These notes correspond to chapter 3 of Jehle and Reny.

1 Introduction to producer theory

We now turn from consumer behavior to producer behavior. We will examine producer behavior in isolation for now, leaving the study of partial and general equilibrium for later in the course. We begin with a discussion of the producer's objectives and develop some assumptions underlying producer behavior. We will then discuss the producer's problem and study the specific case of cost and supply for a production technology that produces a single output.

Generally speaking we will consider our producers to be firms, although it should be noted that the theory developed applies equally to all types of producers, whether they are called firms, production units, families, etc. There may be additional assumptions/restrictions that one desires to make when discussing firms that are not controlled by agents that no not have identical preferences. The theory developed here applies directly to a firm composed of a single individual or agents with identical preferences. Whether these assumptions are applicable for other models of firms, such as partnerships, corporations with owners and managers, etc., is a decision each researcher needs to make on his or her own.

The firm is an interesting economic "agent". There are many potential questions one can ask about the firm. Such questions could be:

1. Who owns the firm? Does the identity of the owner change the firm's objectives?
2. Who manages the firm? If the owner and the manager are different agents, how does this affect the firm's behavior?
3. How is the firm organized? Do different organizational forms of the firm promote or inhibit efficiency?
4. What can the firm do?

These are interesting and important questions that an individual could turn into a long and fruitful academic career, and by no means is this an exhaustive list. Our primary focus will be on the 4^{th} item, What can the firm do? In particular, we will begin by assuming that the firm can transform inputs into outputs. The firm's goal is to make profits – specifically, the firm's goal is to *maximize* profits. It is possible that the firm has other goals – many firms set profit targets or sales targets (in $) or sales targets (in quantity of items sold) or wish to maximize sales. We will focus on profit maximization as (1) it provides a close approximation to the goals of many firms; (2) it is consistent with utility maximization if the firm is controlled by a single individual or agents with identical preferences; and (3) it allows us to use the tools developed in the study of consumer theory to solve the firm's problem.

Now, if you were to start a business one thing you would want to know is exactly *how* your firm would transform inputs into outputs (and ultimately profits). In the theory constructed, the *how* is assumed to be simply a black-box of production. We do not know how, just that there is some technology that exists that allows the firm to transform inputs into outputs. Thus, our view of the theory is fairly accurately represented by the Underpants Gnomes in South Park.[1] The Underpants Gnomes have a business plan:

- Phase 1: Collect underpants
- Phase 2: (Gnome shrugs shoulders, suggesting he does not know)
- Phase 3: Profit

Well, they skip the stage where they turn the inputs into outputs, but for the most part this is dead on. Our firms transform inputs (underpants, phase 1) into outputs to make profits (phase 3), and the exact process is unknown (phase 2). We will be slightly (but not much) more formal than shrugging our shoulders when discussing the production process, but will simply specify it as a "production function", and will list certain properties of that function. Again, should you attempt to enter the business world, most lending

[1] Episode 217, titled Gnomes, originally aired on 12/16/1998.

agents will want a little more than a shrug of the shoulders or that you have some unspecified production function. However, for general theoretic purposes the generic "production function" will suffice.

While the Underpants Gnomes portrayal of the theory is a little simplistic, it is important to note that the theory does not discuss the *how* of production. It is also important to note that given some minimum assumptions about firms, which in essence is what we will discuss, the economy will be able to obtain a competitive equilibrium outcome (we will discuss this later in the course). Thus, there is beauty in the simplicity of the theory in that it allows for equilibrium to be achieved with minimal assumptions.

2 Production

When we talk of production we generally mean the production process. Every firm has some production process that it uses, and this production process limits how much it can produce given a set of inputs. In general, firms have a production possibility set, $Y \subset \mathbb{R}^m$, and each vector $y \in \mathbb{R}^m$ represents a particular production plan. When we are discussing the case of multiple outputs it will be convenient to denote any $y_i < 0$ as an input and $y_i > 0$ as an output. In the case of a single output (which is the bulk of our discussion), we will denote inputs as x_i for $i = 1, .., n$ and the output as y, with $x_i \geq 0$ and $y \geq 0$. In the single output case we will use a production function to represent the production process, such that $y = f(x)$ where $f(x) : \mathbb{R}^n_+ \to \mathbb{R}_+$.

Throughout our discussion of production we will assume that $f(x)$ is continuous, strictly increasing, and strictly quasiconcave on $\mathbb{R}^n_+$ and $f(0) = 0$. Continuity simply means that small changes in input result in small changes in output. By assuming $f(x)$ is strictly increasing we are assuming that adding more of ALL inputs will increase output. The quasiconcavity assumption literally means that any convex combination of two input vectors will produce at least as much as either of the original vectors.

As with consumer theory we will (at times) assume that the production function is differentiable. When it is differentiable we define $\frac{\partial f(x)}{\partial x_i}$ as the marginal product of input i. Given our assumptions it will generally be the case that $\frac{\partial f(x)}{\partial x_i} > 0$. For any output level y there will be certain input combinations that exactly produce y. The collection of these input vectors is called the isoquant, which the book denotes $Q(y)$.

In consumer theory we had the marginal rate of substitution which was the rate at which the consumer traded off one good for the other to remain on the same indifference curve. In producer theory we have the marginal rate of technical substitution (MRTS), which is the rate at which the firm can trade off inputs to remain at the same output level. By definition we have:

$$MRTS_{ij}(x) = \frac{\partial f(x)/\partial x_i}{\partial f(x)/\partial x_j} \tag{1}$$

Example: Consider the production function:

$$f(x_1, x_2) = Ax_1^{\alpha} x_2^{\beta} \tag{2}$$

We then have that the marginal product of x_1 and x_2 are:

$$\frac{\partial f(x_1, x_2)}{\partial x_1} = A\alpha x_1^{\alpha-1} x_2^{\beta} \tag{3}$$

$$\frac{\partial f(x_1, x_2)}{\partial x_2} = A\beta x_1^{\alpha} x_2^{\beta-1} \tag{4}$$

