These notes correspond to chapter 5 of Jehle and Reny.

1 General Equilibrium

In Wealth of Nations Adam Smith posited his invisible hand theorem. In essence, it states that each individual in the economy, acting in his own self interest, will maximize society's welfare through interdependent market actions. The idea is that individual's will do what they please, and if what they please is valuable, then the market will reward them; if not, the market will not reward them and then these individuals may opt to do something that is more valuable to society (or not – they can choose to remain rewarded at a low rate if they so desire).

There are a few basic questions that theorists ask when it comes to modelling. The first is a question of existence, as in does an equilibrium exist in Smith's vision (or a simplified version of it) of a market economy? The second is a question of uniqueness, as in is this the only equilibrium? After uniqueness one can ask about how the equilibrium is selected (if it is not unique) or how stable the equilibrium is (if it is unique – or perhaps even if it is not). The focus of this chapter is on existence of an equilibrium. We will begin with a diagrammatic discussion of exchange in a two-consumer, two-good world.

2 Exchange Economy

In the previous chapter we discussed the notion of partial equilibrium, where we looked at a market in isolation. In general equilibrium we will consider multiple goods (usually two for simplicity, keeping in mind that the discussion extends to more than two) and that market prices are determined endogenously, or within the system. Throughout our discussion of the exchange economy one key assumption is that all exchanges that are made must be voluntary, so that all agents involved in exchange must agree to the exchange. If any agent or group of agents disagrees, then the exchange will not take place.

We begin with an economy of no production so that goods simply exist. A typical story told is that this is Robinson Crusoe on an island who simply receives goods that exist in nature. In order to have exchange we would need a second individual, so we then assume that Friday is there with him. All consumers have preferences over available bundles, and all care only about their own well-being.[1] Each agent is endowed with a nonnegative amount of each good. Thus, there is private ownership of goods (which is an oft-neglected assumption – without private ownership the model falls apart) and this private ownership is respected. Since there is no forced exchange of goods, the only method through which goods may be exchanged is through voluntary exchange. The question then becomes where does a system of voluntary exchange come to rest, or what is the equilibrium of this system.

To begin assume only two consumers (1 and 2) and two goods A and B – the book uses 1 and 2 for goods as well but it makes bookkeeping a little difficult). Let $e^1 = \left(e_A^1, e_B^1\right)$ be consumer 1's endowment of goods A and B respectively and $e^2 = \left(e_A^2, e_B^2\right)$ be consumer 2's endowment of goods A and B. The total endowment in the economy is $e^1 + e^2 = \left(e_A^1 + e_A^2, e_B^1 + e_B^1\right)$.

We can represent this economy in the Edgeworth box (shown in Figure 1). Units of x_A are measured along the horizontal axes and those of x_B are measured along the vertical axes. The length of the horizontal axis is equal to $e_A^1 + e_A^2$ while the length of the vertical axes are equal to $e_B^1 + e_B^2$. The lower left (southwest) corner is consumer 1's origin while the upper right (northeast) corner is consumer 2's. Consumer 1's preferences increase as we move northeast while consumer 2's preferences increase as we move southwest (in essence, consumer 2 is "upside down").

Each point in the box represents an amount of x_A and x_B for both consumers. Thus one point determines 4 pieces (x_A^1, x_A^2, x_B^1, and x_B^2). Since the length and width of the box represent the total endowments of goods A and B, every point inside the box (and on the edges) is a feasible allocation. No points outside the box are feasible.

Finally, we need consumers to have preferences over the goods. We assume that both consumers have standard convex indifference curves. There is only one indifference curve for each consumer that passes

[1] We could add that they care about others but this greatly complicates the math. The same basic result will occur if people care about others' utilities, only it takes more time to get to the result which is why we assume individuals care only about their own well-being.

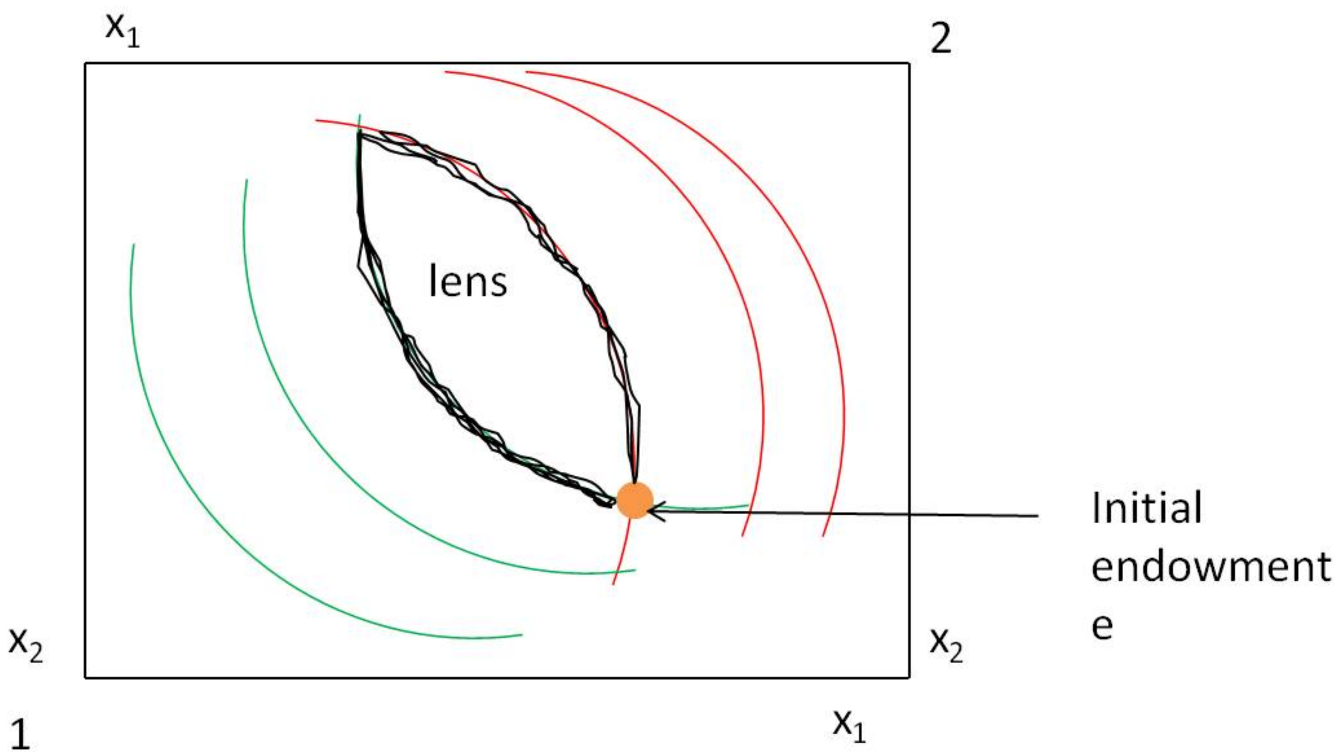

Figure 1: An Edgeworth box with the lens labeled.

