u) = \min_{x \in \mathbb{R}^n_+} px \text{ subject to } u(x) \geq \overline{u} \tag{46}$$

The solution to the expenditure minimization problem is known as the Hicksian demand function, of $x^h(p,u)$. If we find $x^h(p,u)$, then we know that $e(p,u) = p \cdot x^h(p,u)$ because this is the bundle which minimizes expenditure for utility level u and prices p. What these demand functions tell us is how purchases change when we hold utility constant and there is a price change of one good. Let's use $u(x) = x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3}$ with $\alpha_1 + \alpha_2 + \alpha_3 = 1$ to find the Hicksian demands.

Set up the Lagrangian:

$$\mathcal{L}(x_1, x_2, x_3, \lambda) = p_1 x_1 + p_2 x_2 + p_3 x_3 + \lambda\left[u - x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3}\right] \tag{47}$$

Differentiating:

$$\frac{\partial \mathcal{L}}{\partial x_1} = p_1 - \lambda \alpha_1 x_1^{\alpha_1 - 1} x_2^{\alpha_2} x_3^{\alpha_3} = 0 \tag{48}$$

$$\frac{\partial \mathcal{L}}{\partial x_2} = p_2 - \lambda \alpha_2 x_2^{\alpha_2 - 1} x_1^{\alpha_1} x_3^{\alpha_3} = 0 \tag{49}$$

$$\frac{\partial \mathcal{L}}{\partial x_3} = p_3 - \lambda \alpha_3 x_3^{\alpha_3 - 1} x_2^{\alpha_2} x_1^{\alpha_1} = 0 \tag{50}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = u - x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3} = 0 \tag{51}$$

We find that:

$$\frac{p_1}{\alpha_1 x_1^{\alpha_1 - 1} x_2^{\alpha_2} x_3^{\alpha_3}} = \frac{p_2}{\alpha_2 x_2^{\alpha_2 - 1} x_1^{\alpha_1} x_3^{\alpha_3}} \tag{52}$$

$$\frac{\alpha_2 x_1}{p_2} = \frac{\alpha_1 x_2}{p_1} \tag{53}$$

$$x_1 = \frac{\alpha_1 x_2 p_2}{p_1 \alpha_2} \tag{54}$$

We also have:

$$\frac{p_3}{\alpha_3 x_3^{\alpha_3 - 1} x_2^{\alpha_2} x_1^{\alpha_1}} = \frac{p_2}{\alpha_2 x_2^{\alpha_2 - 1} x_1^{\alpha_1} x_3^{\alpha_3}} \tag{55}$$

$$\frac{\alpha_2 x_3}{p_2} = \frac{\alpha_3 x_2}{p_3} \tag{56}$$

$$x_3 = \frac{\alpha_3 x_2 p_2}{p_3 \alpha_2} \tag{57}$$

Plugging into the utility constraint we have:

$$u - \left(\frac{\alpha_1 x_2 p_2}{p_1 \alpha_2}\right)^{\alpha_1} x_2^{\alpha_2} \left(\frac{\alpha_3 x_2 p_2}{p_3 \alpha_2}\right)^{\alpha_3} = 0 \quad (58)$$

$$\left(\frac{\alpha_1 x_2 p_2}{p_1 \alpha_2}\right)^{\alpha_1} x_2^{\alpha_2} \left(\frac{\alpha_3 x_2 p_2}{p_3 \alpha_2}\right)^{\alpha_3} = u \quad (59)$$

$$x_2^{\alpha_1+\alpha_2+\alpha_3} \left(\frac{\alpha_1 p_2}{p_1 \alpha_2}\right)^{\alpha_1} \left(\frac{\alpha_3 p_2}{p_3 \alpha_2}\right)^{\alpha_3} = u \quad (60)$$

$$u \left(\frac{p_1 \alpha_2}{\alpha_1 p_2}\right)^{\alpha_1} \left(\frac{p_3 \alpha_2}{\alpha_3 p_2}\right)^{\alpha_3} = x_2 \quad (61)$$

$$u \frac{p_1^{\alpha_1} \alpha_2^{\alpha_1} p_3^{\alpha_3} \alpha_2^{\alpha_3}}{\alpha_1^{\alpha_1} p_2^{\alpha_1} \alpha_3^{\alpha_3} p_2^{\alpha_3}} = x_2 \quad (62)$$

$$u \frac{p_1^{\alpha_1} p_3^{\alpha_3} \alpha_2^{\alpha_1+\alpha_3}}{p_2^{\alpha_1+\alpha_3} \alpha_1^{\alpha_1} \alpha_3^{\alpha_3}} = x_2 \quad (63)$$

$$u \frac{p_1^{\alpha_1} p_3^{\alpha_3} \alpha_2^{\alpha_1+\alpha_3}}{p_2^{1-\alpha_2} \alpha_1^{\alpha_1} \alpha_3^{\alpha_3}} * \frac{\alpha_2^{\alpha_2}}{\alpha_2^{\alpha_2}} = x_2 \quad (64)$$

$$u \frac{p_1^{\alpha_1} p_3^{\alpha_3} \alpha_2^{\alpha_1+\alpha_2+\alpha_3}}{p_2^{1-\alpha_2} \alpha_1^{\alpha_1} \alpha_3^{\alpha_3} \alpha_2^{\alpha_2}} = x_2 \quad (65)$$

$$u \frac{p_1^{\alpha_1} p_3^{\alpha_3} \alpha_2 p_2^{\alpha_2 - 1}}{\alpha_1^{\alpha_1} \alpha_3^{\alpha_3} \alpha_2^{\alpha_2}} = x_2 \quad (66)$$