The MRTS is then:

$$MRTS_{12} = \frac{\frac{\partial f(x_1,x_2)}{\partial x_1}}{\frac{\partial f(x_1,x_2)}{\partial x_2}} \tag{5}$$

$$MRTS_{12} = \frac{A\alpha x_1^{\alpha-1} x_2^{\beta}}{A\beta x_1^{\alpha} x_2^{\beta-1}} \tag{6}$$

$$MRTS_{12} = \frac{\alpha x_2}{\beta x_1} \tag{7}$$

2.1 Elasticity of substitution

The elasticity of substitution between inputs i and j is denoted σ_{ij}. Holding all other inputs and the level of output constant, we define the elasticity of substitution as the percentage change in input proportions, $\frac{x_j}{x_i}$, associated with a 1 percent change in the MRTS between them. Mathematically, the elasticity of substitution is defined as:

$$\sigma_{ij} = \frac{d\ln(x_j/x_i)}{d\ln(f_i(x)/f_j(x))} = \frac{d(x_j/x_i)}{x_j/x_i}\frac{f_i(x)/f_j(x)}{d(f_i(x)/f_j(x))} \tag{8}$$

where f_i and f_j are the marginal products of inputs i and j. As long as the production function is quasiconcave, $\sigma_{ij} \geq 0$. As $\sigma_{ij} \to 0$ it becomes more difficult to substitute inputs; as $\sigma_{ij} \to \infty$ it becomes easier to substitute inputs. We can look at the CES production function (which we also saw in consumer theory) and examine how it captures many of our production functions as the parameter ρ changes. Recall that the CES function is:

$$y = (x_1^\rho + x_2^\rho)^{1/\rho} \tag{9}$$

for $\rho < 1$ but not $\rho = 0$. We have already shown (in a homework) that the elasticity of substitution for this function is:

$$\sigma = \frac{1}{1-\rho} \tag{10}$$

However, let us take a look at the following results. We want to see what the CES production function becomes as (1) $\rho = 1$, (2) $\rho \to 0$, and (3) $\rho \to -\infty$. The quick answers are (1) linear production function (perfect substitutes), (2) Cobb-Douglas production function, and (3) Leontief production function (perfect complements, or $\min(x)$). Now think about what σ is in each of these cases. In (1) $\sigma = \infty$, (2) $\sigma = 1$, and (3) $\sigma = 0$. Again, it is important to note what you are assuming when you impose certain restrictions on your model.

For the case when $\rho = 1$, we have:

$$\begin{aligned} y &= (x_1^\rho + x_2^\rho)^{1/\rho} && (11)\\ y &= (x_1 + x_2)^1 && (12)\\ y &= x_1 + x_2 && (13) \end{aligned}$$

For the case where $\rho \to 0$ we have $\frac{1}{\rho} \to \infty$, so this is not as simple as (1). We can however take $\ln(y)$ to get $\widetilde{y} = \ln(y) = \frac{\ln[x_1^\rho + x_2^\rho]}{\rho}$. Now, differentiating the numerator and denominator separately with respect to ρ (using L'Hopital's rule) we get:

$$\begin{aligned} \frac{d\ln[x_1^\rho + x_2^\rho]}{d\rho} &= \frac{(x_1^\rho \ln x_1 + x_2^\rho \ln x_2)}{x_1^\rho + x_2^\rho} && (14)\\ \frac{d\rho}{d\rho} &= 1 && (15) \end{aligned}$$

Now we want to find the limit as $\rho \to 0$:

$$\begin{aligned} \lim_{\rho\to 0} \frac{\frac{(x_1^\rho \ln x_1 + x_2^\rho \ln x_2)}{x_1^\rho + x_2^\rho}}{1} &= \ln x_1 + \ln x_2 && (16)\\ \ln x_1 + \ln x_2 &= \ln(x_1 x_2) && (17)\\ e^{\ln(x_1 x_2)} &= x_1 x_2 && (18) \end{aligned}$$

That recovers our original specification and shows that as $\rho \to 0$, the CES production function becomes a Cobb-Douglas production function.

For the case where $\rho \to -\infty$ it becomes more complicated. The key is determining what it means for $u(x_1, x_2) = \min\{x_1, x_2\}$. It means that $u(x_1, x_2) = x_1$ if $x_1 \leq x_2$. So does $\lim_{\rho\to-\infty}[\alpha_1 x_1^\rho + \alpha_2 x_2^\rho]^{1/\rho} = x_1$

if $x_1 \leq x_2$? Letting $\rho < 0$ we have:

$$\begin{aligned} x_1^\rho &\leq x_1^\rho + x_2^\rho && (19)\\ x_1 &\leq \left(x_1^\rho + x_2^\rho\right)^{1/\rho} && (20)\\ x_1 &\geq u(x_1, x_2) && (21) \end{aligned}$$

Note that the inequality flips because $\rho < 0$. Then we also have (because $x_1 \leq x_2$ and $\rho < 0$):

$$\begin{aligned} x_1^\rho + x_2^\rho &\leq x_1^\rho + x_1^\rho && (22)\\ x_1^\rho + x_2^\rho &\leq 2x_1^\rho && (23)\\ \left(x_1^\rho + x_2^\rho\right)^{1/\rho} &\geq 2^{1/\rho} x_1 && (24)\\ u(x) &\geq 2^{1/\rho} x_1 && (25) \end{aligned}$$

Now, we have:

$$\begin{aligned} \lim_{\rho \to -\infty} x_1 &= x_1 && (26)\\ \lim_{\rho \to -\infty} 2^{1/\rho} x_1 &= x_1 && (27) \end{aligned}$$

But we have $x_1 \geq u(x) \geq 2^{1/\rho} x_1$, so by the squeezing theorem we have:

$$\lim_{\rho \to -\infty} u(x) = x_1 \quad (28)$$

when $x_1 \leq x_2$.

2.2 Returns to scale

Oftentimes managers would like to know how output responds to the different levels of inputs used. In the **short run** we assume that at least one input is fixed and so we cannot vary that input. We refer to how output varies in the short run as "returns to variable proportions". In the **long run** the firm can vary all inputs and we can classify production functions by their returns to scale. When discussing returns to scale we want to vary all inputs by the same proportion and then see how output responds. We classify production functions according to their global returns to scale by using the following definition:

Definition 1 *A production function $f(x)$ has the property of (globally):*

1. *Constant returns to scale if $f(tx) = tf(x)$ for all $t > 0$ and all x*
2. *Increasing returns to scale if $f(tx) > tf(x)$ for all $t > 1$ and all x*
3. *Decreasing returns to scale if $f(tx) < tf(x)$ for all $t > 1$ and all x.*

Consider what the above definition means. If a production function has constant returns to scale then doubling all inputs will lead to exactly doubling output. If a production function has increasing returns to scale then doubling all inputs will lead to more than doubling of the output. If a production function has decreasing returns to scale then doubling all inputs will lead to less than a doubling of output.