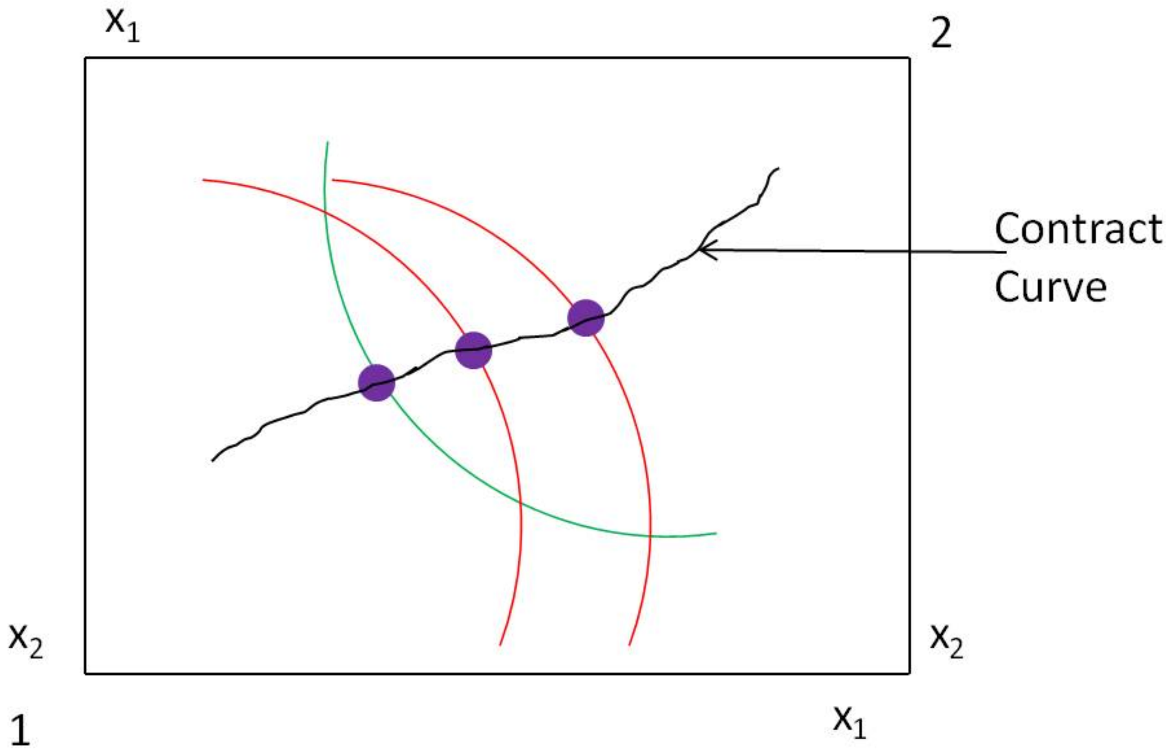

Figure 2: The contract curve for an Edgeworth box.

through each point (we cannot violate transitivity). The line CC is the set of points such that the consumers' indifference curves are tangent to each other. This line is called the contract curve (shown in Figure 2). At any point off of the contract curve the consumers' indifference curves cut through each other.

Suppose that there are initial endowments e^1 and e^2. The question is which allocations of goods A and B to consumers 1 and 2 are equilibria in this exchange economy? First, the allocations must be feasible so they must be inside the box. Second, consider points beneath either consumer's indifference curve along which e (the set of initial endowments) lies. No consumer would voluntarily agree to an exchange that leads to any of those points because they would make themselves strictly worse off and no utility maximizing consumer (remember, here we are considering only consumers who care to maximize their own utility) would agree to such an exchange.

Now consider the "lens" created by the intersection points of the indifference curves along which e lies. At any point along the lens both consumers are at least as well off as they were before and one of them will be strictly better off. As we move along consumer 1's indifference curve this leaves consumer 1 at the same utility but increases the utility of consumer 2. The opposite is true if we move along consumer 2's indifference curve – consumer 2 is left at the same utility while consumer 1's utility increases.

Now consider a point inside the lens at which the two consumers' indifference curves intersect (are NOT tangent). This point is preferred by both consumers to the initial endowment as they are both better off since they are both on higher indifference curves. However, is it an equilibrium? No, because we can now create a new lens and repeat the process. It is not until the consumers reach a point along the contract curve (at which their indifference curves are tangent) within the lens that the system comes to rest. So any point along the contract curve within the lens created by the indifference curves of the initial endowment is an equilibrium in this exchange economy – once one of those points is reached there are no remaining mutually beneficial trades (nor are there any trades that can be made that will make one person strictly better off while leaving the other at the same utility level). So equilibria in this simple exchange economy are Pareto efficient. Thus, we have answered our first question about the existence of equilibria and we have already discovered one important implication of our model of exchange, that all potential equilibria are Pareto efficient.

Consider many consumers and many goods. Let the set of consumers be $i = 1, ..., I$ and suppose there are n goods. Each consumer i has a preference relation, $\succsim^i$, and receives a nonnegative vector of endowments of each of the n goods, $e^i = \left(e^i_1, ..., e^i_n\right)$. The collection $\check{E} = \left(\succsim^i, e^i\right)_{i=1,...,I}$ is an exchange economy.

The conditions that characterize equilibrium in this economy are as follows. The total amount of goods assigned to individuals cannot exceed the endowment (or demand must equal supply). If $e \equiv \left(e^1, ..., e^I\right)$ is the vector of endowments and $x \equiv \left(x^1, ..., x^I\right)$ is an allocation vector where $x^1 \equiv \left(x^i_1, ..., x^i_n\right)$ is a consumer's consumption bundle, then the set of feasible allocations is given by:

$$F(e) \equiv \left\{ x | \sum_i x^i = \sum_i e^i \right\} \tag{1}$$

Definition 1 *A feasible allocation $x \in F(e)$ is Pareto efficient if there is no other feasible allocation $y \in F(e)$ such that $y^i \succsim^i x^i$ for all consumers i with at least one preference strict.*

If $x \in F(e)$ is not Pareto efficient then we can find $y \in F(e)$ such that at least one person is made strictly better off and the others remain at least at their level of utility under x. The person could essentially state "If we rearrange endowments in this manner (y), then everyone will be at least as well off as they were before and I will be strictly better off". Or that person could arrange a series of trades to reach that state (y) since declaring "Here's how to make me better off and leave you at the same level of utility" is unlikely to win the others over. Once at a Pareto efficient allocation it is impossible to move to another allocation without making at least one consumer worse off.

So any Pareto efficient allocation is a potential equilibrium (as it was with only two consumers and two goods) but not all Pareto efficient allocations will be equilibria in the economy with initial endowments e. In the 2 consumer, 2 good case only those allocations on the contract curve within the lens were potential equilibria in that system. It is similar in the I consumer, n good case, although the picture is quite a bit messier. However, with more consumers there are additional restrictions. Consider the case where a particular consumer has endowment e^i and receives a consumption bundle x^i from a Pareto efficient allocation x where $x^i \succ^i e^i$. Clearly the consumer prefers x^i to e^i but why might the consumer still not agree to this allocation? If they can find another consumer j such that i can trade with j and do even better without leaving j any worse off then he was under x^j then the two of you can block the exchange.

Definition 2 *Let $S \subset I$ denote a coalition of consumers. We say that S blocks $x \in F(e)$ if there is an allocation y such that:*

1. $\sum_{i \in S} y^i = \sum_{i \in S} e^i$
2. $y^i \succsim^i x^i$ *for all $i \in S$ with at least one preference strict*

Think about what this definition means. If all of the consumers in S get together and divide up their endowment then they can come up with an allocation that leaves all of them at least as well off as they were under x but at least one person is strictly better off. Basically, there can be no groups of consumers who break off from the economy and decide to work on their own. An allocation is blocked if there is some group that can break off and do better by not being part of the allocation. An allocation is unblocked if no coalition can block it. For the 2 good, 2 consumer case we stated that any allocation along the contract curve within the lens was an equilibrium. Since there are only two consumers neither can form a coalition to block any of those potential equilibria. Thus we will require an equilibrium to be unblocked in the multiple consumer case.

Definition 3 *The core of an exchange economy with endowment e, denoted $C(e)$, is the set of all unblocked feasible allocations.*

Thus far we have assumed that coalitions can form costlessly and that there are no transactions costs in the market. The question is whether a competitive market can correctly organize and align the consumer demands to reach an equilibrium allocation.

3 GE and competitive markets

In the prior section we looked at equilibrium in an exchange economy. Now we turn our attention to that of a perfectly competitive market, where all transactions occur in impersonal markets. Consumers are concerned with their own well-being and both consumers and producers are insignificant on every market and do not affect the market price. Equilibrium in a market occurs when buyers' demands equal sellers' supplies. Equilibrium in the market system occurs when all markets are in equilibrium.

Consider an economy with I consumers who are each endowed with a nonnegative vector e^i of n goods. For simplicity assume no production and that each consumer's preferences can be represented by a utility function u^i that is continuous, strongly increasing, and strictly quasiconcave on $\mathbb{R}_+^n$.

In a competitive market the prices $p = (p_1, ..., p_n)$ are taken as given by all agents in the market. Each consumer solves:

$$\max_{x^i \in \mathbb{R}_+^n} u^i\left(x^i\right) \text{ s.t. } px^i \leq pe^i \tag{2}$$

Instead of now receiving an income equal to w, the consumer's income is equal to the market value of his endowment. The solution to the consumer's problem is a set of demands $x^i\left(p, pe^i\right)$ which depends on market prices and the consumer's income.

