These notes correspond to chapter 2 of Jehle and Reny.

1 Uncertainty

Until now we have considered our consumer's making decisions in a world with perfect certainty. However, we can extend the consumer theory model to include situations where the outcome of a decision is uncertain. When we discuss uncertainty, we mean that there is a probability distribution over possible outcomes that could occur. The consumer knows both the different possible outcomes that could occur as well as the probability distribution over those outcomes.[1] As with our model of the consumer in a world of perfect certainty, we can model the consumer's preferences in a world of uncertainty.

First we must specify is what precisely the consumer has preferences over. In the world of certainty the consumer had preferences over goods, or bundles of goods. When uncertainty is present, the consumer has preferences over gambles.[2] A gamble is simply a set of outcomes, $A = \{a_1, a_2, ..., a_n\}$ with a probability distribution over the set of outcomes. A simple gamble assigns a probability p_i to each of the outcomes, where $p_i \geq 0$ and $\sum_{i=1}^{n} p_i = 1$. A simple gamble is denoted G_S.

Let's say that $A = \{-5, 0, 5\}$ with $p_i = \frac{1}{3}$ for $i = 1, 2, 3$. That would be a simple gamble. A compound gamble would include a gamble as part of the simple gamble. Again, consider $A = \{-5, 0, 5\}$. A compound gamble might be $\left(\frac{1}{3} \circ -5, \frac{1}{3} \circ \left(\frac{1}{9} \circ -5, \frac{4}{9} \circ 0, \frac{4}{9} \circ 5\right), \frac{1}{3} \circ 5\right)$. Let G denote the entire space of gambles.

As with the standard consumer model, there are certain assumptions that we make as to how a consumer relates one gamble to another. With the standard problem we assumed completeness, reflexivity, transitivity, continuity, monotonicity, and convexity. With uncertainty we make similar assumptions.

Axiom 1 *Completeness. For any two distinct gambles g and g' in G, either $g \succsim g'$ or $g' \succsim g$.*

Axiom 2 *Reflexivity. For every gamble g in G, $g \succsim g$.*

Axiom 3 *Transitivity. For any three gambles g, g', and g'' in G, if $g \succsim g'$, and $g' \succsim g''$, then $g \succsim g''$.*

We now assume that the elements of A are ordered such that $a_1 \succsim a_2 \succsim ... \succsim a_n$. For any gamble g, $(\alpha \circ a_1, (1-\alpha) \circ a_n) \succsim g$ when $\alpha = 1$ and $g \succsim (\alpha \circ a_1, (1-\alpha) \circ a_n)$ when $\alpha = 0$.

Axiom 4 *Continuity. For any gamble g in G, there is some probability, $\alpha \in [0,1]$, such that $g \sim (\alpha \circ a_1, (1-\alpha) \circ a_n)$.*

Axiom 5 *Monotonicity. For all probabilities $\alpha, \beta \in [0,1]$,*

$$(\alpha \circ a_1, (1-\alpha) \circ a_n) \succsim (\beta \circ a_1, (1-\alpha) \circ a_n)$$

if and only if $\alpha \geq \beta$.

Consider the continuity and monotonicity axioms and the following $A = \{\$1000, \$10, death\}$. If we consider $g = \$10$ with certainty then there must be a gamble over \$1000 and death such that $g \sim (\alpha \circ \$1000, (1-\alpha) \circ death)$ with $\alpha < 1$. This is not as implausible as it seems, if $\alpha = .9999999999$ then the individual could very well be indifferent to the \$10 with certainty and the gamble over \$1000 and death.

Axiom 6 *Substitution. If $g = \left(p_1 \circ g^1, ..., p_k \circ g^k\right)$ and $h = \left(p_1 \circ h^1, ..., p_k \circ h^k\right)$ are in G, and if $h^i \sim g^i$, then $h \sim g$.*

Finally, the consumer is concerned only with the final set of outcomes and probability distribution over those outcomes, and is not concerned with whether the gamble is simple or compound.