Consider the CES production function. Suppose that we want to determine whether it has increasing, decreasing, or constant returns to scale. We have:

$$\begin{aligned} f(x) &= \left(x_1^\rho + x_2^\rho\right)^{1/\rho} && (29)\\ f(tx) &= \left((tx_1)^\rho + (tx_2)^\rho\right)^{1/\rho} && (30)\\ f(tx) &= \left(t^\rho x_1^\rho + t^\rho x_2^\rho\right)^{1/\rho} && (31)\\ f(tx) &= \left(t^\rho \left(x_1^\rho + x_2^\rho\right)\right)^{1/\rho} && (32)\\ f(tx) &= t\left(x_1^\rho + x_2^\rho\right)^{1/\rho} && (33)\\ f(tx) &= tf(x) && (34) \end{aligned}$$

Regardless of the value of ρ we have that the CES production function has constant returns to scale. However, recall that there are only certain values of ρ that are permissible, so this only applies for those parameter values of ρ.

While knowing the global returns to scale is important, we might also want to be able to determine a local measure of returns to scale (particularly if the production function cannot be classified according to the global returns to scale). To determine local returns to scale we use the *elasticity of scale*, which is defined as:

Definition 2 *The elasticity of scale at the point x is defined as:*

$$\mu(x) \equiv \lim_{t \to 1} \frac{d \ln[f(tx)]}{d \ln(t)} = \frac{\sum_{i=1}^{n} f_i(x) x_i}{f(x)} \tag{35}$$

Returns to scale are classified as locally constant, increasing, or decreasing as $\mu(x)$ is equal to, greater than, or less than one.

We can use the example in the book so that:

$$y = k\left(1 + x_1^{-\alpha} x_2^{-\beta}\right)^{-1} \tag{36}$$

where $\alpha > 0$, $\beta > 0$, and $0 \leq y \leq k$. If we calculate:

$$\mu(x) = \frac{\sum_{i=1}^{n} f_i(x) x_i}{f(x)} \tag{37}$$

we have:

$$f_1(x) = (-1) k \left(1 + x_1^{-\alpha} x_2^{-\beta}\right)^{-2} (-\alpha) \left(x_1^{-\alpha-1} x_2^{-\beta}\right) \tag{38}$$

$$f_1(x) = \alpha k \left(1 + x_1^{-\alpha} x_2^{-\beta}\right)^{-2} \left(x_1^{-\alpha-1} x_2^{-\beta}\right) \tag{39}$$

Now, multiplying by x_1 and dividing by $f(x)$ we have:

$$\frac{f_1(x) x_1}{f(x)} = \frac{\alpha k \left(1 + x_1^{-\alpha} x_2^{-\beta}\right)^{-2} \left(x_1^{-\alpha-1} x_2^{-\beta}\right) x_1}{k\left(1 + x_1^{-\alpha} x_2^{-\beta}\right)^{-1}} \tag{40}$$

$$\frac{f_1(x) x_1}{f(x)} = \alpha \left(1 + x_1^{-\alpha} x_2^{-\beta}\right)^{-1} \left(x_1^{-\alpha} x_2^{-\beta}\right) \tag{41}$$

For x_2 we have a similar result where:

$$\frac{f_2(x) x_2}{f(x)} = \beta \left(1 + x_1^{-\alpha} x_2^{-\beta}\right)^{-1} \left(x_1^{-\alpha} x_2^{-\beta}\right) \tag{42}$$

So that the sum of those terms is:

$$\frac{\sum_{i=1}^{n} f_i(x) x_i}{f(x)} = \alpha \left(1 + x_1^{-\alpha} x_2^{-\beta}\right)^{-1} \left(x_1^{-\alpha} x_2^{-\beta}\right) + \beta \left(1 + x_1^{-\alpha} x_2^{-\beta}\right)^{-1} \left(x_1^{-\alpha} x_2^{-\beta}\right) \tag{43}$$

$$\frac{\sum_{i=1}^{n} f_i(x) x_i}{f(x)} = (\alpha + \beta) \left(1 + x_1^{-\alpha} x_2^{-\beta}\right)^{-1} \left(x_1^{-\alpha} x_2^{-\beta}\right) \tag{44}$$

That is somewhat messy and unintuitive, so we can use $y = k\left(1 + x_1^{-\alpha} x_2^{-\beta}\right)^{-1}$ and $\frac{k}{y} - 1 = x_1^{-\alpha} x_2^{-\beta}$ to find:

$$\mu(y) = (\alpha + \beta) \left(1 + \frac{k}{y} - 1\right)^{-1} \left(\frac{k}{y} - 1\right) \tag{45}$$

$$\mu(y) = (\alpha + \beta) \left(\frac{y}{k}\right) \left(\frac{k}{y} - 1\right) \tag{46}$$

$$\mu(y) = (\alpha + \beta) \left(1 - \frac{y}{k}\right) \tag{47}$$

If we look at how returns vary as output varies, when $y = 0$ we have $\mu(y) = (\alpha + \beta) > 0$. As $y \to k$ we have $\mu(y) \to 0$. If $\alpha + \beta > 1$, then we have that the production function exhibits constant returns to scale for $y = k\left[1 - \frac{1}{\alpha+\beta}\right]$. We can find that by setting $\mu(y) = 1$ and solving for y. If $y < k\left[1 - \frac{1}{\alpha+\beta}\right]$ then there are increasing returns (so at low levels of output) and we can show this by showing that $\mu(y) > 1$ if $y < k\left[1 - \frac{1}{\alpha+\beta}\right]$. If $y > k\left[1 - \frac{1}{\alpha+\beta}\right]$ then there are decreasing returns to scale (at high levels of output). While it is easy to get lost in the math, it is important to remember what you are trying to use the math to show – in this case we are showing that this particular production has returns to scale which vary with output.

3 Cost

The firm's cost function is similar to the consumer's expenditure function. The cost function tells the firm exactly what the minimum level of expenditure is needed to achieve a target level of output, just like the consumer's expenditure function told the consumer exactly what minimum level of expenditure was needed to achieve a target level of utility. So, the firm's cost minimization problem is of the same type as the consumer's expenditure minimization problem, only we use different names.

The basic problem the firm has is to choose its input levels to minimize cost. We assume that the firm operates in a perfectly competitive market (or price-taker market) for inputs, that is the firm's choice of input level does not affect the price of the input. We let w_i be the price of input i, and assume $w_i > 0$ for all $i = 1, ..., n$. The firm's production possibilities are determined by the production function $f(x)$ and assume that the target level of output is denoted by $y > 0$.