Theorem 4 *If u^i is continuous, strongly increasing, and strictly quasiconcave on $\mathbb{R}_+^n$, then for each $p >> 0$ the consumer's problem has a unique solution, $x^i\left(p, pe^i\right)$. In addition, $x^i\left(p, pe^i\right)$ is continuous in p on $\mathbb{R}_{++}^n$.*

We know that the solution is unique because the budget set is bounded and because u^i is strictly quasiconcave. The total endowment of each of the goods can be viewed as the amount of each good that is supplied on the market. Note that this amount is fixed. Taking that view each individual can be considered a consumer and a producer in the market. Market demand is the sum of each individual consumer's demands and market supply for a good is the sum of each individual's endowment. Since demand for each good depends on the prices of every good as well as the consumer's endowment of every good the markets for all goods are interdependent. We can characterize a market by its excess demand function, or by how much demand exceeds supply in each market. The excess demand vector is the n-vector of all excess demand functions.

Definition 5 *The excess demand function for market k is the real valued function:*

$$z_k(p) \equiv \sum\nolimits_i x_k^i\left(p, pe^i\right) - \sum\nolimits_i e_k^i \tag{3}$$

Aggregate excess demand is the vector-valued function:

$$z(p) \equiv \left(z_1(p), ..., z_n(p)\right) \tag{4}$$

Theorem 6 *If for each consumer i, u^i is continuous, strongly increasing, and strictly quasiconcave, then for all $p >> 0$,*

1. *Continuity: $z(\cdot)$ is continuous in p*
2. *Homogeneity: $z(tp) = z(p)$ for all $t > 0$*
3. *Walras' law: $p \cdot z(p) = 0$*

Since $x^i(\cdot)$ is continuous, then $z(\cdot)$ will be continuous. Also, $x^i(\cdot)$ is homogeneous of degree zero in prices, so that $z_k(p)$ is homogeneous of degree zero in prices. The last property is Walras' law. This says that the sum of the market value of excess demands will always be equal. Essentially what this means is that if we have two markets then the market value of the excess demand for one good, $p_1 z_1(p)$, is exactly equal to the negative of the market value of the excess demand is $-p_2 z_2(p)$. This means that in a two market world if the market for one good is in equilibrium then the market for the other good must also be in equilibrium (meaning no excess supply or demand). With n markets, if $n-1$ markets are in equilibrium then the n^{th} market is as well.

If a single market is in equilibrium, so that $z_k(p) = 0$, then we can say that this is a partial equilibrium. If $z(p) = 0$, or when quantity supplied equals quantity demanded in every market, then there is a general equilibrium.

Definition 7 *A vector $p^* \in \mathbb{R}^n_{++}$ is called a Walrasian equilibrium if $z(p^*) = 0$.*

Does a Walrasian equilibrium exist? The following set of conditions on aggregate excess demand guarantee a Walrasian equilibrium price vector exists.

Theorem 8 *Suppose $z(p)$ satisfies the following three conditions*

1. *$z(\cdot)$ is continuous on $\mathbb{R}^n_{++}$.*
2. *$p \cdot z(p) = 0$ for all $p >> 0$*
3. *If $\{p^m\}$ is a sequence of price vectors in $\mathbb{R}^n_{++}$ converging to $\overline{p} \neq 0$, and $\overline{p}_k = 0$ for some good k, then for some good k' with $\overline{p}'_k = 0$, the associated sequence of excess demands in the market for good k', $\{z_{k'}(p^m)\}$, is unbounded above*

Then there is a price vector $p^* >> 0$ such that $z(p^*) = 0$.

The third condition is the only new condition that we see. If the prices of some but not all goods are arbitrarily close to zero, then the excess demand for at least one of those goods is arbitrarily high. Now we just need to see when condition 3 holds as this is important for establishing that the price vector $z(p^*) = 0$ exists.

Theorem 9 *If each consumer's utility function is continuous, strongly increasing, and strictly quasiconcave on $\mathbb{R}^n_+$, and if the aggregate demand endowment of each good is strictly positive (i.e. $\sum_{i=1}^I e^i >> 0$), then aggregate excess demand satisfies conditions 1-3 of the previous theorem.*

All we need to ensure that our 3 conditions hold is the assumptions we have already made about each consumer's utility function and a positive endowment of each good.

Theorem 10 *(Existence) If each consumer's utility function is continuous, strongly increasing, and strictly quasiconcave on $\mathbb{R}^n_+$, and $\sum_{i=1}^I e^i >> 0$, then there exists at least one price vector, $p^* >> 0$, such that $z(p^*) = 0$.*

This theorem gives us existence of an equilibrium. Note that only relative prices matter in this setting because $z(p^*) = z(tp^*)$ for all $t > 0$. So we can scale one price how we want to. Typically we normalize one price to 1 and then use that good as the monetary measure of the economy. Now we define a Walrasian equilibrium allocation.

Definition 11 *Let p^* be a Walrasian equilibrium for some economy with initial endowments e, and let*

$$x(p^*) \equiv \left(x^1\left(p^*, p^* \cdot e^1\right), ..., x^I\left(p^*, p^* \cdot e^I\right)\right) \tag{5}$$

where component i gives the n-vector of goods demanded and received by consumer i at prices p^*. Then $x(p^*)$ is called a Walrasian equilibrium allocation, or WEA.

For a competitive equilibrium we will need to find both the Walrasian equilibrium allocation AND the price vector that equilibrates supply and demand.

Let's work through the example in the book here. Consider a two person economy where each individual's utility function is given by:

$$u^i(x_1, x_2) = x_1^\rho + x_2^\rho \text{ for } i = 1, 2 \tag{6}$$

Suppose there is one unit of endowment of each good and let individual 1 have all of good 1 and individual 2 have all of good 2, so that $e^1 = (1, 0)$ and $e^2 = (0, 1)$ with $e = (1, 1)$. We have worked with this utility function before and so we "know" that:

$$x_j^i\left(p, y^i\right) = \frac{p_j^{\frac{1}{1-\rho}} y^i}{p_1^{\frac{\rho}{\rho-1}} + p_2^{\frac{\rho}{\rho-1}}} \tag{7}$$

where the j's represent goods and the i's represent an individual, so that p_j is the price of good j while y^i is consumer i's income. Thus, we have one portion of the solution, the demands. Now we also need to find the price vector.

In this setting income is NOT simply given to the consumer but depends upon his or her endowment and the value of that endowment. For consumer 1 we have $y^1 = p_1 * 1 + p_2 * 0 = p_1$. For consumer 2 we have $y^2 = p_1 * 0 + p_2 * 1 = p_2$. We can normalize one of these prices, so if we multiply each price by $\frac{1}{p_2}$ we then have $\overline{p}_1 = \frac{p_1}{p_2}$ and $\overline{p}_2 = 1$. Now we have to equilibrate demand and supply, so that:

$$x_1^1\left(\overline{p}^*, \overline{p}^* \cdot e^1\right) + x_1^2\left(\overline{p}^*, \overline{p}^* \cdot e^2\right) = e_1^1 + e_1^2 \tag{8}$$

You might ask why we are only equilibrating supply and demand for one market – recall that from Walras' law if supply and demand are equal in $N-1$ markets, then supply and demand are equal in the N^{th} market. Since we only have 2 markets, we only need to check one supply and demand condition. We have:

$$\frac{\overline{p}_1^{\frac{1}{\rho-1}} y^1}{\overline{p}_1^{\frac{\rho}{\rho-1}} + \overline{p}_2^{\frac{\rho}{\rho-1}}} + \frac{\overline{p}_1^{\frac{1}{\rho-1}} y^2}{\overline{p}_1^{\frac{\rho}{\rho-1}} + \overline{p}_2^{\frac{\rho}{\rho-1}}} = 1 + 0 \tag{9}$$