To clarify, we are finding the Hicksian demands when solving the expenditure minimization problem, so

$$x_2^h(p,u) = u \frac{p_1^{\alpha_1} p_3^{\alpha_3} \alpha_2 p_2^{\alpha_2 - 1}}{\alpha_1^{\alpha_1} \alpha_3^{\alpha_3} \alpha_2^{\alpha_2}} \quad (67)$$

We can then find that:

$$x_1^h(p,u) = u \frac{p_2^{\alpha_2} p_3^{\alpha_3} \alpha_1 p_1^{\alpha_1 - 1}}{\alpha_1^{\alpha_1} \alpha_3^{\alpha_3} \alpha_2^{\alpha_2}} \quad (68)$$

$$x_3^h(p,u) = u \frac{p_1^{\alpha_1} p_2^{\alpha_2} \alpha_3 p_3^{\alpha_3 - 1}}{\alpha_1^{\alpha_1} \alpha_3^{\alpha_3} \alpha_2^{\alpha_2}} \quad (69)$$

As we did with the indirect utility function, we have a set of results with the expenditure function:

Theorem 8 *If $u(\cdot)$ is continuous and strictly increasing, then $e(p,u)$ is:*

1. *Zero when u takes on the lowest level of utility in U*
2. *Continuous on its domain $\mathbb{R}^n_{++} \times U$*
3. *For all $p >> 0$, strictly increasing and unbounded above in u.*
4. *Increasing in p*
5. *Homogeneous of degree 1 in p.*
6. *Concave in p*
7. *If $u(\cdot)$ is also stricly quasiconcave, we have Shephard's lemma: $e(p,u)$ is differentiable in p at (p^0, u^0) with $p^0 >> 0$, and*

$$\frac{\partial e(p^0, u)}{\partial p_i} = x_i^h(p^0, u) \qquad i = 1, ..., n \quad (70)$$

Again, we will take these without proof although the text has proofs. Expenditure is zero when utility is at its lowest possible level – there is no need to spend any money to achieve that level. As u increases, expenditure must increase (holding prices constant). Also, expenditure is unbounded as utility increases. If prices (or one price) increase, then expenditure does not decrease. There are also homogeneity, concavity, and continuity results. Finally, we can derive the Hicksian demand functions directly from the expenditure function. Recall that the expenditure function is essentially px, or $p_1x_1 + p_2x_2 + ... + p_nx_n$. Thus we can simply differentiate the expenditure function to find the Hicksian demands. As an example, consider the following expenditure function:

$$e(p,u) = \frac{up_1^{\alpha_1}p_2^{\alpha_2}p_3^{\alpha_3}}{\alpha_1^{\alpha_1}\alpha_2^{\alpha_2}\alpha_3^{\alpha_3}} \tag{71}$$

where $\alpha_1 + \alpha_2 + \alpha_3 = 1$. The expenditure function is zero when $u = 0$ (from a Cobb-Douglas this is the lowest level of utility we can have). It is strictly increasing in u and unbounded above in u. It is increasing in any price. For homogeneity of degree 1 in prices we have:

$$e(tp,u) = \frac{u(tp_1)^{\alpha_1}(tp_2)^{\alpha_2}(tp_3)^{\alpha_3}}{\alpha_1^{\alpha_1}\alpha_2^{\alpha_2}\alpha_3^{\alpha_3}} \tag{72}$$

$$e(tp,u) = \frac{ut^{\alpha_1}p_1^{\alpha_1}t^{\alpha_2}p_2^{\alpha_2}t^{\alpha_3}p_3^{\alpha_3}}{\alpha_1^{\alpha_1}\alpha_2^{\alpha_2}\alpha_3^{\alpha_3}} \tag{73}$$

$$e(tp,u) = \frac{t^{\alpha_1+\alpha_2+\alpha_3}up_1^{\alpha_1}p_2^{\alpha_2}p_3^{\alpha_3}}{\alpha_1^{\alpha_1}\alpha_2^{\alpha_2}\alpha_3^{\alpha_3}} \tag{74}$$

$$e(tp,u) = te(p,u) \tag{75}$$

For Shephard's lemma, we have:

$$\frac{\partial e(p,u)}{\partial p_i} = \frac{u\alpha_i p_i^{\alpha_i-1}p_j^{\alpha_j}p_k^{\alpha_k}}{\alpha_i^{\alpha_i}\alpha_j^{\alpha_j}\alpha_k^{\alpha_k}} = x_i^h(p,u) \tag{76}$$

4.2.1 Relating $v(p,y)$ and $e(p,u)$

There is a relationship between $v(p,y)$ and $e(p,u)$. If we fix p and y and let $u = v(p,y)$. As per the definition of $v(p,y)$, u is the maximum utility that can be attained when prices are p and income is y. Also, if the consumer wishes to achieve utility level u at prices p, then the consumer will be able to achieve that level of utility with expenditure y. But the expenditure minimization function tells us the LEAST amount of expenditure needed to achieve utility level u at prices p. Thus, we would need:

$$e(p,u) \leq y \tag{77}$$

$$e(p,v(p,y)) \leq y \tag{78}$$

We can perform a similar analysis if we fix p and u. We know that at prices p and utility u, we will need:

$$v(p,y) \geq u \tag{79}$$

$$v(p,e(p,u)) \geq u \tag{80}$$

Now we have a theorem explaining the relationship between $e(p,u)$ and $v(p,y)$.