Axiom 7 *Reduction to simple gambles. For any gamble $g \in G$, if $(p_1 \circ a_1, ..., p_n \circ a_n)$ is the simple gamble induced by g, then $(p_1 \circ a_1, ..., p_n \circ a_n) \sim g$.*

Consider our compound gamble $\left(\frac{1}{3} \circ -5, \frac{1}{3} \circ \left(\frac{1}{9} \circ -5, \frac{4}{9} \circ 0, \frac{4}{9} \circ 5\right), \frac{1}{3} \circ 5\right)$. Note that this gamble has $\frac{1}{3} \circ -5, \frac{1}{27} \circ -5, \frac{4}{27} \circ 0, \frac{4}{27} \circ 5, \frac{1}{3} \circ 5$, which would be equivalent to the simple gamble $\left(\frac{10}{27} \circ -5, \frac{4}{27} \circ 0, \frac{13}{27} \circ 5\right)$. This, the compound gamble can be reduced to a simple gamble. Again, these are axioms, and they may or may not be true.

[1] If the consumer is uncertain over either of those then we would be discussing the concepts of ambiguity or Knightian uncertainty. We will not consider those concepts here.

[2] Note that other texts use the word "lotteries" in place of "gambles".

1.1 Von Neumann-Morgenstern Utility

As with consumer theory under certainty, life becomes much easier if preferences can be represented by a utility function. This will of course be the case, which is why we make some of these assumptions. Suppose that $u : G \rightarrow \mathbb{R}$ is a utility function representing $\succsim$ on G. For every i, u assigns a particular number $u(a_i)$ to the degenerate gamble $(1 \circ a_i)$. Let $u(a_i)$ denote the utility of outcome a_i.

Definition 8 *The utility function $u : G \rightarrow \mathbb{R}$ has the expected utility property , if for every $g \in G$,*

$$u(g) = \sum_{i=1}^{n} p_i u(a_i)$$

where $(p_1 \circ a_1, ..., p_n \circ a_n)$ is the simple gamble induced by g.

If u has the expected utility property, then $u(p_1 \circ a_1, ..., p_n \circ a_n) = \sum_{i=1}^{n} p_i u(a_i)$. An individual is an expected utility maximizer if the individual always chooses the gamble with the highest expected utility.

Theorem 9 *Let preferences $\succsim$ over gambles in G satisfy axioms 1-7. There there exists a utility function $u : G \rightarrow \mathbb{R}$ representing $\succsim$ on G such that u has the expected utility property.*

The text contains this proof – we will not go through it. The important thing is that preferences can be represented by a utility function. When we have a utility function representing $\succsim$ on G we will refer to it as the vN-M utility function. We will work through the example though.

Let $A = \{\$10, \$4, -\$2\}$, with $\$10 \succ \$4 \succ -\$2$. We need to find which simple gambles over \$10 and $-\$2$ are indifferent to each of the outcomes. Suppose we have:

$$\begin{aligned} \$10 &\sim (1 \circ \$10, 0 \circ -\$2) \\ \$4 &\sim (.6 \circ \$10, .4 \circ -\$2) \\ -\$2 &\sim (0 \circ \$10, 1 \circ -\$2) \end{aligned}$$

which gives $u(\$10) \equiv 1$, $u(\$4) \equiv .6$, and $u(-\$2) \equiv 0$. We can now rank other gambles on A. Suppose we have:

$$\begin{aligned} g_1 &= (.2 \circ \$4, .8 \circ \$10) \\ g_2 &= (.07 \circ -\$2, .03 \circ 4, .9 \circ \$10) \end{aligned}$$

Then we have:

$$\begin{aligned} u(g_1) &= .2u(\$4) + .8u(\$10) \\ u(g_2) &= .07u(-\$2) + .03u(\$4) + .9u(\$10) \end{aligned}$$

We then have $u(g_1) = 0.92$ and $u(g_2) = 0.918$, so $g_1 \succ g_2$. Note that $\$4 \sim (.6 \circ \$10, .4 \circ -\$2)$ in our example, but that $.6u(\$10) + .4u(-\$2) = 5.2$. So even though this gamble pays off an average of \$5.2, the individual prefers \$4 with certainty over the gamble. This is only for the example, and different individuals may have different preferences.