But we know that $\overline{p}_2 = 1$, $y^2 = 1$ and $y^1 = \overline{p}_1$. Substituting in we have:

$$\begin{aligned}
\frac{\overline{p}_1^{\frac{1}{\rho-1}} * \overline{p}_1}{\overline{p}_1^{\frac{\rho}{\rho-1}} + 1^{\frac{\rho}{\rho-1}}} + \frac{\overline{p}_1^{\frac{1}{\rho-1}} * 1}{\overline{p}_1^{\frac{\rho}{\rho-1}} + 1^{\frac{\rho}{\rho-1}}} &= 1 && (10)\\
\frac{\overline{p}_1^{\frac{\rho}{\rho-1}} + \overline{p}_1^{\frac{1}{\rho-1}}}{\overline{p}_1^{\frac{\rho}{\rho-1}} + 1} &= 1 && (11)\\
\overline{p}_1^{\frac{\rho}{\rho-1}} + \overline{p}_1^{\frac{1}{1-\rho}} &= \overline{p}_1^{\frac{\rho}{\rho-1}} + 1 && (12)\\
\overline{p}_1^{\frac{1}{1-\rho}} &= 1 && (13)\\
\overline{p}_1 &= 1 = \overline{p}_2 && (14)
\end{aligned}$$

so that an equilibrium allocation is given by:

$$\begin{aligned}
x_1^1 &= \frac{1}{2}, x_2^1 = \frac{1}{2} - \text{consumer 1's demands} && (15)\\
x_1^2 &= \frac{1}{2}, x_2^2 = \frac{1}{2} - \text{consumer 2's demands} && (16)\\
p_1 &= p_2 - \text{prices} && (17)
\end{aligned}$$

In this economy the consumers simply exchange half of their endowment for half of the other person's endowment. If we vary the endowments we can find more equilibrium allocations. For instance, now let $e^1 = \left(1, \frac{1}{4}\right)$ and $e^2 = \left(0, \frac{3}{4}\right)$. Note that the total endowment and the demand functions are the same. However, $y^2 = \frac{3}{4}$ and $y^1 = \overline{p}_1 + \frac{1}{4}$. Setting supply equal to demand for good 1 again we have:

$$\begin{aligned}
\frac{\overline{p}_1^{\frac{1}{\rho-1}} y^1}{\overline{p}_1^{\frac{\rho}{\rho-1}} + \overline{p}_2^{\frac{\rho}{\rho-1}}} + \frac{\overline{p}_1^{\frac{1}{\rho-1}} y^2}{\overline{p}_1^{\frac{\rho}{\rho-1}} + \overline{p}_2^{\frac{\rho}{\rho-1}}} &= 1 && (18)\\
\frac{\overline{p}_1^{\frac{1}{\rho-1}} \left(\overline{p}_1 + \frac{1}{4}\right)}{\overline{p}_1^{\frac{\rho}{\rho-1}} + 1} + \frac{\overline{p}_1^{\frac{1}{\rho-1}} \frac{3}{4}}{\overline{p}_1^{\frac{\rho}{\rho-1}} + 1} &= 1 && (19)\\
\overline{p}_1^{\frac{\rho}{\rho-1}} + \frac{1}{4}\overline{p}_1^{\frac{1}{\rho-1}} + \overline{p}_1^{\frac{1}{\rho-1}} \frac{3}{4} &= \overline{p}_1^{\frac{\rho}{\rho-1}} + 1 && (20)\\
\overline{p}_1^{\frac{1}{\rho-1}} &= 1 && (21)\\
\overline{p}_1 &= 1 && (22)
\end{aligned}$$

Now we have that relative prices are the same as before, but the equilibrium allocation will be different. To see this note that in order for both consumers to consume $\frac{1}{2}$ of each good that consumer 1 would have to trade $\frac{1}{2}$ of good 1 to consumer 2 for $\frac{1}{4}$ of good 2. But this cannot happen because the goods must be traded at a rate of 1 for 1. To determine the amounts consumed, simply plug the prices and income back into the demand functions:

$$x_1^1\left(p^*, y\left(p^*, e\right)\right) = \frac{\overline{p}_1^{\frac{1}{\rho-1}} y^1}{\overline{p}_1^{\frac{\rho}{\rho-1}} + \overline{p}_2^{\frac{\rho}{\rho-1}}} \tag{23}$$

$$x_1^1\left(p^*, y\left(p^*, e\right)\right) = \frac{1 * \left(1 + \frac{1}{4}\right)}{1+1} \tag{24}$$

$$x_1^1\left(p^*, y\left(p^*, e\right)\right) = \frac{\frac{5}{4}}{2} \tag{25}$$

$$x_1^1\left(p^*, y\left(p^*, e\right)\right) = \frac{5}{8} \tag{26}$$

This means that consumer 2 consumes $x_1^2 = \frac{3}{8}$ (since there is one unit available in the economy). Since consumer 2's initial endowment was $e^2 = \left(0, \frac{3}{4}\right)$ and the price ratio as $1:1$, we know that the consumer had to give up $\frac{3}{8}$ of his good 2 to receive $\frac{3}{8}$ of good 1. Thus, the equilibrium is:

$$x_1^1 = \frac{5}{8}, x_1^2 = \frac{3}{8} \text{ – consumption of good 1} \tag{27}$$

$$x_2^1 = \frac{5}{8}, x_2^2 = \frac{3}{8} \text{ – consumption of good 2} \tag{28}$$

$$p_1^* = p_2^* = 1 \text{ – equilibrium prices} \tag{29}$$

Note that the initial endowment shifts the equilibrium allocation away from a 50/50 split of the endowments but this is because consumer 1 is better off consuming his endowment of $\left(1, \frac{1}{4}\right)$ than he is consuming the allocation $\left(\frac{1}{2}, \frac{1}{2}\right)$. If we let $\rho = \frac{1}{2}$ we can see this:

$$u\left(1, \frac{1}{4}\right) = 1^{1/2} + \frac{1}{4}^{1/2} = \frac{3}{2} = 1.5 \tag{30}$$

$$u\left(\frac{1}{2}, \frac{1}{2}\right) = \frac{1}{2}^{1/2} + \frac{1}{2}^{1/2} = \sqrt{2} \approx 1.4 \tag{31}$$

The closer ρ gets to 1 the further apart these utilities become while the closer ρ gets to 0 the closer they become, though the consumer still prefers his endowment regardless of the value of ρ as long as $0 < \rho < 1$. Thus, consumer 1 can "block" the allocation where he receives $\left(\frac{1}{2}, \frac{1}{2}\right)$ and simply consume his endowment unless consumer 2 agrees to a different exchange. Consumer 2 will agree to exchange $\frac{3}{8}$ of his endowment of good 2 for $\frac{3}{8}$ of consumer 1's endowment of good 1 because this exchange makes consumer 2 better off. To see this, let $\rho = \frac{1}{2}$ again and we have:

$$u\left(0, \frac{3}{4}\right) = 0^{1/2} + \frac{3}{4}^{1/2} = \frac{\sqrt{3}}{2} \approx 0.866 \tag{32}$$

$$u\left(\frac{3}{8}, \frac{3}{8}\right) = \frac{3}{8}^{1/2} + \frac{3}{8}^{1/2} = \frac{\sqrt{3}\sqrt{8}}{4} \approx 1.22 \tag{33}$$

Again, as ρ gets closer to 1 this gap closes and as ρ gets closer to 0 this gap widens.

Now that we see there are different equilibrium allocations depending on initial endowments, we can describe the set of Walrasian equilibrium allocations.

Definition 12 *For any economy with endowments e, let $W(e)$ denote the set of Walrasian equilibrium allocations.*

Theorem 13 *Consider an exchange economy $\left(u^i, e^i\right)_{i \in I}$. If each consumer's utility function , u^i, is continuous and strictly increasing on $\mathbb{R}^n_+$, then every Walrasian equilibrium allocation is in the core. That is,*

$$W(e) \subset C(e) \tag{34}$$

Using this theorem, we have the First Fundamental Theorem of Welfare Economics.

Theorem 14 *(First Fundamental Theorem of Welfare Economics) Given that $W(e) \subset C(e)$, every Walrasian equilibrium allocation is Pareto-efficient.*

Now, what has actually happened in all of this discussion? Consumers act to maximize utility given initial endowments and their own preferences. They do not know the preferences of other consumers (they could, but they do not have to). Exchange is voluntary, so that every individual in the economy must agree to the exchange and there can be no blocking coalitions. Letting exchange take place leads to an equilibrium price vector and allocation such that there are no exchanges that can take place that will make at least one individual better off while leaving all other individuals at least as well off as they were before. All of this is accomplished without any intervention by third parties. Thus, Adam Smith's invisible hand theorem is at work, as all individuals are acting to maximize their own utility (which may or may not include other people's utilities as arguments of the function) and this leads to society achieving an efficient allocation.