Theorem 9 *Let $v(p,y)$ and $e(p,u)$ be the indirect utility function and expenditure function for some consumer whose utility function is strictly increasing and continuous. Then for all $p >> 0$, $y \geq 0$, and $u \in U$:*

1. $e(p,v(p,y)) = y$
2. $v(p,e(p,u)) = u$

Again, we forgo the proof and use an example. We will use

$$v(p,y) = \frac{\alpha_1^{\alpha_1}\alpha_2^{\alpha_2}\alpha_3^{\alpha_3}y}{p_1^{\alpha_1}p_2^{\alpha_2}p_3^{\alpha_3}} \tag{81}$$

$$e(p,u) = \frac{up_1^{\alpha_1}p_2^{\alpha_2}p_3^{\alpha_3}}{\alpha_1^{\alpha_1}\alpha_2^{\alpha_2}\alpha_3^{\alpha_3}} \tag{82}$$

So:

$$e(p,v(p,y)) = \frac{\left(\frac{\alpha_1^{\alpha_1}\alpha_2^{\alpha_2}\alpha_3^{\alpha_3}y}{p_1^{\alpha_1}p_2^{\alpha_2}p_3^{\alpha_3}}\right)p_1^{\alpha_1}p_2^{\alpha_2}p_3^{\alpha_3}}{\alpha_1^{\alpha_1}\alpha_2^{\alpha_2}\alpha_3^{\alpha_3}} \tag{83}$$

$$e(p,v(p,y)) = y \tag{84}$$

Also:

$$v(p,e(p,u)) = \frac{\alpha_1^{\alpha_1}\alpha_2^{\alpha_2}\alpha_3^{\alpha_3}\left(\frac{up_1^{\alpha_1}p_2^{\alpha_2}p_3^{\alpha_3}}{\alpha_1^{\alpha_1}\alpha_2^{\alpha_2}\alpha_3^{\alpha_3}}\right)}{p_1^{\alpha_1}p_2^{\alpha_2}p_3^{\alpha_3}} \tag{85}$$

$$v(p,e(p,u)) = u \tag{86}$$

Theorem 10 *Given that $u(\cdot)$ is strictly increasing, continuous, and strictly quasiconcave, the following relations hold between the Marshallian and Hicksian demand functions when $p >> 0$, $y \geq 0$, and $u \in U$.*

1. $x_i(p,y) = x_i^h(p,v(p,y))$
2. $x_i^h(p,u) = x_i(p,e(p,u))$

This first relationship states that the Marshallian demand for prices p and income y are identical to the Hicksian demand at prices p when utility is $v(p,y)$. Alternatively, the Hicksian demands at p and u are equal to the Marshallian demands when prices are p and expenditure is given by $y = e(p,u)$.

As an example:

$$x_i^h(p,u) = \frac{u\alpha_i p_i^{\alpha_i-1}p_j^{\alpha_j}p_k^{\alpha_k}}{\alpha_i^{\alpha_i}\alpha_j^{\alpha_j}\alpha_k^{\alpha_k}} \tag{87}$$

$$x_i^h(p,v(p,y)) = \frac{\left(\frac{y\alpha_i^{\alpha_i}\alpha_j^{\alpha_j}\alpha_k^{\alpha_k}}{p_i^{\alpha_i}p_j^{\alpha_j}p_k^{\alpha_k}}\right)\alpha_i p_i^{\alpha_i-1}p_j^{\alpha_j}p_k^{\alpha_k}}{\alpha_i^{\alpha_i}\alpha_j^{\alpha_j}\alpha_k^{\alpha_k}} \tag{88}$$

$$x_i^h(p,v(p,y)) = \frac{\alpha_i y p_i^{\alpha_i-1}}{p_i^{\alpha_i}} \tag{89}$$

$$x_i^h(p,v(p,y)) = \frac{\alpha_i y}{p_i} \tag{90}$$

$$x_i^h(p,v(p,y)) = x_i(p,y) \tag{91}$$

Also:

$$x_i(p,y) = \frac{\alpha_i y}{p_i} \tag{92}$$

$$x_i(p,e(p,u)) = \frac{\alpha_i\left(\frac{up_i^{\alpha_i}p_j^{\alpha_j}p_k^{\alpha_k}}{\alpha_i^{\alpha_i}\alpha_j^{\alpha_j}\alpha_k^{\alpha_k}}\right)}{p_i} \tag{93}$$

$$x_i(p,e(p,u)) = \frac{u\alpha_i p_i^{\alpha_i-1}p_j^{\alpha_j}p_k^{\alpha_k}}{\alpha_i^{\alpha_i}\alpha_j^{\alpha_j}\alpha_k^{\alpha_k}} \tag{94}$$

$$x_i(p,e(p,u)) = x_i^h(p,u) \tag{95}$$

5 Properties of consumer demand

In principles and intermediate microeconomics you typically study demand functions first and then (perhaps) utility or consumer theory. Here we have started with the utility function and used the utility function to derive demand functions. Right now we are concerned with individual demand functions. When you studied them at the undergraduate level, we simply stated things like "The Law of Demand states that there is an inverse relationship between the price of a good and its quantity demanded". You were usually asked to take that on faith, and intuitively it makes sense – holding everything else constant, if you raise the price of a good its quantity demanded should fall. Now, we will see that the demand functions used in the undergraduate classes were a direct result of the assumptions that we have been making about our preference relation $\succsim$, our utility function $u(\cdot)$, and our feasible set.