Definition 3 *The cost function for the firm facing fixed input prices $w >> 0$ and required output $y \in f\left(\mathbb{R}^n_+\right)$ is defined as the minimum value function:*

$$c(w, y) \equiv \min_{x \in \mathbb{R}^n_+} \; wx \; s.t. \; f(x) \geq y \tag{48}$$

If $x(w, y)$ solves the cost minimization problem, then:

$$c(w, y) = w \cdot x(w, y) \tag{49}$$

We can structure this as a constrained maximization problem and use Lagrange's method. We then have:

$$\min_{x \in \mathbb{R}^n_+} \; \mathcal{L}(x, \lambda) = wx + \lambda(y - f(x)) \tag{50}$$

Assuming that the firm uses a positive amount of all inputs (so that $x_i > 0$ for all inputs $i = 1, ..., n$), we then have an interior solution. The first-order conditions to the problem hold with equality so that:

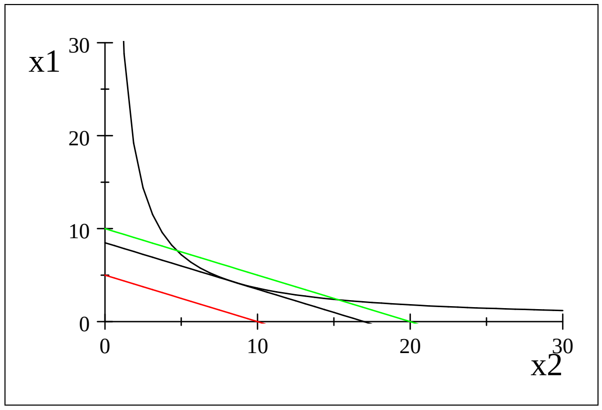
$$\frac{\partial \mathcal{L}}{\partial x_i} = w_i - \lambda^* \frac{\partial f(x^*)}{\partial x_i} = 0 \text{ for } i = 1, ..., n \tag{51}$$

$$w_i = \lambda^* \frac{\partial f(x^*)}{\partial x_i} \tag{52}$$

If we take the ratio of the first-order conditions for two inputs i and j we have:

$$\frac{w_i}{w_j} = \frac{\partial f(x^*)/\partial x_i}{\partial f(x^*)/\partial x_j} \tag{53}$$

Recall that the marginal rate of technical substitution is equal to the ratio of the marginal products for those two inputs, so that at the optimal (interior) solution to the cost minimization problem we have that the $MRTS_{ij} = \frac{w_i}{w_j}$. This is the same type of result we had in the consumer's problem, and simply means that the slope of the isoquant must equal the ratio of the input prices at the optimal interior solution.



We can see from the figure that, at an interior solution, the isoquant must be tangent to the slope of the price ratio (technically this is the isocost curve, meaning "same cost") when cost is minimized. In the picture the cost of the red line is too low, the cost of the green line is too high, and the cost of the black line is tangent to the isoquant.

The solution to the cost minimization problem is a vector of inputs, $x(w,y)$. If $w >> 0$ and $f(x)$ is strictly quasiconcave then $x(w,y)$ will be unique. This solution is called the conditional input demand because it is conditional on the level of output being chosen, and that level of output may or may not be profit-maximizing. Given $x(w,y)$ we can find the cost function as:

$$c(w,y) = w \cdot x(w,y) \tag{54}$$

Let's work an example with $f(x) = x_1^{1/3} x_2^{1/3}$. We have:

$$\min_{x_1 \geq 0, x_2 \geq 0} \mathcal{L}(x_1, x_2, \lambda) = w_1 x_1 + w_2 x_2 + \lambda \left(y - x_1^{1/3} x_2^{1/3} \right) \tag{55}$$

We know that both $x_1 > 0$ and $x_2 > 0$ as long as $y > 0$ because if either $x_1 = 0$ or $x_2 = 0$ then $f(x) = 0$. We can find the first-order conditions and set them equal to zero:

$$\frac{\partial \mathcal{L}}{\partial x_1} = w_1 - \lambda^* \frac{1}{3} (x_1^*)^{-2/3} (x_2^*)^{1/3} = 0 \tag{56}$$

$$\frac{\partial \mathcal{L}}{\partial x_2} = w_2 - \lambda^* \frac{1}{3} (x_1^*)^{1/3} (x_2^*)^{-2/3} = 0 \tag{57}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = y - x_1^{1/3} x_2^{1/3} = 0 \tag{58}$$

Now, we have:

$$w_1 = \lambda^* \frac{1}{3} (x_1^*)^{-2/3} (x_2^*)^{1/3} \tag{59}$$

$$w_2 = \lambda^* \frac{1}{3} (x_1^*)^{1/3} (x_2^*)^{-2/3} \tag{60}$$

and taking the ratio we have:

$$\frac{w_1}{w_2} = \frac{\lambda^* \frac{1}{3} (x_1^*)^{-2/3} (x_2^*)^{1/3}}{\lambda^* \frac{1}{3} (x_1^*)^{1/3} (x_2^*)^{-2/3}} \tag{61}$$

$$\frac{w_1}{w_2} = \frac{x_2^*}{x_1^*} \tag{62}$$

$$x_1^* \frac{w_1}{w_2} = x_2^* \tag{63}$$

Substituting into the constraint we have:

$$
\begin{aligned}
y &= x_1^{1/3} x_2^{1/3} && (64)\\
y &= \left(x_1^*\right)^{1/3}\left(x_1^* \frac{w_1}{w_2}\right)^{1/3} && (65)\\
y &= \left(x_1^*\right)^{2/3}\left(\frac{w_1}{w_2}\right)^{1/3} && (66)\\
y^3 &= \left(x_1^*\right)^{2} \frac{w_1}{w_2} && (67)\\
y^3 \frac{w_2}{w_1} &= \left(x_1^*\right)^{2} && (68)\\
\sqrt{y^3 \frac{w_2}{w_1}} &= x_1^* && (69)
\end{aligned}
$$

We can then find x_2^* as:

$$
\begin{aligned}
x_1^* \frac{w_1}{w_2} &= x_2^* && (70)\\
\sqrt{y^3 \frac{w_2}{w_1}} \frac{w_1}{w_2} &= x_2^* && (71)\\
\sqrt{y^3 \frac{w_1}{w_2}} &= x_2^* && (72)
\end{aligned}
$$

Note that these conditional input demand functions correspond to intuition – if the price of the input increases, then its own quantity used falls, while the quantity used of the other input increases. We can then find the cost function $c(w, y)$:

$$
\begin{aligned}
c(w, y) &= w_1 x_1 + w_2 x_2 && (73)\\
c(w, y) &= w_1 \sqrt{y^3 \frac{w_2}{w_1}} + w_2 \sqrt{y^3 \frac{w_1}{w_2}} && (74)\\
c(w, y) &= \sqrt{y^3 w_2 w_1} + \sqrt{y^3 w_2 w_1} && (75)\\
c(w, y) &= 2\sqrt{y^3 w_2 w_1} && (76)
\end{aligned}
$$

Now that we have gone through an example we can list some properties of a cost function and of the conditional input demand functions.