It seems fairly basic now but 100 years ago it was uncertain what the necessary assumptions were that would generate this type of outcome. We have developed a very basic model that leads the economy to efficient outcomes through voluntary exchange. However, note that the equilibrium allocation obtained is simply Pareto efficient, and there is nothing about equity or fairness or any other consideration at the equilibrium allocation. Now, the question is, Supposing that we can identify the equilibrium allocation we would like to achieve, can a market system achieve it? The answer is yes, and the result is the Second Fundamental Theorem of Welfare Economics.

Theorem 15 *Consider an exchange economy $\left(u^i, e^i\right)_{i \in I}$ with aggregate endowment $\sum_{i=1}^{I} e^i >> 0$, and with each utility function continuous, strongly increasing, and strictly quasiconcave on $\mathbb{R}^n_+$. Suppose that $\overline{x}$ is a Pareto efficient allocation for $\left(u^i, e^i\right)_{i \in I}$, and that endowments are redistributed so that the new endowment vector is $\overline{x}$. Then $\overline{x}$ is a competitive equilibrium allocation of the resulting exchange economy $\left(u^i, \overline{x}^i\right)_{i \in I}$.*

Thus, any Pareto optimal point can be reached by "simply" redistributing the endowments correctly and letting the market work. This argument can be used by government intervention types, and the justification will be that there are different points along the boundary of the utility possibilities set that yield different SOCIETAL utility. Thus, a point with one individual having everything above a subsistence level, which is Pareto optimal, may not be viewed as having as high a societal utility as a point where everyone has the same amount of all goods, which may also be Pareto optimal. Thus, a normative judgement is made that ranks the different Pareto optimal equilibria, and the idea is to find the one where societal utility is the largest. Be aware that this result also relies on the same assumptions of the model that says that every competitive equilibrium is Pareto optimal – one cannot argue that the assumptions that yield the First Fundamental Theorem of Welfare Economics are not met in the "real world" and so this requires that the First Fundamental Theorem of Welfare Economics be dismissed while simultaneously arguing for redistribution based upon the results of the Second Fundamental Theorem of Welfare Economics. There may be good reasons for redistribution policy, but those reasons will have to be established on their own merits if the assumptions of the model are not met.

4 GE and production

Now we introduce firms into the market. But having a firm in the market means that we must make some modifications. First, the firm makes profits, but the firm is not an entity that consumes goods so the firm has no need for profits. Thus, the profits of a firm must be transferred back to the consumers. For most firms there is an easily seen distinction between inputs and outputs, but in the market there is less of a distinction. We will not make a distinction about whether a good is an input or output for all firms, but let negative quantities denote inputs and positive quantities denote outputs.

4.1 Producers

Suppose there is a fixed number of firms J, with $j = 1, ..., J$. Let $y^j \in \mathbb{R}^n$ be a production plan for firm j. Here we have that if $y_k^j < 0$ the good is an input and if $y_k^j > 0$ it is an output. We make the following assumptions about production possibility sets. We assume that firm profits are bounded from below by zero and that output requires some inputs. We also assume that the production set is closed and bounded. We also assume strong convexity, which rules out constant and increasing returns to scale technologies and ensures that there is a maximum profit for the firm.

The firm's problem is to maximize profit by choosing a production plan given a price vector $p \geq 0$. We have:

$$\max_{y^j \in Y^j} \; p \cdot y^j \tag{35}$$

Note that if a good is an input it appears as a cost (because $y_k^j < 0$) and if it is an output it appears as a revenue (because $y_k^j > 0$). Let $y^j(p)$ denote the firm's supply function which can be found from the profit maximization problem.[2]

Theorem 16 *If Y^j has profits that are bounded from below by zero, output requires some inputs, the production set is closed, bounded, and has strong convexity, then for every price vector $p >> 0$ the solution to the firm's problem is unique and denoted by $y^j(p)$. Moreover, $y^j(p)$ is continuous on $\mathbb{R}^n_{++}$. and $\Pi_j(p)$ is well-defined and continuous on $\mathbb{R}^n_+$.*

Note that the profit function is homogeneous of degree 1 in commodity prices and each output supply and input demand function is homogeneous of degree zero in prices.

Now, consider aggregate production. Suppose that there are no externalities (or synergies) in production between firms, and we define the aggregate production possibilities set as:

$$Y \equiv \left\{ y | y = \sum_j y^j, \text{ where } y^j \in Y^j \right\} \tag{36}$$

Essentially the aggregate production set is the sum of all the individual production sets. This aggregate production set will have the same properties as the individual production sets. Suppose we wish to maximize aggregate profits. If $p >> 0$ then there will be a unique maximum of aggregate profit.

Theorem 17 *For any prices $p \geq 0$, we have*

$$p \cdot \overline{y} \geq p \cdot y \text{ for all } y \in Y \tag{37}$$

if and only if for some $\overline{y}^j \in Y^j$, $j \in J$, we may write $\overline{y} = \sum_j \overline{y}^j$, and

$$p \cdot \overline{y}^j \geq p \cdot y^j \text{ for all } y^j \in Y^j,\ j \in J \tag{38}$$

Thus, a production plan maximizes aggregate profit if and only if it maximizes individual firm profits. This is what is meant by there are no externalities or synergies across firms.

4.2 Consumers

Consumers are as they were in the pure exchange economy with one small modification. They still have a utility function u^i which is continuous, strongly increasing, and strictly quasiconcave. They still consume nonnegative amounts of all goods. Note that they can still supply some goods to the market if they have a strictly positive endowment. For instance, if we give a consumer an amount of time $T > 0$, the consumer can supply t units to the market and "consume" (as leisure, usually) $1 - t$ units.

The one modification is that the consumer's income now changes because the consumer receives a share (possibly 0) of each firm's profits. Denote consumer i's share of firm j's profits as θ^{ij}, where $0 \leq \theta^{ij} \leq 1$ and

[2] Though technically this $y^j(p)$ contains the output supply function and input demand functions.

that, $\sum_i \theta^{ij} = 1$ for all $j \in J$, or that the total share of profits for each firm sums to 1. The consumer now receives income from his endowment as well as from the firm's profit so the consumer's budget constraint is:

$$p \cdot x^i \leq p \cdot e^i + \sum_j \theta^{ij} \Pi^j (p) \tag{39}$$

If $m^i (p) = p \cdot e^i + \sum_j \theta^{ij} \Pi^j (p)$, the consumer's problem is:

$$\max_{x^i \in \mathbb{R}^n_+} u^i \left(x^i\right) \text{ s.t. } p \cdot x^i \leq m^i (p) \tag{40}$$

Since the firm earns a nonnegative profit the consumer's income is nonnegative because $p \geq 0$ and $e^i \geq 0$.

Theorem 18 *If each Y^j has profits that are bounded from below by zero, output requires some inputs, the production set is closed, bounded, and has strong convexity, and if u^i is continuous, strongly increasing, and strictly quasiconcave, then a solution to the consumer's problem exists and is unique for all $p >> 0$. Denoting the solution by $x^i \left(p, m^i (p)\right)$, and $x^i \left(p, m^i (p)\right)$ is continuous in p on $\mathbb{R}^n_{++}$. In addition, $m^i (p)$ is continuous on $\mathbb{R}^n_+$.*

4.3 Equilibrium

Once again we define an excess demand function for each good and an aggregate excess demand function for the economy. Excess demand for good k is:

$$z_k (p) \equiv \sum_i x^i_k \left(p, m^i (p)\right) - \sum_j y^j_k (p) - \sum_i e^i_k \tag{41}$$

with aggregate excess demand vector:

$$z (p) \equiv (z_1 (p), ..., z_n (p)) \tag{42}$$

A Walrasian equilibrium price vector $p^* >> 0$ clears all markets so that $z (p^*) = 0$.

Theorem 19 *Consider the economy $\left(u^i, e^i, \theta^{ij}, Y^j\right)_{i \in I, j \in J}$. If each Y^j has profits that are bounded from below by zero, output requires some inputs, the production set is closed, bounded, and has strong convexity, and if u^i is continuous, strongly increasing, and strictly quasiconcave, and $y + \sum_i e^i >> 0$ for some aggregate production vector $y \in \sum_j Y^j$, then there exists at least one price vector $p^* >> 0$, such that $z (p^*) = 0$.*

This is very similar to the theorem when there was no production. Now, however, it is not the endowment that has to be strictly greater than zero but the aggregate production. Once again we can normalize one price because excess demand is homogeneous of degree zero.