Two of the most basic concepts are *relative prices* and *real income*. Economists are more concerned with relative prices rather than actual prices, as consumers care about the quantity of money only in terms of the amount of goods and services a particular amount of money can buy (people have little utility for the actual good "money", other than that it serves as a medium of exchange by which they can purchase goods). Thus, we can discuss prices in terms of relative prices – namely, we can fix the price of one good (call it the numeraire) and then denominate all other goods in that numeraire. Economists also discuss purchasing power in terms of real income. If one individual has \$10,000 and the other has \$100,000 then we tend to think that the person with \$100,000 is better off than the one with \$10,000. This is true if prices are the same, but if the individual with \$10,000 faces a price vector that is $\frac{1}{100}^{th}$ of the price vector that the individual with \$100,000 (so that the person with \$100,000 faces prices that are 100x higher), then the person with \$10,000 will be better off than the person with \$100,000 because the person with \$10,000 can purchase more goods. If consumer preferences are complete, transitive, reflexive, strictly monotonic, and strictly convex, then Marshallian demand functions are homogeneous of degree zero (which essentially means that if you increase all prices and wealth by the same proportion the consumer's Marshallian demand does not change) and budget balancedness holds (the budget constraint holds with equality).

5.1 Income and substitution effects

While finding the solution to the UMP or the EMP is an important step, many economists focus on what happens when something changes in the economic system. We will begin by discussing price changes in Hicksian demand, as Hicksian demand satisfies the law of demand (price increases, quantity demanded decreases) while Walrasian demand may or may not. However, Hicksian demand is a function of an unobservable variable, utility. Walrasian demand, however, is a function of the observable variables (or at least variables that we might be able to observe) price and wealth (or income).

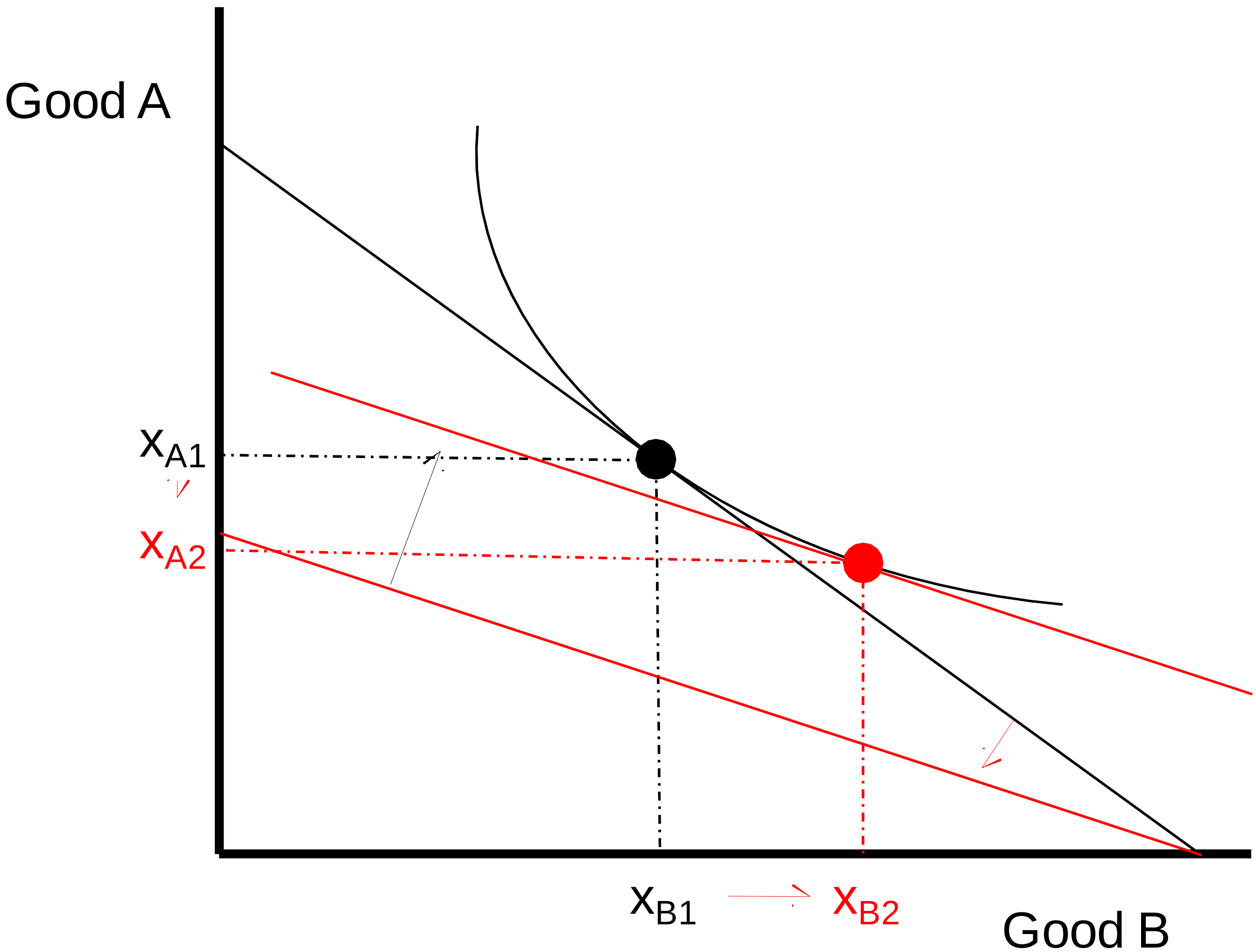
We have that the own-price derivatives of Hicksian demand are nonpositive because Hicksian demand follows the compensated law of demand. This means