Theorem 4 *(Properties of the cost function) If f is continuous and strictly increasing, then $c(w, y)$ is:*

1. *Zero when $y = 0$*
2. *Continuous on its domain*
3. *For all $w >> 0$, strictly increasing and unbounded above in y*
4. *Increasing in w*
5. *Homogeneous of degree one in w*
6. *Concave in w*
7. *If f is strictly quasiconcave, then we have Shephard's lemma: $c(w, y)$ is differentiable in w at $\left(w^0, y^0\right)$ whenever $w^0 >> 0$, and*
$$\frac{\partial c\left(w^0, y^0\right)}{\partial w_i} = x_i\left(x^0, y^0\right), \text{ for } i = 1, ..., n \qquad (77)$$

Consider our cost function from the example:

$$c(w,y) = 2\sqrt{y^3 w_2 w_1} \tag{78}$$

We can see that if $y = 0$, $c(w,0) = 0$. This means that if output is equal to 0, then cost is equal to 0. We can also see that the function is continuous. For strictly increasing in y, we see that:

$$\frac{\partial c(w,y)}{\partial y} = 2\sqrt{w_2 w_1}\frac{3}{2}y^{1/2} \tag{79}$$

$$\frac{\partial c(w,y)}{\partial y} = 3\sqrt{w_2 w_1 y} \tag{80}$$

which is strictly greater than 0 as long as w_2, w_1, and y are strictly greater than 0. So that if output increases, cost increases. We can find the derivative with respect to each w_i:

$$\frac{\partial c(w,y)}{\partial w_i} = 2\sqrt{y^3 w_j}\frac{1}{2}w_i^{-1/2} \tag{81}$$

$$\frac{\partial c(w,y)}{\partial w_i} = \sqrt{\frac{y^3 w_j}{w_i}} > 0 \tag{82}$$

So that if any input price increases, then cost increases as well. For homogeneous of degree one in w we have:

$$c(w,y) = 2\sqrt{y^3 w_2 w_1} \tag{83}$$

$$c(tw,y) = 2\sqrt{y^3 t w_2 t w_1} \tag{84}$$

$$c(tw,y) = 2\sqrt{y^3 t^2 w_2 w_1} \tag{85}$$

$$c(tw,y) = 2t\sqrt{y^3 w_2 w_1} \tag{86}$$

$$c(tw,y) = tc(w,y) \tag{87}$$

So that if input prices increase by a certain percentage then cost increases by the same percentage. For concave in w, we find the second derivative with respect to w. We have:

$$\frac{\partial c(w,y)}{\partial w_i} = \sqrt{\frac{y^3 w_j}{w_i}} \tag{88}$$

$$\frac{\partial^2 c(w,y)}{\partial w_i^2} = -\frac{1}{2}\sqrt{y^3 w_j} w_i^{-3/2} \tag{89}$$

$$\frac{\partial^2 c(w,y)}{\partial w_i^2} = -\frac{1}{2}\sqrt{\frac{y^3 w_j}{w_i^3}} < 0 \text{ for all } i = 1, ..., n \tag{90}$$

For Shephard's lemma we simply take the first partial derivative with respect to each input price to find the conditional input demands:

$$\frac{\partial c(w,y)}{\partial w_i} = x_i(w,y) \tag{91}$$

$$x_i(w,y) = \sqrt{\frac{y^3 w_j}{w_i}} \tag{92}$$

Now we have some properties of conditional input demands.

Theorem 5 *Suppose that the production function is continuous, strictly increasing, and strictly quasiconcave, with $f(0) = 0$ and that the associated cost function is twice continuously differentiable. Then*

1. *$x(w,y)$ is homogeneous of degree zero in w*

2. *The substitution matrix, defined and denoted:*

$$\sigma^*(w,y) = \begin{pmatrix} \frac{\partial x_1(w,y)}{\partial w_1} & \cdots & \frac{\partial x_1(w,y)}{\partial w_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial x_n(w,y)}{\partial w_1} & & \frac{\partial x_n(w,y)}{\partial w_n} \end{pmatrix} \tag{93}$$

is symmetric and negative semidefinite. In particular the negative semidefinite property implies that $\frac{\partial x_i(w,y)}{\partial w_i} \leq 0$ *for all* i.

Using our cost function $c(w,y) = 2\sqrt{y^3 w_1 w_2}$ we know that the conditional input demand is:

$$x_i(w,y) = \sqrt{\frac{y^3 w_j}{w_i}} \tag{94}$$

For homogeneous of degree zero in w we have:

$$x_i(w,y) = \sqrt{\frac{y^3 w_j}{w_i}} \tag{95}$$

$$x_i(tw,y) = \sqrt{\frac{y^3 t w_j}{t w_i}} \tag{96}$$

$$x_i(tw,y) = \sqrt{\frac{y^3 w_j}{w_i}} \tag{97}$$

$$x_i(tw,y) = x_i(w,y) \tag{98}$$

So that if we increase all input prices by the same percentage we do not change the conditional factor demands. We can see that:

$$x_i(w,y) = \sqrt{\frac{y^3 w_j}{w_i}} \tag{99}$$

$$\frac{\partial x_i(w,y)}{\partial w_i} = -\frac{1}{2}\sqrt{\frac{y^3 w_j}{w_i^3}} < 0 \tag{100}$$

because this is just property 6 of the cost function and $\frac{\partial x_i(w,y)}{\partial w_i} = \frac{\partial^2 c(w,y)}{\partial w_i^2}$

3.1 Short run costs

Recall that the short run is a time period in which at least one input level is fixed. We can define the short run cost function as:

Definition 6 *Let the production function be* $f(z)$, *where* $z \equiv (x, \overline{x})$. *Suppose that* x *is a subvector of variable inputs and* $\overline{x}$ *is a subvector of fixed inputs. Let* w *and* $\overline{w}$ *be the associated input prices for the variable and fixed inputs respectively. The short run, or restricted, total cost function is defined as:*

$$sc(w, \overline{w}, y; x) \equiv \min_x \; w \cdot x + \overline{w} \cdot \overline{x} \; s.t. \; f(x, \overline{x}) \geq y \tag{101}$$

If $x(w, \overline{w}, y, \overline{x})$ *solves this minimization problem, then*

$$sc(w, \overline{w}, y; x) = w \cdot x(w, \overline{w}, y, \overline{x}) + \overline{w} \cdot \overline{x} \tag{102}$$

The optimized cost of the variable inputs, $w \cdot x(w, \overline{w}, y, \overline{x})$, *is called total variable cost. The cost of the fixed inputs,* $\overline{w} \cdot \overline{x}$, *is called total fixed cost.*

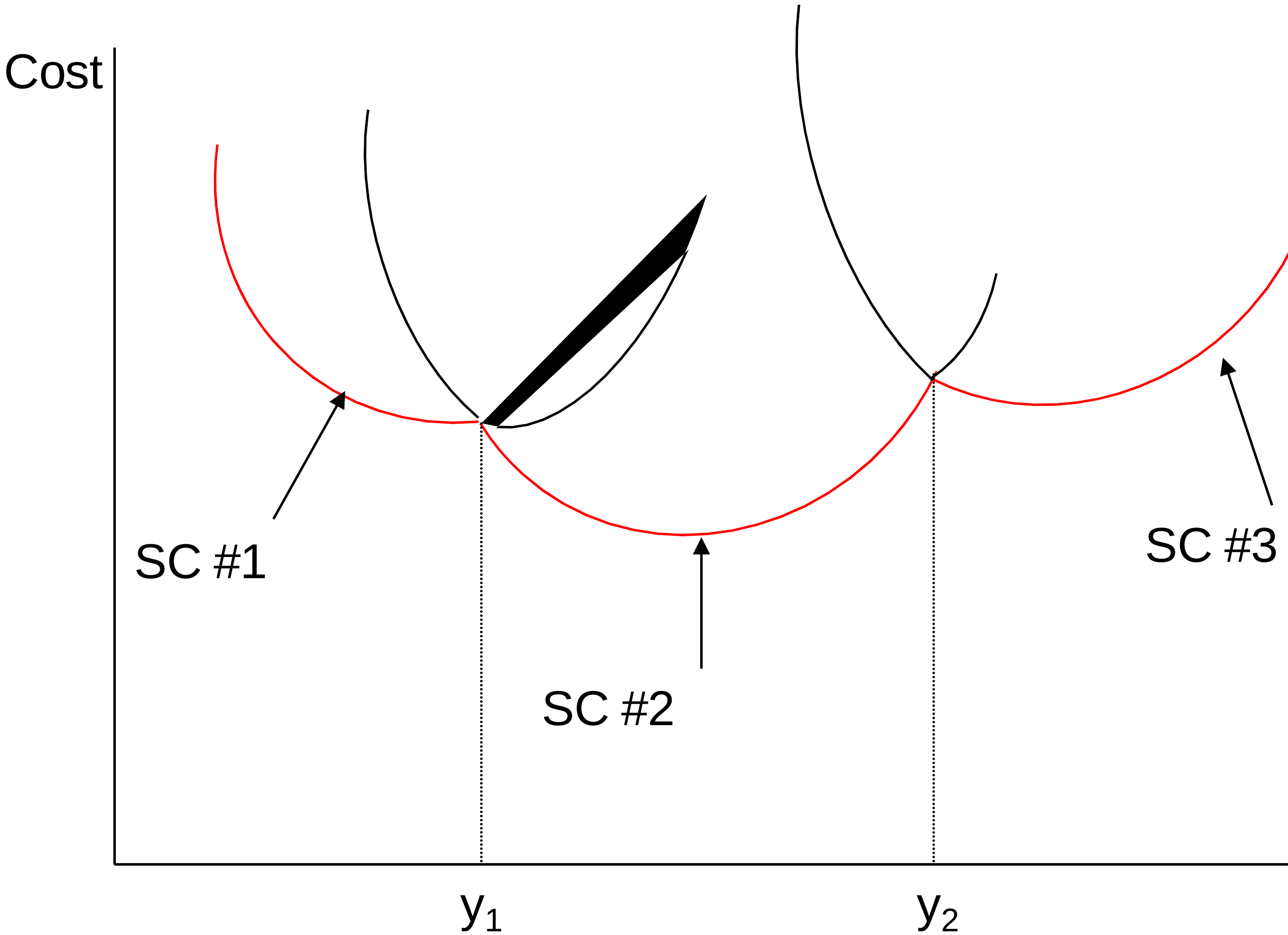


Figure 1: The long run cost curve is the lower envelope of the short run cost curves.

Note that the difference between long run cost and short run cost is that some inputs are fixed and so are NOT choice variables. What we can show is that the short run cost function is at least as high as the long run cost function, or:

$$sc\left(w, \overline{w}, y; x\right) = c\left(w, \overline{w}, y\right) \tag{103}$$

Also, for every level of output we know that the long run and short run cost functions will be equal at some level of fixed inputs. Using this information we can show that the long run cost curve is the lower envelope of the short run cost curves. In Figure 1 we have three short run cost curves. The red portion of each of those curves traces out the long run cost curve. Essentially what is being done is that before incurring any fixed cost a producer would like to determine which "plant size" to use for his particular scale of operations (output level). Up to y_1 the firm would use sc_1. Between y_1 and y_2 the firm would use sc_2. After output level y_2 the firm would use sc_3. If we had all the possible short run cost curves then we would have a smoother long run cost curve.

4 Competitive Firm

We now consider a firm that operates in perfectly competitive output and input markets. The perfectly competitive assumption means that the firm exerts no impact on the price of either output or inputs, thus taking the price as given (leading to the alternative designation of "price taker"). Typically we assume that the firms are small relative to the size of the market.

4.1 Profit maximization

Define profit as the difference between total revenue and total cost. The competitive price of the output good is p. Revenues are then $R(y) = py$. Suppose that the firm wishes to sell output level y^0. Let x^0 be a feasible vector of inputs that can produce y^0. Then the firm's cost will be $w \cdot x^0$. The firm must decide both the level of output as well as the level of inputs to use.