1.1.1 Ordinal vs. cardinal rankings

In our standard consumer theory model we only cared about whether or not a utility function preserved the order of preferences. Thus, if two utility functions, such as $u(x) = x_1 x_2$ and $v(x) = \ln x_1 + \ln x_2$ kept the order of preferences the same, regardless of the utility value the functions generate, they were viewed as equivalent utility functions. This is not so when we move to expected utility functions. If we let $A = \{a, b, c\}$ with $a \succ b \succ c$ and our assumptions about $\succsim$ are satisfied, we must have an $\alpha \in (0, 1)$ such that:

$$b \sim (\alpha \circ a, (1 - \alpha) \circ c)$$

The number α means something here – it is not really a free parameter. If we double α or cut it in half this changes the decision-maker's preferences. If we have u as a vN-M utility function which represents $\succsim$ and satisfies the expected utility property then we have:

$$\begin{aligned}
u(b) &= \alpha u(a) + (1-\alpha) u(c) \\
u(b) - u(c) &= \alpha u(a) - \alpha u(c) \\
u(b) - u(c) &= \alpha u(a) - \alpha \left(\frac{u(b) - \alpha u(a)}{1-\alpha} \right) \\
u(b) - u(c) &= \frac{(1-\alpha)\alpha u(a) - \alpha u(b) + \alpha^2 u(a)}{1-\alpha} \\
u(b) - u(c) &= \frac{\alpha u(a) - \alpha^2 u(a) - \alpha u(b) + \alpha^2 u(a)}{1-\alpha} \\
\frac{1-\alpha}{\alpha} &= \frac{u(a) - u(b)}{u(b) - u(c)}
\end{aligned}$$

Note that the ratio of the differences of the utilities are determined by α and vice versa. So we can only make changes to our vN-M utility function which preserve this equality. So not all positive monotonic transformations of expected utility functions are permissible. What we find is that only positive affine transformations are permissible.

Theorem 10 *Suppose that the vN-M utility function $u(\cdot)$ represents $\succsim$. Then the vN-M utility function $v(\cdot)$ represents those same preferences if and only if for some scalar α and some scalar $\beta > 0$:*

$$v(g) = \alpha + \beta u(g)$$

for all gambles g.

While we do not have as much freedom in choosing expected utility functions as we do in choosing standard utility functions, there is still no particular meaning to the particular value an expected utility function generates. As we can see, there are many expected utility functions which will represent preferences, and the numeric output of:

$$\begin{aligned}
v(g) &= 1 + u(g) \\
y(g) &= 2 + u(g)
\end{aligned}$$

will differ, but $u(g)$, $v(g)$, and $y(g)$ will all represent the same preferences over gambles.

1.2 Risk Aversion

We have already touched upon risk aversion when constructing an example for our expected utility function. We have $A = \{\$10, \$4, -\$2\}$ and we stated that:

$$\$4 \sim (.6 \circ \$10, .4 \circ -\$2)$$

This led to the agent considering \$4 with certainty as indifferent to the gamble $(.6 \circ \$10, .4 \circ -\$2)$ which has an expected value of \$5.2. Thus, the agent is willing to sacrifice some expected payment in favor of receiving a lesser payment with certainty. This is essentially the definition of a risk averse agent.

A point of caution is in order here. You need to remember to distinguish between the *expected value* of a gamble and the *expected utility* of a gamble. The expected value of a gamble is simply the weighted average of the outcomes of the gamble, where the weights are given by the probabilities with which the outcomes occur. For $(.6 \circ \$10, .4 \circ -\$2)$, the expected value is $.6 * 10 + .4 * -\$2 = \$5.2$. The expected utility is the weighted average of the *utility* of the outcomes, so for $(.6 \circ \$10, .4 \circ -\$2)$ we would have $.6 * u(10) + .4 * u(-\$2) = .6$ (according to our example). These are two distinct concepts that sound alike, much like a convex function and a convex set are two distinct concepts even though they both contain the term "convex". If we let $u(E(g))$ be the utility of the expected value of the gamble and $u(g)$ be the utility of the gamble, we can define risk aversion as follows:

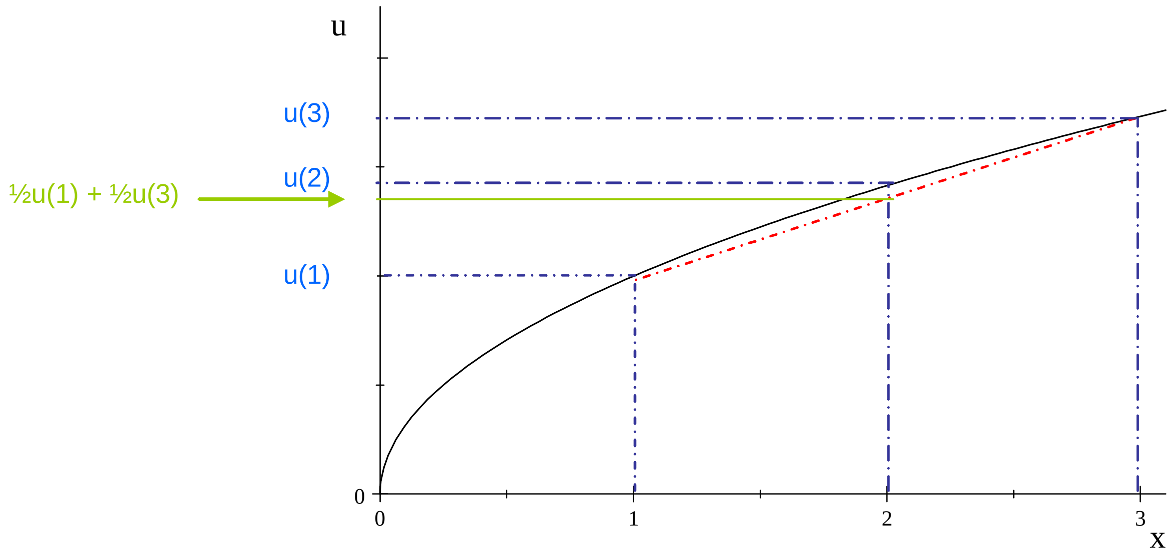


Figure 1: An individual with a Bernoulli utility function $u(x) = \sqrt{x}$.

Definition 11 *Let $u(\cdot)$ be an individual's vN-M utility function for gambles over nonnegative levels of wealth. Then for the simple gamble $g = (p_1 \circ w_1, ..., p_n \circ w_n)$, the individual is said to be:*

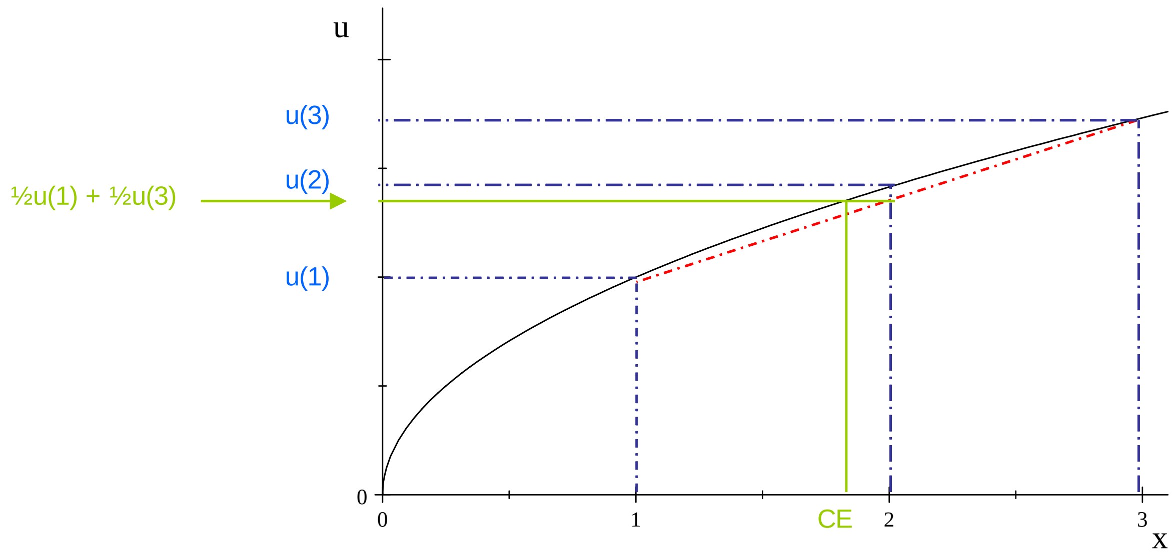
1. *risk averse at g if $u(E(g)) > u(g)$*
2. *risk neutral at g if $u(E(g)) = u(g)$*
3. *risk loving at g if $u(E(g)) < u(g)$*

If for every nondegenerate simple gamble, g, the individual is, for example, risk averse at g, then the individual is said to be risk averse.