$$\frac{\partial x_i^h(p,u)}{\partial p_i} \leq 0.$$

Recall that with a Hicksian demand change we are determining how much quantity demanded falls when price increases by keeping the consumer on the same indifference curve (or at the same utility level).Given that our indifference curves are downward sloping, it is necessarily the case that Hicksian demand decreases (if we have a differentiable utility function and are at an interior solution) as the figure above shows or remains at zero (if we have a utility function that is nondifferentiable and are at an interior solution – we stay at the same point, think of perfect complements – or if we are at a corner solution). We can also show this mathematically as we have:

$$\begin{aligned} \frac{\partial e(p,u)}{\partial p_i} &= x_i^h(p,u) \\ \frac{\partial^2 e(p,u)}{\partial p_i \partial p_i} &= \frac{\partial x_i^h(p,u)}{\partial p_i} \leq 0 \end{aligned}$$

This is because the expenditure function is a concave function.


Good A
x_{A1}
x_{A2}
x_{B1}
x_{B2}
Good B

Now consider the cross-price derivative of Hicksian demand $x_i^h(p,u)$ with respect to the price of good k, p_k. If $\frac{\partial x_i^h(p,u)}{\partial p_k} \leq 0$ then goods i and k are complements or complementary goods, because as the price of good k increases the Hicksian demand for good i decreases. Thus we are consuming less of good k and less of good i when p_k increases. If $\frac{\partial x_i^h(p,u)}{\partial p_k} \geq 0$ then goods i and k are substitutes because as the price of good k increases the Hicksian demand for good i increases. Note that if the cross-price derivative is equal to zero then the goods could be classified as either substitutes or complements. However, consider what it means if the cross-price derivative truly is zero – a change in p_k has no effect on the Hicksian demand for good i. Thus the two goods could be classified as independent. We know that there must be at least one good which which has a nonpositive substitution effect for any specific good in the economy. To see this, consider the 2-good case. If the price of good k increases, then the consumption of good k will decrease (unless the consumer is at a corner solution) because Hicksian demand follows the compensated law of demand. Now, if the consumer is to remain at the same utility level, and he is consuming less of good k, then he must consume more of good i.

5.1.1 Decomposing Hicksian demand changes

The purpose of using Hicksian demand is because Hicksian demand follows the compensated law of demand. But, we cannot observe Hicksian demand because one of its arguments is unobservable (utility level). We can exploit the relationship between Hicksian demand and Walrasian demand to obtain information on price effects.[3]

Proposition 11 *(The Slutsky Equation) Suppose that $u(\cdot)$ is a continuous utility function representing locally nonsatiated preference relation $\succsim$ defined on $X = \mathbb{R}_+^N$. Then for all (p,y) and $u = v(p,y)$ we have*

$$\frac{\partial x_i^h(p,u)}{\partial p_k} = \frac{\partial x_i(p,y)}{\partial p_k} + \frac{\partial x_i(p,y)}{\partial y} x_k(p,y) \text{ for all } i,k.$$

Or, rewriting in terms of the cross-price effect of the Walrasian demand:

$$\frac{\partial x_i(p,y)}{\partial p_k} = \frac{\partial x_i^h(p,u)}{\partial p_k} + \frac{\partial x_i(p,y)}{\partial y} x_k(p,y) \text{ for all } i,k.$$

Proof. We know that $x_i^h(p,u) = x_i(p, e(p,u))$ at the optimal solution to the consumer's problem. We can differentiate with respect to p_k and evaluate at $\overline{p}$ and $\overline{u}$.

Statement	Reason
1. $\frac{\partial x_i^h(\overline{p},\overline{u})}{\partial p_k} = \frac{\partial x_i(\overline{p},e(\overline{p},\overline{u}))}{\partial p_k} + \frac{\partial x_i(\overline{p},e(\overline{p},\overline{u}))}{\partial y}\frac{\partial e(\overline{p},\overline{u})}{\partial p_k}$	1. Chain rule for differentiation
2. $\frac{\partial x_i^h(\overline{p},\overline{u})}{\partial p_k} = \frac{\partial x_i(\overline{p},e(\overline{p},\overline{u}))}{\partial p_k} + \frac{\partial x_i(\overline{p},e(\overline{p},\overline{u}))}{\partial y} x_k^h(\overline{p},\overline{u})$	2. Earlier result on relation of $e(p,u)$ to $x^h(p,u)$
3. $x_k^h(\overline{p},\overline{u}) = x_k(\overline{p}, e(\overline{p},\overline{u})) = x_k(\overline{p},\overline{y})$	3. Earlier result on relation of $x_k^h(p,u)$ to $x_k(p,y)$
4. $e(\overline{p},\overline{u}) = \overline{y}$	4. Earlier result on relation of $e(p,u)$ to y
5. $\frac{\partial x_i^h(p,u)}{\partial p_k} = \frac{\partial x_i(p,y)}{\partial p_k} + \frac{\partial x_i(p,y)}{\partial y} x_k(p,y)$	5. Substitution