4.3.1 Robinson Crusoe economy

When we allow for production in the economy the simplest economy will take the form of one individual, with that individual acting as both consumer and producer. In essence, consider the story of Robinson Crusoe, shipwrecked on an island. The individual is endowed with an amount of time T. He can either choose to use his time for leisure or for productive value. Production in this case consists of gathering coconuts, which also provide Crusoe with utility. Crusoe all receives all profits from producing and selling coconuts. Let k denote the number of hours that Crusoe spends as labor and let y be the number of coconuts. The production set for the firm and the economy are the same since there is only one firm. Assume that:

$$Y = \{(-k, y) | 0 \leq k \leq b, \text{ and } 0 \leq y \leq k^\alpha\} \tag{43}$$

with $b > 0$ and $\alpha \in (0, 1)$. This production set says that it takes k hours to produce k^α units of y, or that if $k = 2$ then it would take 2 hours to produce 2^α coconuts. The parameter b bounds the production set.

Let Crusoe's utility function be:

$$u (h, y) = h^{1-\beta} y^\beta \tag{44}$$

where $\beta \in (0,1)$. Here, h denotes the amount of hours of leisure that Crusoe consumes and y is the amount of coconuts. Assume Crusoe has $T > 0$ units of time and 0 coconuts, so that his endowment is $e = (T, 0)$. Assume that $b > T$ so that we do not have to worry about a corner solution where $h = b \leq T$.

Let $p > 0$ be the price of coconuts and $w > 0$ be the price per hour of Crusoe's time. Crusoe's budget constraint is:

$$py + wh \leq wT + \pi \tag{45}$$

where π is the profit of the firm. Start with finding the firm's output supply function and input demand function. We know that $y = k^{\alpha}$ because the firm does not want to pay for hours that are unproductive. The firm chooses $k \geq 0$ to maximize:

$$py - wk \tag{46}$$
$$pk^{\alpha} - wk \tag{47}$$

If $\alpha < 1$ then we will not have $h = 0$ as a profit-maximizing vector (we will have an interior solution). Finding the firm's first-order condition we have:

$$\alpha p k^{\alpha-1} - w = 0 \tag{48}$$
$$k^{\alpha-1} = \frac{w}{\alpha p} \tag{49}$$
$$k = \left(\frac{w}{\alpha p}\right)^{\frac{1}{\alpha-1}} \tag{50}$$
$$k = \left(\frac{\alpha p}{w}\right)^{\frac{1}{1-\alpha}} \tag{51}$$

Since $y = k^{\alpha}$, we have:

$$y = \left(\frac{\alpha p}{w}\right)^{\frac{\alpha}{1-\alpha}} \tag{52}$$

Firm profits are:

$$\pi(w,p) = p\left(\frac{\alpha p}{w}\right)^{\frac{\alpha}{1-\alpha}} - w\left(\frac{\alpha p}{w}\right)^{\frac{1}{1-\alpha}} \tag{53}$$

We can simplify this to:

$$\pi(w,p) = p\left(\frac{\alpha p}{w}\right)^{\frac{\alpha}{1-\alpha}} - w\left(\frac{\alpha p}{w}\right)^{\frac{1}{1-\alpha}} \tag{54}$$
$$\pi(w,p) = p^{\frac{1}{1-\alpha}}\frac{w}{\alpha}\left(\frac{\alpha}{w}\right)^{\frac{1}{1-\alpha}} - w\frac{\alpha}{\alpha}\left(\frac{\alpha p}{w}\right)^{\frac{1}{1-\alpha}} \tag{55}$$
$$\pi(w,p) = \frac{w}{\alpha}\left(\frac{\alpha p}{w}\right)^{\frac{1}{1-\alpha}} - w\frac{\alpha}{\alpha}\left(\frac{\alpha p}{w}\right)^{\frac{1}{1-\alpha}} \tag{56}$$
$$\pi(w,p) = w\left(\frac{\alpha p}{w}\right)^{\frac{1}{1-\alpha}}\left(\frac{1}{\alpha} - \frac{\alpha}{\alpha}\right) \tag{57}$$
$$\pi(w,p) = \left(\frac{1-\alpha}{\alpha}\right) w \left(\frac{\alpha p}{w}\right)^{\frac{1}{1-\alpha}} \tag{58}$$

so that if $p > 0$ and $w > 0$ profits we will have $\pi(p,w) > 0$.

Now look at the consumer's problem. Crusoe's budget constraint is now:

$$py + wh \leq wT + \pi(w,p) \tag{59}$$

With the utility function $u(y,h) = h^{1-\beta}y^{\beta}$, we have the following problem:

$$\max_{y \geq 0, h \geq 0} h^{1-\beta}y^{\beta} \text{ s.t. } py + wh = wT + \pi(w,p) \tag{60}$$

Solving for y in the budget constraint (since it holds with equality), we have:

$$y = \frac{wT + \pi(w,p) - wh}{p} \tag{61}$$

so that the consumer now simply maximizes utility which is a function of h:

$$\max_{h \geq 0} \; h^{1-\beta} \left(\frac{wT + \pi(w,p) - wh}{p} \right)^{\beta} \tag{62}$$

The first-order condition (which holds with equality because the consumer will choose a positive amount of h and y) is:

$$\begin{aligned}
(1-\beta) h^{1-\beta-1} \left(\frac{wT + \pi(w,p) - wh}{p} \right)^{\beta} + h^{1-\beta} \beta \left(\frac{wT + \pi(w,p) - wh}{p} \right)^{\beta-1} \left(-\frac{w}{p} \right) &= 0 \qquad (63) \\
(1-\beta) h^{-\beta} \left(\frac{wT + \pi(w,p) - wh}{p} \right) + h^{1-\beta} \beta \left(-\frac{w}{p} \right) &= 0 \qquad (64) \\
(1-\beta) h^{-\beta} (wT + \pi(w,p) - wh) - h^{1-\beta} \beta w &= 0 \qquad (65) \\
(1-\beta) h^{-\beta} (wT + \pi(w,p) - wh) &= h^{1-\beta} \beta w \qquad (66) \\
(1-\beta)(wT + \pi(w,p) - wh) &= \frac{h^{1-\beta} \beta w}{h^{-\beta}} \qquad (67) \\
(1-\beta)(wT + \pi(w,p) - wh) &= h \beta w \qquad (68) \\
(1-\beta)(wT + \pi(w,p)) - (1-\beta) wh &= h \beta w \qquad (69) \\
(1-\beta)(wT + \pi(w,p)) &= h \beta w + (1-\beta) wh \qquad (70) \\
(1-\beta)(wT + \pi(w,p)) &= hw (\beta + 1 - \beta) \qquad (71) \\
(1-\beta)(wT + \pi(w,p)) &= hw \qquad (72) \\
\frac{(1-\beta)(wT + \pi(w,p))}{w} &= h \qquad (73)
\end{aligned}$$

So that we have:

$$\begin{aligned}
y &= \frac{wT + \pi(w,p) - wh}{p} \qquad (74) \\
y &= \frac{wT + \pi(w,p) - w \left(\frac{(1-\beta)(wT + \pi(w,p))}{w} \right)}{p} \qquad (75) \\
py &= wT + \pi(w,p) - w \left(\frac{(1-\beta)(wT + \pi(w,p))}{w} \right) \qquad (76) \\
py &= wT + \pi(w,p) - (1-\beta)(wT + \pi(w,p)) \qquad (77) \\
py &= \beta (wT + \pi(w,p)) \qquad (78) \\
y &= \frac{\beta (wT + \pi(w,p))}{p} \qquad (79)
\end{aligned}$$

Now, we need to use the market equilibrium condition to find the price vector. But we can impose $p^* = 1$ because we can normalize one price and then we only need to solve one market clearing condition to find w^*. We want to find:

$$\begin{aligned}
h + k &= T \qquad (80) \\
\frac{(1-\beta)(wT + \pi(w,p))}{w} + \left(\frac{\alpha p}{w} \right)^{\frac{1}{1-\alpha}} &= T \qquad (81) \\
\frac{(1-\beta)(wT + \pi(w,1))}{w} + \left(\frac{\alpha * 1}{w} \right)^{\frac{1}{1-\alpha}} &= T \qquad (82) \\
\frac{(1-\beta)(wT + \pi(w,1))}{w} + \left(\frac{\alpha * 1}{w} \right)^{\frac{1}{1-\alpha}} &= T \qquad (83)
\end{aligned}$$