Suppose the firm wishes to maximize profits. The firm then solves the following problem:

$$\max_{(x,y)\geq 0} \; py - w \cdot x \text{ s.t. } f(x) \geq y \tag{104}$$

where $f(x)$ is a production function that is continuous, strictly increasing, and strictly quasiconcave. But we know that $f(x) = y$, so we can simply substitute in for the output level y and solve:

$$\max_{x \in \mathbb{R}^n_+} \; pf(x) - w \cdot x \tag{105}$$

If we assume an interior solution, then the first-order conditions at the optimal level of inputs are:

$$p\frac{\partial f(x^*)}{\partial x_i} = w_i \text{ for } i = 1, ..., n \tag{106}$$

The term on the left is the marginal revenue product (MRP), which is the rate at which revenue increases with an additional unit of input i. At the optimum we have that MRP is equal to the wage rate, or, if we take the ratio of the first-order conditions for two inputs, that:

$$\frac{\partial f(x^*)/\partial x_i}{\partial f(x^*)/\partial x_j} = \frac{w_i}{w_j} \text{ for all } i, j \tag{107}$$

or that we have the familiar condition that the MRTS is equal to the ratio of the wages.

We could also structure the firm's problem by using the firm's cost function. We would have:

$$\max_{y \geq 0} \; py - c(w, y) \tag{108}$$

If production actually occurs, so that $y^* > 0$, then the first-order condition holds with equality and we have:

$$p - \frac{dc(w, y^*)}{dy} = 0 \tag{109}$$

This is the familiar result from principles of microeconomics that in perfectly competitive markets price equals marginal cost.

We can now discuss the firm's profit function and output supply function.

Definition 7 *The firm's profit depends only on input and output prices and is defined as the maximum-value function:*

$$\pi(p, w) \equiv \max_{(x,y)\geq 0} \; py - w \cdot x \;\; s.t. \;\; f(x) \geq y \tag{110}$$

The profit function is useful only if a maximum profit actually exists. For example, if a firm has a production function with increasing returns to scale, we can show that there is no maximum value for profits. To do this assume that x' and $y' = f(x')$ do maximize profits at p and w. Then we have:

$$f(tx') > tf(x') \text{ for all } t > 1 \tag{111}$$

because of increasing returns. Now:

$$\begin{aligned} f(tx') &> tf(x') && (112)\\ pf(tx') &> ptf(x') && (113)\\ pf(tx') - wx' &> ptf(x') - wx' && (114)\\ pf(tx') - wx' &> pf(x') - wx && (115)\end{aligned}$$

Thus, the firm could always increase profit by increasing the proportion of inputs it uses if it has increasing returns to scale production. When the profit function exists it has the following properties.

Theorem 8 *(Properties of the profit function) If f is continuous, strictly increasing, strictly quasiconcave, and has $f(0) = 0$, then for $p \geq 0$ and $w \geq 0$, the profit function $\pi(p, w)$, when well-defined, is continuous and*

1. *Increasing in p*
2. *Decreasing in w*
3. *Homogeneous of degree one in (p, w)*
4. *Convex in (p, w)*
5. *Differentiable in $(p, w) >> 0$. Moreover, we have:*

$$\begin{aligned} \frac{\partial \pi(p,w)}{\partial p} &= y(p,w) && (116)\\ -\frac{\partial \pi(p,w)}{\partial w_i} &= x_i(p,w) \text{ for } i = 1, ..., n && (117)\end{aligned}$$

As an example, let's consider the following problem:

$$\max_{y \geq 0} \; py - 2\sqrt{y^3 w_2 w_1} \qquad (118)$$

We have:

$$\begin{aligned} p - 2\sqrt{w_2 w_1}\frac{3}{2}\sqrt{y} &= 0 && (119)\\ 3\sqrt{w_2 w_1 y} &= p && (120)\\ w_2 w_1 y &= \frac{p^2}{9} && (121)\\ y &= \frac{p^2}{9w_2 w_1} && (122)\end{aligned}$$

Substituting this in for y we have:

$$\begin{aligned} \pi(p,w) &= p\frac{p^2}{9w_2w_1} - 2\sqrt{\left(\frac{p^2}{9w_2w_1}\right)^3 w_2w_1} && (123)\\ \pi(p,w) &= \frac{p^3}{9w_2w_1} - 2\sqrt{\frac{p^6}{729w_2^3w_1^3}w_2w_1} && (124)\\ \pi(p,w) &= \frac{p^3}{9w_2w_1} - 2\frac{p^3}{27w_2w_1} && (125)\\ \pi(p,w) &= \left(\frac{1}{9w_2w_1} - \frac{2}{27w_2w_1}\right)p^3 && (126)\\ \pi(p,w) &= \left(\frac{1}{27w_2w_1}\right)p^3 && (127)\end{aligned}$$

Given our profit function we can test each of the properties. It is certainly increasing in p as:

$$\frac{\partial \pi(p,w)}{\partial p} = \frac{1}{9w_2w_1}p^2 > 0 \tag{128}$$

so that profit is increasing if price increases. It is decreasing in w as:

$$\frac{\partial \pi(p,w)}{\partial w_i} = \frac{p^3}{27w_j}(-1)\,w_i^{-2} \tag{129}$$

$$\frac{\partial \pi(p,w)}{\partial w_i} = -\frac{p^3}{27w_jw_i^2} < 0 \tag{130}$$

This simply means that profit is decreasing in input prices. It is homogeneous of degree one in (p,w) so that:

$$\pi(p,w) = \left(\frac{1}{27w_2w_1}\right)p^3 \tag{131}$$

$$\pi(tp,tw) = \left(\frac{1}{27tw_2tw_1}\right)(tp)^3 \tag{132}$$

$$\pi(tp,tw) = \frac{t^3p^3}{27t^2w_2w_1} \tag{133}$$

$$\pi(tp,tw) = \frac{tp}{27w_2w_1} \tag{134}$$

$$\pi(tp,tw) = t\pi(p,w) \tag{135}$$

So that if output price and input prices are increased in the same proportion then profit increases by that proportion. We won't take the total derivative to show that the profit function is convex, but we can show that:

$$\frac{\partial \pi(p,w)}{\partial p} = y(p,w) \tag{136}$$

$$\pi(p,w) = \left(\frac{1}{27w_2w_1}\right)p^3 \tag{137}$$

$$\frac{\partial \pi(p,w)}{\partial p} = \frac{3p^2}{27w_2w_1} \tag{138}$$

$$\frac{\partial \pi(p,w)}{\partial p} = \frac{p^2}{9w_2w_1} \tag{139}$$

which is just $y(p,w)$. And then:

$$-\frac{\partial \pi(p,w)}{\partial w_i} = x_i(p,w) \tag{140}$$

$$\pi(p,w) = \left(\frac{1}{27w_2w_1}\right)p^3 \tag{141}$$

$$-\frac{\partial \pi(p,w)}{\partial w_i} = -(-1)\frac{p^3}{27w_jw_i^2} \tag{142}$$

$$-\frac{\partial \pi(p,w)}{\partial w_i} = \frac{p^3}{27w_jw_i^2} \tag{143}$$

Using our result from above we had that:

$$x_i(w,y) = \sqrt{\frac{y^3w_j}{w_i}} \tag{144}$$

Substituting in for y we have:

$$x_i(w, y(p, w)) = \sqrt{\frac{\left(\frac{p^2}{9w_2w_1}\right)^3 w_j}{w_i}} \quad (145)$$

$$x_i(p, w) = \sqrt{\frac{\frac{p^6}{729w_j^3w_i^3} w_j}{w_i}} \quad (146)$$

$$x_i(p, w) = \sqrt{\frac{p^6 w_j}{729w_j^3w_i^4}} \quad (147)$$

$$x_i(p, w) = \frac{p^3}{27w_jw_i^2} \quad (148)$$

Now we turn to properties of the output supply and input demand functions. Note that these are no longer conditional input demand functions because we are solving the profit maximization problem.

Definition 9 *(Properties of output supply and input demand functions) Let $\pi(p, w)$ be a twice continuously differentiable profit function for some competitive firm. Then, for all $p > 0$ and $w >> 0$, when $\pi(p, w)$ is well-defined:*

1. *The output supply function is homogeneous of degree zero in (p, w)*
2. *The own-price effect of the output supply function, $\partial y(p, w)/\partial p$, is nonnegative.*
3. *The input demand functions are homogeneous of degree zero in (p, w)*
4. *The own-price effects of the input demand functions are nonpositive, $\partial x_i(p, w)/\partial w_i$, for all $i = 1, ..., n$*
5. *The substitution matrix:*

$$\begin{pmatrix} \frac{\partial y(p,w)}{\partial p} & \frac{\partial y(p,w)}{\partial w_1} & \cdots & \frac{\partial y(p,w)}{\partial w_n} \\ \frac{-\partial x_1(p,w)}{\partial p} & \frac{-\partial x_1(p,w)}{\partial w_1} & \cdots & \frac{-\partial x_1(p,w)}{\partial w_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{-\partial x_n(p,w)}{\partial p} & \frac{-\partial x_n(p,w)}{\partial w_1} & \cdots & \frac{-\partial x_n(p,w)}{\partial w_n} \end{pmatrix} \quad (149)$$

is symmetric and positive negative semidefinite.

For the homogeneity results we have:

$$y(p, w) = \frac{p^2}{9w_2w_1} \quad (150)$$

$$y(tp, tw) = \frac{(tp)^2}{9tw_2tw_1} \quad (151)$$

$$y(tp, tw) = \frac{t^2p^2}{9t^2w_2w_1} \quad (152)$$

$$y(tp, tw) = y(p, w) \quad (153)$$

and:

$$x_i(p, w) = \frac{p^3}{27w_jw_i^2} \quad (154)$$

$$x_i(tp, tw) = \frac{(tp)^3}{27tw_j(tw_i)^2} \quad (155)$$

$$x_i(tp, tw) = \frac{t^3p^3}{27tw_jt^2w_i^2} \quad (156)$$

$$x_i(tp, tw) = x_i(p, w) \quad (157)$$

Again, these results simply mean that if you change all prices in the same proportion that the output supply and input demand functions remain the same. We can also see the first derivatives of $y(p,w)$ and $x_i(p,w)$ have the correct signs:

$$\begin{aligned} y(p,w) &= \frac{p^2}{9w_2w_1} && (158) \\ \frac{\partial y(p,w)}{\partial p} &= \frac{2p}{9w_2w_1} > 0 && (159) \end{aligned}$$

and:

$$\begin{aligned} x_i(p,w) &= \frac{p^3}{27w_jw_i^2} && (160) \\ \frac{\partial x_i(p,w)}{\partial w_i} &= \frac{-2p^3}{27w_jw_i^3} < 0 && (161) \end{aligned}$$

The first result simply means that output supply follows the law of supply, while the second means if the price of an input increases the firm will use less of it, provided all other input prices are positive. Finally, we have that:

$$\begin{aligned} \frac{\partial y(p,w)}{\partial w_i} &= -\frac{\partial x_i(p,w)}{\partial p} && (162) \\ \frac{p^2}{9w_jw_i^2}(-1) &= -\frac{3p^2}{27w_jw_i^2} && (163) \\ \frac{-p^2}{9w_jw_i^2} &= \frac{-p^2}{9w_jw_i^2} && (164) \end{aligned}$$

4.2 Short run profit function

We can also discuss the short run profit function, which is simply the profit function when some inputs are fixed.

Definition 10 *Let the production function be $f(x,\overline{x})$ where x is a subvector of variable inputs and $\overline{x}$ is a subvector of fixed inputs. Let w and $\overline{w}$ be the associated input prices for variable and fixed inputs, respectively. The short run, or restricted, profit function is defined as*

$$\pi(p,w,\overline{w},\overline{x}) \equiv \max_{y,x} \; py - w\cdot x - \overline{w}\cdot\overline{x} \;\; s.t. \;\; f(x,\overline{x}) \geq y \qquad (165)$$

the solutions $y(p,w,\overline{w},\overline{x})$ and $x(p,w,\overline{w},\overline{x})$ are called the short run, or restricted, output supply and variable input demand functions.

For all $p>0$ and $w>>0$, $\pi(p,w,\overline{w},\overline{x})$ where well-defined, is continuous in p and w, increasing in p, decreasing in w, and convex in p and w. If $\pi(p,w,\overline{w},\overline{x})$ is twice continuously differentiable, $y(p,w,\overline{w},\overline{x})$ and $x(p,w,\overline{w},\overline{x})$ are homogeneous of degree zero in (p,w), $\frac{\partial y(p,w,\overline{w},\overline{x})}{\partial p} \geq 0$ and $\frac{\partial x_i(p,w,\overline{w},\overline{x})}{\partial w_i} \leq 0$ and the substitution matrix is symmetric and positive semidefinite.