■

For the Walrasian demand, the change in quantity of good i with respect to a change in the price of good k is known as the Total Effect of the change in price of good k. The total effect is decomposed into the Substitution Effect $\left(\frac{\partial x_i^h(p,u)}{\partial p_k}\right)$ and the Income (or Wealth) Effect $\left(\frac{\partial x_i(p,y)}{\partial y} x_k(p,y)\right)$. The Substitution Effect is the change in quantity demanded of good i due to the fact that good i is now relatively more (less) expensive if the price of another good (say good k) increases (decreases) when the prices of all other goods stay the same. Thus, if the price of a good k increases, we would expect that a consumer would purchase more of a second good i because i is now a relatively less expensive substitute (unless of course the goods are complements). The Income Effect is the change in quantity demanded of good i due to the fact that

[3] For a recent reference on using the Slutsky equation in empirical work, see Fisher, Shively, and Buccola (2005). Activity Choice, Labor Allocation, and Forest Use in Malawi. *Land Economics*, Vol. 81:4

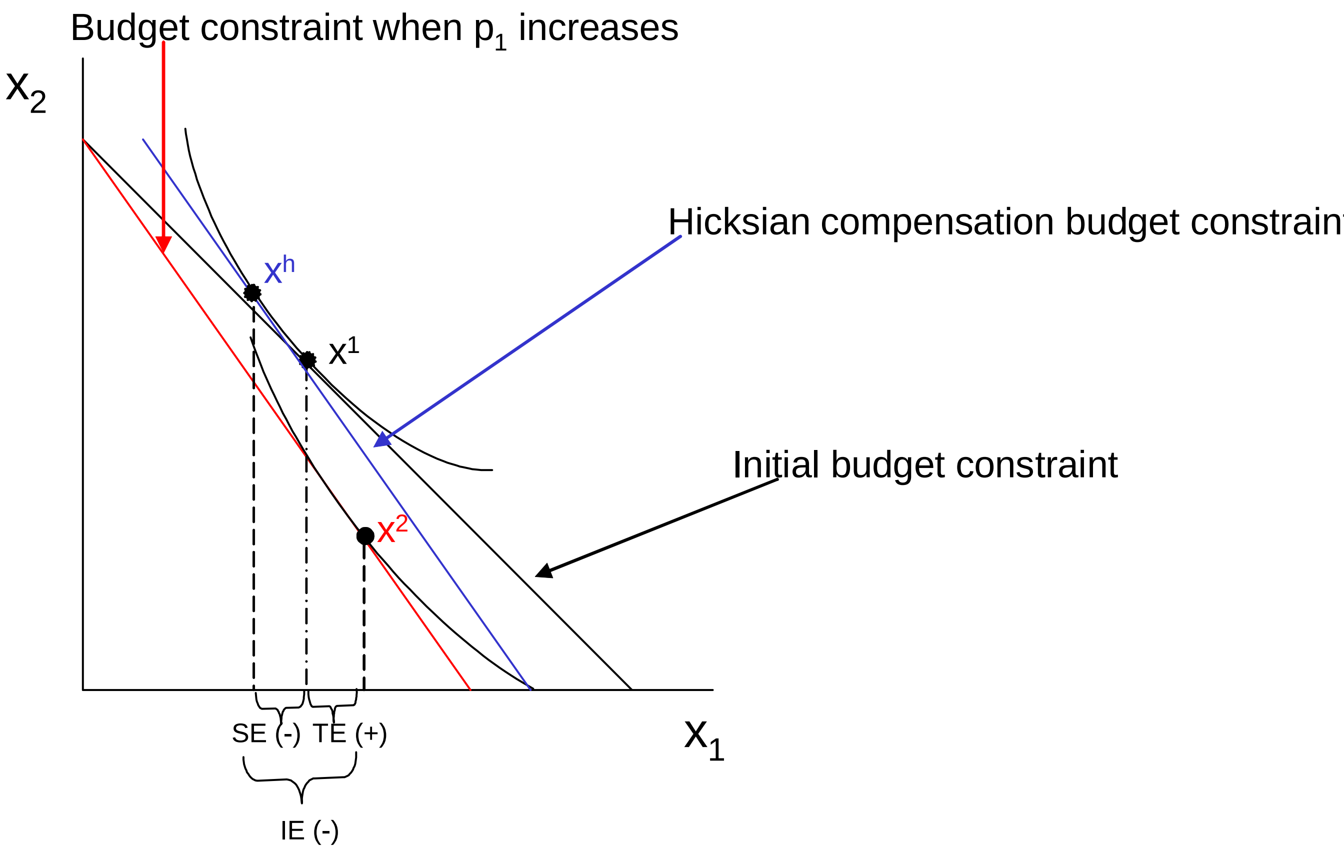


Figure 1: Decomposing the effect of a price change on a Giffen good.

the consumer has control over how he spends his wealth. There need be no actual change in y for there to be an income effect, but if the price of good k increases, then the consumer may not just decide to reduce consumption of good k at the rate of the price increase. For example, if p_k doubles, the consumer may keep consumption of good i the same and simply reduce consumption of good k to $\frac{1}{2}$ its previous level, but is not required to act in this manner. The consumer may cut consumption by more (or less) than $\frac{1}{2}$ and adjust consumption of good i accordingly. The consumer may even increase the amount of good k when a price increase occurs – this is the case of a Giffen good, and it occurs because the Income Effect overwhelms the Substitution Effect. Consider the Slutsky equation for a change in the own-price of a good:

$$\frac{\partial x_i(p,y)}{\partial p_i} = \frac{\partial x_i^h(p,u)}{\partial p_i} + \frac{\partial x_i(p,y)}{\partial y} x_i(p,y)$$