Each definition of risk attitude implies a particular restriction on our vN-M utility function. If the individual is risk averse we have $u(E(g)) > u(g)$. What this means is that the utility of the expected value of the gamble g (or $u(E(g))$) is greater than the utility of the gamble itself (or $u(g)$). In this case, the individual prefers the certain amount $E(g)$ to the gamble g. Thus, if the individual is risk averse, that individual's vN-M utility function will be concave. If the decision-maker is strictly risk averse, then we have a strictly concave vN-M utility function. If the individual is risk neutral then the individual has a linear vN-M utility function and if risk loving the individual has a convex vN-M utility function.

Figure 1 shows the vN-M utility function $u(x) = \sqrt{x}$. The figure shows that the utility of a certain amount is greater than the utility of a gamble that gives that amount on average. The lottery in the picture is a lottery over the outcomes \$1 and \$3 with probability $\frac{1}{2}$ on each. Given that $u(1) = 1$, $u(2) = \sqrt{2}$, and $u(3) = \sqrt{3}$, we can see that the gamble $g = \left(\frac{1}{2} \circ 1, \frac{1}{2} \circ 3\right)$ over the outcomes \$1 and \$3 has an expected value of \$2, yet the individual's expected utility is only $\frac{1+\sqrt{3}}{2} < \sqrt{2}$. Strict concavity implies that the marginal utility of money is decreasing, so that if an individual has \$2, the utility gain from an additional dollar (to \$3) is less than the utility loss of an additional dollar (to \$1).

Figure **??** shows the vN-M utility function $u(x) = \sqrt{x}$ for the gamble $g = \left(\frac{1}{2} \circ 1, \frac{1}{2} \circ 3\right)$ over the outcomes \$1 and \$3, although the certainty equivalent, CE, has been added to this figure. The certainty equivalent is the sure amount of money that yields the same utility as the expected value of the gamble. To find this, we need to set $u(x) = u(gamble)$. In the example, the expected utility of the gamble is given by



$\frac{1}{2}u(1) + \frac{1}{2}u(3)$, or $\frac{1+\sqrt{3}}{2}$. So $\sqrt{x} = \frac{1+\sqrt{3}}{2}$. We then have:

$$\begin{aligned} \sqrt{x} &= \tfrac{1+\sqrt{3}}{2} \\ x &= \left(\tfrac{1+\sqrt{3}}{2}\right)^2 \\ x &= \tfrac{1+2\sqrt{3}+3}{4} \\ x &= \tfrac{4+2\sqrt{3}}{4} \\ x &= 1+\tfrac{\sqrt{3}}{2} \end{aligned} \tag{1}$$

So we have that the certainty equivalent is $CE = 1+\frac{\sqrt{3}}{2} \approx 1.866$, and $u\left(1+\frac{\sqrt{3}}{2}\right) = \frac{1+\sqrt{3}}{2}$. (Note that those are TWO DIFFERENT NUMBERS). So the individual is indifferent between 1.866 with certainty and the gamble $g = \left(\frac{1}{2} \circ 1, \frac{1}{2} \circ 3\right)$. Be careful to distinguish between the terms *expected value* and *expected utility* in this context, as it is easy to gloss over the particular terms. The *expected value* is simply the weighted average (with the weights given by the specific lottery) of the actual outcomes, while the *expected utility* is the weighted average (with the weights given by the specific lottery) of the UTILITY of those outcomes.

Definition 12 *The certainty equivalent of any simple gamble g over wealth levels is an amount of wealth, CE, offered with certainty, such that $u(g) \equiv u(CE)$. The risk premium is an amount of wealth, P, such that $u(g) \equiv u(E(g) - P)$.*

In the example above, the risk premium is equal to $E(g) - CE$, or $\$2 - \$\left(1+\frac{\sqrt{3}}{2}\right) = \$\left(1-\frac{\sqrt{3}}{2}\right) \approx \$0.134$.

1.2.1 Measuring risk aversion

We oftentimes would like to know not only what a particular individual's risk attitude is but also the degree of that risk attitude. Given two risk averse individuals, how might we determine which is more risk averse than the other? Since we know that a risk averse individual has a concave vN-M utility function, looking at the second derivative of that utility function is a natural starting point. We know that $u''(w) < 0$ if the individual is risk averse (because the function is concave), but can the actual value obtained from looking only at the second derivative allow us to compare two individual's risk aversion levels? The answer is no, because we saw earlier that vN-M utility functions are NOT unique. There is no difference (for us) between $u(x) = \ln x$ and $v(x) = 7\ln x$. However, $u''(x) = -\frac{1}{x^2}$ while $v''(x) = -\frac{7}{x^2}$. However, we can take the ratio of the second derivative to the first derivative to find a measure of risk aversion.