Now we have:

$$\pi(p,w) = \left(\frac{1-\alpha}{\alpha}\right) w \left(\frac{\alpha p}{w}\right)^{\frac{1}{1-\alpha}} \tag{84}$$

$$\pi(1,w) = \left(\frac{1-\alpha}{\alpha}\right) w \left(\frac{\alpha}{w}\right)^{\frac{1}{1-\alpha}} \tag{85}$$

so:

$$\frac{(1-\beta)\left(wT + \left(\frac{1-\alpha}{\alpha}\right) w \left(\frac{\alpha}{w}\right)^{\frac{1}{1-\alpha}}\right)}{w} + \left(\frac{\alpha}{w}\right)^{\frac{1}{1-\alpha}} = T \tag{86}$$

$$\frac{(1-\beta) wT}{w} + \frac{(1-\beta)\left(\frac{1-\alpha}{\alpha}\right) w \left(\frac{\alpha}{w}\right)^{\frac{1}{1-\alpha}}}{w} + \left(\frac{\alpha}{w}\right)^{\frac{1}{1-\alpha}} = T \tag{87}$$

$$(1-\beta) T + (1-\beta)\left(\frac{1-\alpha}{\alpha}\right)\left(\frac{\alpha}{w}\right)^{\frac{1}{1-\alpha}} + \left(\frac{\alpha}{w}\right)^{\frac{1}{1-\alpha}} = T \tag{88}$$

$$(1-\beta)\left(\frac{1-\alpha}{\alpha}\right)\left(\frac{\alpha}{w}\right)^{\frac{1}{1-\alpha}} + \left(\frac{\alpha}{w}\right)^{\frac{1}{1-\alpha}} = \beta T \tag{89}$$

$$\left(\frac{\alpha}{w}\right)^{\frac{1}{1-\alpha}}\left(\frac{(1-\beta)(1-\alpha)}{\alpha} + 1\right) = \beta T \tag{90}$$

$$\left(\frac{\alpha}{w}\right)^{\frac{1}{1-\alpha}}\left(\frac{(1-\beta)(1-\alpha)}{\alpha} + \frac{\alpha}{\alpha}\right) = \beta T \tag{91}$$

$$\left(\frac{\alpha}{w}\right)^{\frac{1}{1-\alpha}}\left(\frac{(1-\beta)(1-\alpha)+\alpha}{\alpha}\right) = \beta T \tag{92}$$

$$\left(\frac{\alpha}{w}\right)^{\frac{1}{1-\alpha}}\left(\frac{1-\beta(1-\alpha)}{\alpha}\right) = \beta T \tag{93}$$

$$\left(\frac{\alpha}{w}\right)^{\frac{1}{1-\alpha}} = \frac{\alpha\beta T}{1-\beta(1-\alpha)} \tag{94}$$

$$\frac{\alpha}{w} = \left(\frac{\alpha\beta T}{1-\beta(1-\alpha)}\right)^{1-\alpha} \tag{95}$$

$$\frac{w}{\alpha} = \left(\frac{1-\beta(1-\alpha)}{\alpha\beta T}\right)^{1-\alpha} \tag{96}$$

$$w = \alpha\left(\frac{1-\beta(1-\alpha)}{\alpha\beta T}\right)^{1-\alpha} \tag{97}$$

Note that $w > 0$ because (1) $0 < (1-\alpha) < 1$ so that (2) $0 < \beta(1-\alpha) < 1$ so that (3) $0 < 1-\beta(1-\alpha) < 1$.

Checking to make sure that the market for h clears:

$$
\begin{aligned}
(1-\beta)\left(\frac{1-\alpha}{\alpha}\right)\left(\frac{\alpha}{\alpha\left(\frac{1-\beta(1-\alpha)}{\alpha\beta T}\right)^{1-\alpha}}\right)^{\frac{1}{1-\alpha}}+\left(\frac{\alpha}{\alpha\left(\frac{1-\beta(1-\alpha)}{\alpha\beta T}\right)^{1-\alpha}}\right)^{\frac{1}{1-\alpha}} &= \beta T \qquad (98)\\
(1-\beta)\left(\frac{1-\alpha}{\alpha}\right)\left(\frac{1}{\left(\frac{1-\beta(1-\alpha)}{\alpha\beta T}\right)^{1-\alpha}}\right)^{\frac{1}{1-\alpha}}+\left(\frac{1}{\left(\frac{1-\beta(1-\alpha)}{\alpha\beta T}\right)^{1-\alpha}}\right)^{\frac{1}{1-\alpha}} &= \beta T \qquad (99)\\
(1-\beta)\left(\frac{1-\alpha}{\alpha}\right)\left(\frac{\alpha\beta T}{1-\beta(1-\alpha)}\right)+\left(\frac{\alpha\beta T}{1-\beta(1-\alpha)}\right) &= \beta T \qquad (100)\\
(1-\beta)(1-\alpha)\frac{\beta T}{1-\beta(1-\alpha)}+\frac{\alpha\beta T}{1-\beta(1-\alpha)} &= \beta T \qquad (101)\\
(1-\beta)(1-\alpha)\frac{T}{1-\beta(1-\alpha)}+\frac{\alpha T}{1-\beta(1-\alpha)} &= T \qquad (102)\\
\frac{(1-\beta-\alpha+\alpha\beta)T+\alpha T}{1-\beta+\alpha\beta} &= T \qquad (103)\\
\frac{(1-\beta+\alpha\beta)T}{1-\beta+\alpha\beta} &= T \qquad (104)\\
T &= T \qquad (105)
\end{aligned}
$$

So that we have an equilibrium price vector of:

$$
(w^*,p^*)=\left(\alpha\left(\frac{1-\beta(1-\alpha)}{\alpha\beta T}\right)^{1-\alpha},1\right) \qquad (106)
$$

We can then find equilibrium demands, equilibrium output supply, input demand, and profit by plugging in the prices.

Example with numbers:

Consider the same example from above only we will use $\alpha=\frac{2}{3}$, $\beta=\frac{1}{2}$, and $T=1$. We want to find the price vector (p^*,w^*) and the allocations $\left(k^*,y^{s*},y^{d*},h^*,\pi^*\right)$. Setting up the firm's profit function we have:

$$
\begin{aligned}
\pi &= py-wk \qquad (107)\\
&\pi=pk^{\frac{2}{3}}-wk \qquad (108)\\
\frac{\partial\pi}{\partial k} &= p\frac{2}{3}k^{-\frac{1}{3}}-w \qquad (109)\\
0 &= p\frac{2}{3}k^{-\frac{1}{3}}-w \qquad (110)\\
\frac{3}{2}\frac{w}{p} &= k^{-\frac{1}{3}} \qquad (111)\\
\frac{27}{8}\frac{w^3}{p^3} &= k^{-1} \qquad (112)\\
k^* &= \frac{8p^3}{27w^3} \qquad (113)
\end{aligned}
$$

Then we have that the supply of coconuts is:

$$\begin{aligned} y &= k^{\frac{2}{3}} && (114)\\ y &= \left(\frac{8p^3}{27w^3}\right)^{2/3} && (115)\\ y &= \left(\frac{2p}{3w}\right)^2 && (116)\\ y^* &= \frac{4p^2}{9w^2} && (117) \end{aligned}$$

So that firm profits are:

$$\begin{aligned} \pi &= p\frac{4p^2}{9w^2} - w\frac{8p^3}{27w^3} && (118)\\ \pi &= \frac{4p^3}{9w^2} - \frac{8p^3}{27w^2} && (119)\\ \pi &= \frac{12p^3}{27w^2} - \frac{8p^3}{27w^2} && (120)\\ \pi^* &= \frac{4p^3}{27w^2} && (121) \end{aligned}$$

Now we set up the consumer's utility maximization problem:

$$\max_{y\geq 0, h\geq 0} \; h^{1/2}y^{1/2} \text{ s.t. } py + wh = w + \frac{4p^3}{27w^2} \tag{122}$$