We know that if p_i increases that the Substitution Effect $\left(\frac{\partial x_i^h(p,u)}{\partial p_i}\right)$ will be negative. However, there is no such restriction on the Total Effect $\left(\frac{\partial x_i(p,y)}{\partial p_i}\right)$ as it may be positive or negative (recall the case of Giffen goods). It will be positive if the Income Effect is more negative than the Substitution Effect (remember, this is an OWN-price equation, so the Hicksian demand must decrease when its own price increases). In this case, we have a Giffen good because $\frac{\partial x_i(p,y)}{\partial p_i} > 0$. Thus, it is an usually large negative income effect that is driving the Giffen good result. Typically one would think that income effects would be positive (we know that $x_i(p,w) \geq 0$, so focus on $\frac{\partial x_i(p,y)}{\partial y}$). This is just the derivative of the Walrasian demand function with respect to wealth, and usually if wealth increases consumers consume more of a good (hence the reason we call these goods "normal goods"). However, if a good is a Giffen good then it must have a wealth effect negative enough to overwhelm the negative substitution effect. Thus any good that is a

Giffen good must be an inferior good. However, this does not mean that all inferior goods are Giffen goods – if the derivative of the Walrasian demand is negative (so that the good is inferior), it is possible that the wealth effect is LESS negative than the substitution effect. In this case, while the good is inferior, its total effect will still be negative. Figure 1 shows the effect of a price change of a Giffen good decomposed into its total, substitution, and income effects. The initial budget constraint is in black and the optimal bundle is represented by x^1. The new budget constraint after an increase in the price of good x_1 is given in red and its optimal consumption bundle is represented by x^2. The blue budget constraint is the budget constraint that returns the consumer to his original utility after the price change and the optimal bundle is represented by x^h. Now, the total effect is simply the change in good x_1 when its price changes, so we compare the quantity of x_1 consumed under the initial budget constraint with the quantity consumed under the budget constraint when p_1 increases. Note that there is an INCREASE in consumption of x_1 when p_1 increases – thus we have a Giffen good (the exact "equation" to find this is quantity of x_1 consumed at bundle x^2 minus quantity of x_1 consumed at bundle x^1). To find the income effect, compare the quantity of x_1 consumed under the new budget constraint with the quantity of x_1 consumed under the budget constraint with the new relative prices that returns the consumer to his initial utility level (the Hicksian compensation budget constraint as it is labeled). Again, to find this take the quantity of x_1 consumed at x^h and subtract the quantity of x_1 consumed at x^2. The substitution effect is simply the change in consumption of x_1 at x^1 to consumption of x_1 at x^h (take the amount of x_1 consumed at x^h and subtract the amount of x_1 consumed at x^1). Note that since this is an own-price effect on Hicksian demand it must be negative. We can do the exact same analysis for good x_2 when the price of good x_1 increases. For good x_2, its total effect is negative, while its substitution effect is positive (only two goods so they must be substitutes) but its income effect is MORE positive than its substitution effect, leading to the negative total effect.[4]

Now, there are a few additional results that rely on the Hessian matrix of the expenditure function $e(p,u)$. If we take the derivative of $e(p,u)$ once with respect to p we will obtain a row vector of length N, where N is the number of goods (we will have one derivative for each of the N goods). Recall that our Hicksian demand without a subscript, $x^h(p,u)$ is really a vector of Hicksian demands, one for each good, or $x^h(p,u) = \begin{bmatrix} x_1^h(p,u) & x_2^h(p,u) \end{bmatrix}$ for the two-good world. Alternatively, we could write $\frac{\partial e(p,u)}{\partial p} = \begin{bmatrix} \frac{\partial e(p,u)}{\partial p_1} & \frac{\partial e(p,u)}{\partial p_2} \end{bmatrix}$. So the Hicksian demand function is nothing more than the gradient of the expenditure function in pure math terms. Note that the vectors are the same because $\frac{\partial e(p,u)}{\partial p} = h(p,u)$. The Hessian matrix is simply an $N \times N$ matrix of second partial derivatives. For our two-good world, we would have:

$$\frac{\partial^2 e(p,u)}{\partial p^2} = \sigma(p,u) = \begin{bmatrix} \frac{\partial^2 e(p,u)}{\partial p_1 \partial p_1} & \frac{\partial^2 e(p,u)}{\partial p_1 \partial p_2} \\ \frac{\partial^2 e(p,u)}{\partial p_2 \partial p_1} & \frac{\partial^2 e(p,u)}{\partial p_2 \partial p_2} \end{bmatrix} = \begin{bmatrix} \frac{\partial x_1^h(p,u)}{\partial p_1} & \frac{\partial x_1^h(p,u)}{\partial p_2} \\ \frac{\partial x_2^h(p,u)}{\partial p_1} & \frac{\partial x_2^h(p,u)}{\partial p_2} \end{bmatrix}.$$

Now, a proposition:

Proposition 12 *Suppose that $u(\cdot)$ is a continuous utility function representing locally nonsatiated preference relation $\succsim$ on the consumption set $X = \mathbb{R}_+^L$. Suppose also that $x^h(\cdot,u)$ is continuously differentiable at (p,u) and denote its $N \times N$ derivative matrix by $\sigma(p,u)$. Then*

1. $\sigma(p,u) = D_p^2 e(p,u)$
2. $\sigma(p,u)$ *is a negative semidefinite matrix.*
3. $\sigma(p,u)$ *is a symmetric matrix.*
4. $\sigma(p,u)\, p = 0$.