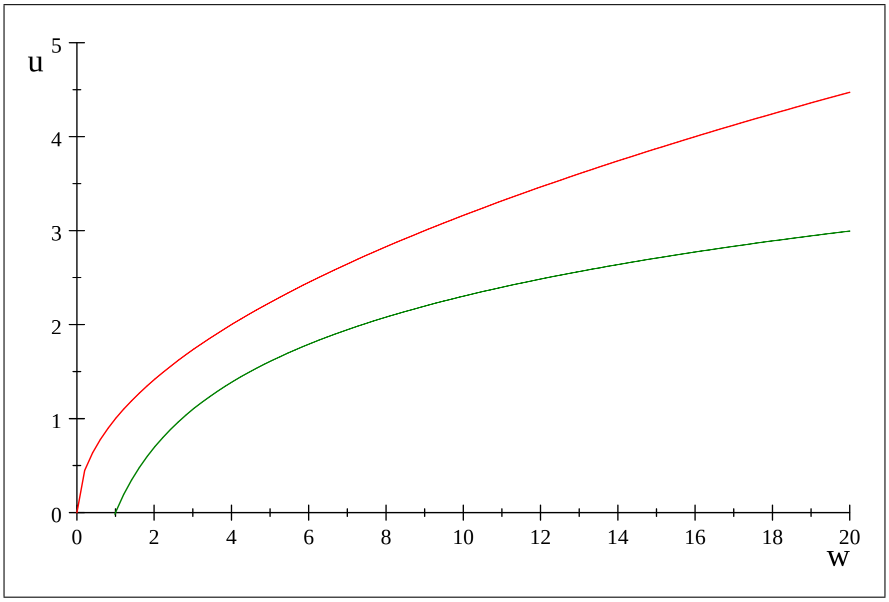
Definition 13 *The Arrow-Pratt measure of absolute risk aversion is:*

$$R_a(w) \equiv -\frac{u''(w)}{u'(w)} \tag{2}$$

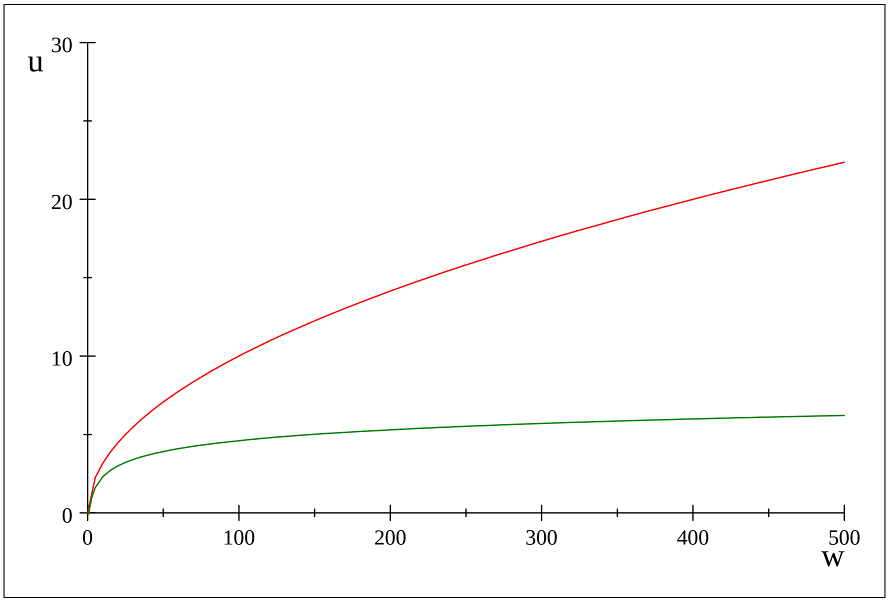
We can then see that $R_a(w) > 0$ if the individual is risk averse and $R_a(w) < 0$ if the individual is risk loving. We can show that individuals with larger Arrow-Pratt measures are more risk averse than those with smaller Arrow-Pratt measures of risk aversion. For instance, consider $u(x) = \ln x$ and $v(x) = \sqrt{x}$. We have:

$$\begin{array}{c} R_a(w) = \frac{1}{x} \text{ if } u(x) \\ R_a(w) = \frac{1}{2x} \text{ if } v(x) \end{array} \tag{3}$$