Solving for y (recall that this is y^d, the amount of coconuts demanded) in the budget constraint (since it holds with equality), we have:

$$\begin{aligned} y &= \frac{w + \frac{4p^3}{27w^2} - wh}{p} && (123)\\ y &= \frac{w - wh}{p} + \frac{4p^2}{27w^2} && (124) \end{aligned}$$

so that the consumer now simply maximizes utility which is a function of h:

$$\max_{h\geq 0} \; h^{1/2}\left(\frac{w - wh}{p} + \frac{4p^2}{27w^2}\right)^{1/2} \tag{125}$$

The first order condition is:

$$\frac{du}{dh} = \frac{1}{2}h^{-1/2}\left(\frac{w-wh}{p}+\frac{4p^2}{27w^2}\right)^{1/2} + h^{1/2}\frac{1}{2}\left(\frac{w-wh}{p}+\frac{4p^2}{27w^2}\right)^{-1/2}\left(-\frac{w}{p}\right) \tag{126}$$

$$0 = \frac{1}{2}h^{-1/2}\left(\frac{w-wh}{p}+\frac{4p^2}{27w^2}\right)^{1/2} + h^{1/2}\frac{1}{2}\left(\frac{w-wh}{p}+\frac{4p^2}{27w^2}\right)^{-1/2}\left(-\frac{w}{p}\right) \tag{127}$$

$$0 = h^{-1/2}\left(\frac{w-wh}{p}+\frac{4p^2}{27w^2}\right)^{1/2} - \frac{w}{p}h^{1/2}\left(\frac{w-wh}{p}+\frac{4p^2}{27w^2}\right)^{-1/2} \tag{128}$$

$$\frac{w}{p}h^{1/2}\left(\frac{w-wh}{p}+\frac{4p^2}{27w^2}\right)^{-1/2} = h^{-1/2}\left(\frac{w-wh}{p}+\frac{4p^2}{27w^2}\right)^{1/2} \tag{129}$$

$$\frac{w}{p}h = \frac{w-wh}{p}+\frac{4p^2}{27w^2} \tag{130}$$

$$\frac{w}{p}h - \frac{4p^2}{27w^2} = \frac{w-wh}{p} \tag{131}$$

$$wh - \frac{4p^3}{27w^2} = w - wh \tag{132}$$

$$2wh = w + \frac{4p^3}{27w^2} \tag{133}$$

$$h^* = \frac{1}{2} + \frac{2p^3}{27w^3} \tag{134}$$

So we now know the consumer's amount of h. We can then find the consumer's amount of y:

$$y = \frac{w - w\left(\frac{1}{2}+\frac{2p^3}{27w^3}\right)}{p} + \frac{4p^2}{27w^2} \tag{135}$$

$$y = \frac{w}{p} - \frac{w}{p}\left(\frac{1}{2}+\frac{2p^3}{27w^3}\right) + \frac{4p^2}{27w^2} \tag{136}$$

$$y = \frac{w}{2p} - \frac{2p^2}{27w^2} + \frac{4p^2}{27w^2} \tag{137}$$

$$y^* = \frac{w}{2p} + \frac{2p^2}{27w^2} \tag{138}$$

We can then check the market equilibrium condition for hours since:

$$h + k = T \tag{139}$$

$$\frac{1}{2} + \frac{2p^3}{27w^3} + \frac{8p^3}{27w^3} = 1 \tag{140}$$

We can normalize the price $p = 1$ so that:

$$\frac{1}{2} + \frac{2}{27w^3} + \frac{8}{27w^3} = 1 \tag{141}$$

$$\frac{10}{27w^3} = \frac{1}{2} \tag{142}$$

$$\frac{20}{27} = w^3 \tag{143}$$

$$\sqrt[3]{\frac{20}{27}} = w \tag{144}$$

$$\frac{\sqrt[3]{20}}{3} = w \tag{145}$$

So that the equilibrium price vector is:

$$(w^*, p^*) = \left(\frac{\sqrt[3]{20}}{3}, 1 \right) \tag{146}$$

We then have the following for the firm:

$$k^* = \frac{8p^3}{27w^3} = \frac{8*1}{27*\left(\frac{\sqrt[3]{20}}{3}\right)^3} = \frac{8}{20} = 0.4 \tag{147}$$

$$y^* = \frac{4p^2}{9w^2} = \frac{4*1}{9*\left(\frac{\sqrt[3]{20}}{3}\right)^2} = \frac{4}{20^{2/3}} \approx 0.543 \tag{148}$$

$$\pi^* = \frac{4p^3}{27w^2} = \frac{4*1}{27*\left(\frac{\sqrt[3]{20}}{3}\right)^2} = \frac{4}{3*20^{2/3}} \approx 0.181 \tag{149}$$

For the consumer we have:

$$h^* = \frac{1}{2} + \frac{2p^3}{27w^3} = \frac{1}{2} + \frac{2*1}{27*\left(\frac{\sqrt[3]{20}}{3}\right)^3} = \frac{10}{20} + \frac{2}{20} = \frac{12}{20} = 0.6 \tag{150}$$

$$\begin{aligned} y^* &= \frac{w}{2p} + \frac{2p^2}{27w^2} = \frac{\frac{\sqrt[3]{20}}{3}}{2*1} + \frac{2*1}{27*\left(\frac{\sqrt[3]{20}}{3}\right)^2} = \frac{\sqrt[3]{20}}{6} + \frac{2}{3*\left(\sqrt[3]{20}\right)^2} \\ &= \frac{\sqrt[3]{20}}{6} + \frac{2}{3*\left(\sqrt[3]{20}\right)^2} = \frac{\sqrt[3]{20}*\left(\sqrt[3]{20}\right)^2+4}{6*\left(\sqrt[3]{20}\right)^2} = \frac{24}{6*\left(\sqrt[3]{20}\right)^2} = \frac{4}{20^{2/3}} = 0.543 \end{aligned} \tag{151}$$

We can see from the numerical example that at $(p^*, w^*) = \left(1, \frac{\sqrt[3]{20}}{3}\right)$, $y^{s*} = y^{d*}$and $h^* + k^* = T$.

4.4 Welfare in production

We will not go through these in detail since they are the same basic theorems as before.

Definition 20 *Let $p^* >> 0$ be a Walrasian equilibrium for the economy $\left(u^i, e^i, \theta^{ij}, Y^j\right)_{i\in I j\in J}$. Then the pair $(x(p^*), y(p^*))$ is a Walrasian equilibrium allocation where $x(p^*)$ denotes the vector $\left(x^1, x^2, ..., x^I\right)$, whose ith entry is the utility-maximizing bundle demanded by consumer i facing prices p^* and income $m^i(p^*)$; and where $y(p^*)$ denotes the vector, $\left(y^1, y^2, ..., y^J\right)$, of profit-maximizing production vectors at prices p^*.*

This definition simply defines the Walrasian equilibrium allocation where all consumers are maximizing utility and all firms are maximizing profit. Now we define a Pareto efficient allocation:

Definition 21 *The feasible allocation (x, y) is Pareto-efficient if there is no other feasible allocation $(\overline{x}, \overline{y})$ such that $u^i\left(\overline{x}^i\right) \geq u^i\left(x^i\right)$ for all $i \in I$ with at least one strict inequality.*

We can now show that the WEA is Pareto-efficient.

Theorem 22 *(First Welfare Theorem with Production) If each u^i is continuous and strictly increasing on $\mathbb{R}^n_+$, then every Walrasian equilibrium allocation is Pareto-efficient.*

Now a restatement of the Second Welfare Theorem

Theorem 23 *Suppose that (i) each u^i is continuous, strongly increasing, and strictly quasiconcave and (ii) each Y^j is has profits that are bounded from below by zero, output requires some inputs, the production set is closed, bounded, and has strong convexity, (iii) $y + \sum_i e^i >> 0$ for some aggregate production vector y, and (iv) the allocation $(\widehat{x}, \widehat{y})$ is Pareto-efficient.*

Then there are income transfers, $T_1, ..., T_I$, satisfying $\sum_i T_i = 0$, and a price vector $\overline{p}$ such that:

1. *$\widehat{x}^i$ maximizes $u^i\left(x^i\right)$ s.t. $\overline{p} \cdot x^i \leq m^i(\overline{p}) + T_i$, $i \in I$*
2. *$\widehat{y}^j$ maximizes $\overline{p} \cdot y^j$ s.t. $y^j \in Y^j$, $j \in J$*