We have already discussed the first result. The second and third results have to do with the fact that since $e(p,u)$ is a twice continuously differentiable concave function, it has a symmetric and negative semidefinite Hessian matrix. The fourth result follows from Euler's formula since $x^h(p,u)$ is homogeneous of degree zero in prices. Homogeneity of degree zero implies that price derivatives of Hicksian demand for

[4] Obtaining the correct sign for these effects may be a little confusing. The key is to take the amount of the good at the NEW bundle, and subtract the amount of the good at the original bundle.

any good i, when weighted by these prices, sum to zero. Euler's formula states if $x^h(p,u)$ is homogeneous of degree zero in prices and wealth, then:

$$\sigma(p,u)\,p = \begin{bmatrix} \frac{\partial x_1^h(p,u)}{\partial p_1} & \frac{\partial x_1^h(p,u)}{\partial p_2} \\ \frac{\partial x_2^h(p,u)}{\partial p_1} & \frac{\partial x_2^h(p,u)}{\partial p_2} \end{bmatrix} * \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} \frac{\partial x_1^h(p,u)}{\partial p_1}p_1 + \frac{\partial x_1^h(p,u)}{\partial p_2}p_2 \\ \frac{\partial x_2^h(p,u)}{\partial p_1}p_1 + \frac{\partial x_2^h(p,u)}{\partial p_2}p_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

As for the terms "symmetric" and "negative semidefinite" matrix, a symmetric matrix is simply a matrix that equals its transpose (to transpose a matrix simply take the first column of the original matrix and make that the first row of the transpose, then take the second column of the matrix and make that the second row of the transpose, etc.). So, for our two-good world, if $\sigma(p,u)$ is our matrix and $\sigma^T(p,u)$ is its transpose,

$$\sigma(p,u) = \sigma^T(p,u)$$

Or

$$\begin{bmatrix} \frac{\partial x_1^h(p,u)}{\partial p_1} & \frac{\partial x_1^h(p,u)}{\partial p_2} \\ \frac{\partial x_2^h(p,u)}{\partial p_1} & \frac{\partial x_2^h(p,u)}{\partial p_2} \end{bmatrix} = \begin{bmatrix} \frac{\partial x_1^h(p,u)}{\partial p_1} & \frac{\partial x_1^h(p,u)}{\partial p_1} \\ \frac{\partial x_2^h(p,u)}{\partial p_2} & \frac{\partial x_2^h(p,u)}{\partial p_2} \end{bmatrix}$$

Notice that the two off diagonal elements are switched. Thus, since $\sigma(p,u)$ is symmetric, $\frac{\partial x_1^h(p,u)}{\partial p_2} = \frac{\partial x_2^h(p,u)}{\partial p_1}$, or using the expenditure function notation, $\frac{\partial^2 e(p,u)}{\partial p_1 \partial p_2} = \frac{\partial^2 e(p,u)}{\partial p_2 \partial p_1}$. Thus, it does not matter which price you use to differentiate with first – the result will be the same. For a refresher on the definition of negative semidefiniteness, take a look at the mathematical appendix. That $\sigma(p,u)$ is negative semidefinite ensures us that the own-price derivatives of the Hicksian demand function are less than or equal to zero (note that the own-price derivatives of the Hicksian demand function are the elements along the diagonal of $\sigma(p,u)$).

5.2 Elasticities

One final concept that we will discuss which is commonly used in economics is elasticity. Elasticity is a unitless measure, and it tells us how responsiveness quantity changes are to changes in prices or income (or, more generally, a "dollar" measure). By definition, elasticities are determined as the ratio of the percentage change of one variable to another. There are three common elasticities in consumer theory: own-price elasticity of demand, cross-price elasticity, and income elasticity.

5.2.1 Price elasticities

The general formula for a price elasticity is:

$$\varepsilon_{ij} = \frac{\partial x_i(p,y)}{\partial p_j} * \frac{p_j}{x_i(p,y)}$$

If we have $i = j$ then we have an own-price elasticity of demand. In general, when $i = j$, $\frac{\partial x_i(p,y)}{\partial p_i} \leq 0$ as most goods are NOT Giffen goods. As such, we typically take the absolute value of own-price elasticity of demand (since it is negative). If $0 \leq \varepsilon_{ii} < 1$ we say that demand is inelastic. This simply means that there is not as large a percentage change in quantity as there is in price. If $\varepsilon_{ij} \geq 1$ then we say that demand is elastic, or that the percentage change in quantity is larger than that of price.

If we have $i \neq j$ then we have a cross-price elasticity of demand – how responsive is the quantity of good i to a change in the price of good j. Note that cross-price elasticity of demand can be positive or negative. If $\varepsilon_{ij} > 0$ then we have substitute goods, as a price increase in good j will cause more of good i to be consumed (consumers substitute away from good j towards good i). If $\varepsilon_{ij} < 0$ then we have complements or complementary goods, as the consumer is now buying less of good i in response to a price increase in good j. If $\varepsilon_{ij} = 0$, then the two goods are independent.

5.2.2 Income elasticity

The general formula for income elasticity is:

$$\eta_i = \frac{\partial x_i(p,y)}{\partial y} * \frac{y}{x_i(p,y)}$$

Again, income elasticity may be positive or negative. If $\eta_i \geq 0$ then we have a normal good as purchases of good i increase (or stay the same) as income increases. If $\eta_i < 0$ then we have an inferior good as consumers are purchasing less of good i despite more income.