Plotting $u(x)$ and $v(x)$ we have:



where $u(x)$ is in green and $v(x)$ is in red (or where $v(x)$ is always above $u(x)$). Now, for any amount of $w > 0$ we have that $R_a^u(w) > R_a^v(w)$. Thus, the individual with $u(x) = \ln x$ is more risk averse than the individual with $v(x) = \sqrt{x}$ because the CE for the individual with $u(x)$ is higher than that for the individual with $v(x)$ (holding the gambles constant, obviously). The picture becomes more pronounced as we look at larger w:



As a last bit of discussion about risk aversion consider how an individual's risk attitude would vary with wealth. One might expect that the gamble $g = \left(\frac{1}{2} \circ \$0, \frac{1}{2} \circ \$500\right)$ is viewed differently by a graduate student than it is by a billionaire, simply because \$500 is not all that much money to a billionaire. We can look at how risk varies with wealth and classify risk attitudes accordingly. If risk aversion is decreasing as wealth increases we have decreasing absolute risk aversion (DARA), while if it is constant we have constant absolute risk aversion (CARA), and if it is increasing we have increasing absolute risk aversion (IARA). Under CARA additional wealth does not alter an individual's risk attitude while under IARA the wealthier the individual is the less likely the individual will accept a small gamble.

Consider the following problem where we assume the individual has decreasing absolute risk aversion (DARA). The individual has an amount w to put into a risky asset. The risky asset has N potential rates of return r_i (the outcomes) with probability p_i for $i = 1, ..., N$. Let β be the amount of wealth to be put into the risky asset, so that final wealth is:

$$(w - \beta) + (1 + r_i)\,\beta = w + \beta r_i \tag{4}$$

What we would like to show is that the optimal amount invested into the risky asset is increasing in w, or $\frac{d\beta^*}{dw} > 0$. To do so we need to find β^*. The individual's problem is to maximize the expected utility of wealth. This problem is:

$$\max_{\beta} \sum_{i=1}^{N} p_i u\left(w + \beta r_i\right) \text{ s.t. } 0 \leq \beta \leq w \tag{5}$$

First, we want to show the conditions under which the amount invested is zero, or $\beta^* = 0$. Differentiating with respect to β we have:

$$\sum_{i=1}^{N} p_i u'\left(w + \beta^* r_i\right) r_i \leq 0 \tag{6}$$

Now this first derivative is less than or equal to zero because the constraint is binding at $\beta^* = 0$ and so the derivative is nonincreasing. We know that $u'(w) > 0$, so that means that:

$$\sum_{i=1}^{N} p_i r_i \leq 0 \tag{7}$$

Thus, the individual will choose not to invest in the risky asset if the expected rate of return is nonpositive.

Now, assume that $\beta^* > 0$ so that the first order condition holds with equality:

$$\sum_{i=1}^{N} p_i u'\left(w + \beta^* r_i\right) r_i = 0 \tag{8}$$

The second order condition is:

$$\sum_{i=1}^{N} p_i u'\left(w + \beta^* r_i\right) r_i^2 < 0 \tag{9}$$

since we are assuming risk aversion. The question is how to proceed from here. If we had a functional form for $u(\cdot)$ we would be able to calculate β^* and find $\frac{d\beta^*}{dw}$. Without that, we need to find how β^* changes when w increases for the general functional form. To do so we need some way to calculate $d\beta^*$ and dw. We can find the total differential of the first-order condition. The total differential of a general function $f(x, y)$ is:

$$df = \frac{\partial f(x, y)}{\partial x} dx + \frac{\partial f(x, y)}{\partial y} dy \tag{10}$$

We have to find the total differential for both sides of:

$$\sum_{i=1}^{N} p_i u'\left(w + \beta^* r_i\right) r_i = 0 \tag{11}$$

For the right hand side we have $df = 0$. For the left hand side we have:

$$df = \sum_{i=1}^{N} p_i u'' \left(w + \beta^* r_i\right) r_i dw + \sum_{i=1}^{N} p_i u'' \left(w + \beta^* r_i\right) r_i^2 d\beta^* \tag{12}$$

Setting our df's equal to each other:

$$\begin{aligned} 0 &= \sum_{i=1}^{N} p_i u'' \left(w + \beta^* r_i\right) r_i dw + \sum_{i=1}^{N} p_i u'' \left(w + \beta^* r_i\right) r_i^2 d\beta^* &(13)\\ \sum_{i=1}^{N} p_i u'' \left(w + \beta^* r_i\right) r_i^2 d\beta^* &= -\sum_{i=1}^{N} p_i u'' \left(w + \beta^* r_i\right) r_i dw &(14)\\ \frac{d\beta^*}{dw} &= \frac{-\sum_{i=1}^{N} p_i u'' \left(w + \beta^* r_i\right) r_i}{\sum_{i=1}^{N} p_i u'' \left(w + \beta^* r_i\right) r_i^2} &(15) \end{aligned}$$

We know that:

$$\sum_{i=1}^{N} p_i u'' \left(w + \beta^* r_i\right) r_i^2 < 0 \tag{16}$$

because this is just the second order condition. Now we need to show that:

$$-\sum_{i=1}^{N} p_i u'' \left(w + \beta^* r_i\right) r_i < 0 \tag{17}$$

How to proceed from here? We know the $p_i > 0$, so focus on $-u'' \left(w + \beta^* r_i\right) r_i$. We also want to be able to use $R_a(w)$ since we know that the individual has DARA. If we take the following identity:

$$-u'' \left(w + \beta^* r_i\right) r_i = -u'' \left(w + \beta^* r_i\right) r_i \tag{18}$$

and multiply the right hand side by a particular form of 1, $\frac{u'(w+\beta^* r_i)}{u'(w+\beta^* r_i)}$, we get:

$$\begin{aligned} -u'' \left(w + \beta^* r_i\right) r_i &= -u'' \left(w + \beta^* r_i\right) r_i * \frac{u' \left(w + \beta^* r_i\right)}{u' \left(w + \beta^* r_i\right)} &(19)\\ -u'' \left(w + \beta^* r_i\right) r_i &= R_a \left(w + \beta^* r_i\right) r_i u' \left(w + \beta^* r_i\right) &(20) \end{aligned}$$

because $R_a \left(w + \beta^* r_i\right) = \frac{-u''(w+\beta^* r_i)}{u'(w+\beta^* r_i)}$. Now, we have:

$$\begin{aligned} R_a(w) &> R_a \left(w + \beta^* r_i\right) \text{ if } r_i > 0 &(21)\\ R_a(w) &< R_a \left(w + \beta^* r_i\right) \text{ if } r_i < 0 &(22) \end{aligned}$$

These are true because when $r_i > 0$ we have $w + \beta^* r_i > w$ and when $r_i < 0$ we have $w > w + \beta^* r_i$ and because we are assuming DARA (so that $R_a(x) < R_a(y)$ when $x > y$). Either way:

$$R_a(w) r_i > R_a \left(w + \beta^* r_i\right) r_i \tag{23}$$

Now, since we know this we have:

$$-u'' \left(w + \beta^* r_i\right) r_i < R_a(w) r_i u' \left(w + \beta^* r_i\right) \tag{24}$$

Taking expectations we have:

$$\begin{aligned} -\sum_{i=1}^{N} p_i u'' \left(w + \beta^* r_i\right) r_i &< \sum_{i=1}^{N} p_i R_a(w) r_i u' \left(w + \beta^* r_i\right) &(25)\\ -\sum_{i=1}^{N} p_i u'' \left(w + \beta^* r_i\right) r_i &< R_a(w) \sum_{i=1}^{N} p_i r_i u' \left(w + \beta^* r_i\right) &(26) \end{aligned}$$

Now, notice that on the right hand side we have the coefficient of absolute risk aversion multiplied by the first order condition. But we know that:

$$\sum_{i=1}^{N} p_i r_i u' (w + \beta^* r_i) = 0 \tag{27}$$

$$-\sum_{i=1}^{N} p_i u'' (w + \beta^* r_i) r_i < 0 \tag{28}$$

Since that term is negative, we have:

$$\frac{d\beta^*}{dw} > 0 \tag{29}$$