Simultaneous games of complete information*

1 Introduction

Why study theory? There are two reasons that I can think of to study theory. The most important is that theory provides a way for a researcher to determine the assumptions he or she views as important, reduce the model of behavior to those assumptions, and then make predictions about behavior *that follow logically* from those assumptions. The key is that predictions follow logically from the assumptions (hence why it is emphasized) – anyone can make statements about what might occur, and just about all of those things may actually come true at some point in time, but the point is to understand what behavior drives us to those predictions. It is completely possible that the researcher neglects one or two important aspects, but the theory can always be revised to incorporate new assumptions. The second reason is that "everyone else does it". For better or worse, the language of economics is basically theoretical and mathematical economics (although that may be changing – slowly). Thus, in order to communicate with other economists it is important to understand that in which they believe. Also, in order to effectively criticize a theoretical model you need to be able to understand it.[1] Simply making a statement like "Well, that's just theory" is not going to win you an argument with a theorist, nor should it.

Now, why study game theory? Most of standard economic theory assumes that individual actors are non-strategic in the sense that their actions have little to no effect on others actions. For example, consider the classic case of a consumer in a market. The consumer has a budget constraint and a utility function, and sees a set of prices, and then chooses the affordable bundle of goods that provides the highest utility. For large numbers of consumers or firms this non-strategic view may be true, but what about when there are only small numbers of either firms or consumers? A classic example is oligopoly theory, which was basically the theory of the kinked demand curve prior to the introduction of game theory (recall in the movie "A Beautiful Mind" that when John Nash is being informed about his receiving the Nobel Prize that the informer tells him his equilibrium concept has been used in antitrust cases). The idea is that when agents are allowed to make strategic decisions, and other agents know that the other agents will be strategic, predictions may change.

2 A normal (or matrix or strategic) form game

The classic example of a simultaneous game with complete information is the Prisoner's Dilemma.

> Two prisoners have been charged with a crime. Each prisoner is strictly interested in his or her own well-being, and less time in jail is better than more time in jail (6 months is better than 10 months). The prisoners are isolated in different holding cells where they will be questioned by the DA. The DA comes in and makes an offer to Prisoner 1. The DA says that if Prisoner 1

*Based on Chapter 1 of Gibbons (1992).

[1] You should also attempt to understand *why* the theory was developed. For instance, a lot of people criticize general equilibrium models without understanding why general equilibrium was developed in the first place. Walras and later Arrow and Debreu and some others I am forgetting set out with showing that markets can be efficient, and provided a nice set of axioms to show the conditions under which this is true. I do not believe any of them ever argued that all the conditions were true in the real world, just that if these conditions held then markets would be efficient. If you think about this in the context of the time, this is extremely important given that the prevailing thought was that some central planner was needed to coordinate activity (check out Samuelson's prediction about how the Soviet Union was supposed to surpass the US in some of the early editions of his text).

confesses and Prisoner 2 confesses that both will be sentenced to 6 months in jail. However, if neither prisoner confesses, then both of you will do the minimum amount of time and be out in 2 months. But if Prisoner 1 confesses and Prisoner 2 does not confess then Prisoner 1 can go home free (spend zero months in jail) and Prisoner 2 gets to spend 10 months in jail. The DA also tells you that Prisoner 2 is offered the same plea bargain and that if he/she confesses and you don't, then you will be sentenced to 10 months in jail and Prisoner 2 will go home free. What should the prisoners do?

Before solving the game we should identify the elements of the game. There are two players, Prisoner 1 and Prisoner 2 (technically there is a third player, the DA, but that player is passive in the sense that his action to make the offer has already been determined). According to the rules of the game, there are only 2 actions that each player can take, "Confess" or "Not Confess". There are 4 outcomes – both prisoners confess, both prisoners do not confess, or one prisoner confesses and the other does not. The payoffs to each of these outcomes are, respectively, $(-6,-6)$, $(-2,-2)$, $(-10,0)$ (when Prisoner 2 confesses but Prisoner 1 does not), and $(0,-10)$ (when Prisoner 1 confesses but Prisoner 2 does not). Now, to analyze a game of this type we can use the matrix (or bi-matrix or normal or strategic) form of the game. I will provide the picture of the normal form of the game and then describe its construction.

		Prisoner 2	
		Confess	Not Confess
Prisoner 1	Confess	$-6,-6$	$0,-10$
	Not Confess	$-10,0$	$-2,-2$

We call Prisoner 1 the row player and list his possible strategies (in this game each action is a strategy for the game) along the rows. Prisoner 2 is the column player and his actions are listed along the columns. Note that in this game both prisoners have the same number of strategies as well as the same strategies, but neither of these need be true. There are 4 outcome cells. In the "Confess, Confess" outcome cell are the numbers -6 and -6, representing the payoff to Prisoner 1 and Prisoner 2 respectively. Each of the other outcome cells also has a pair of payoffs listed in it. It is important to note that the convention is to list the row player's payoff on the left (or first). So all you need to know to create the normal form of a game is who the player's are, what their strategies are, and what the payoffs associated with the play of those strategies are.

Before we get any further, we should formally define some concepts. Each player i has a strategy space, S_i, which is the set of all strategies available to player i. For the prisoners, $S_i = \{Confess, Not\ Confess\}$. Let s_i denote an individual strategy for player i, so that $s_i = Confess$ or $s_i = Not\ Confess$ for the prisoners. We can also write that $s_i \in S_i$. While in this game there is no difference between a strategy and an action, in other games this is not true. Technically, strategies are sets of feasible actions. Consider the game of Chess. On the opening move there are 20 actions that White can take. On White's next move there are a lot more actions that he can take, depending on the action he took at his first move and what exactly Black did. However, White's *strategy* for the game is a complete contingent plan for EVERY POSSIBLE scenario he might encounter. So his strategy might include moving his Kingside pawn on his first move. But then he would also have to determine what actions he will take on his second move for each of the 20 moves that the Black player may make. So after each player moves once White's strategy already has 21 pieces of information (his opening move as well as what his move would be for each of the 20 moves by Black). The number of actions that White has at this stage is ... well, more than 21.

Consider I players in a game. Then $(s_1, s_2, ..., s_I)$ denotes a combination of strategies for each of the I players. This leads to an outcome. Let $u_i(s_1, s_2, ..., s_I)$ be player i's payoff (or utility) function. Formally, we can define a game as (notation is slightly different from Gibbons below – the Γ_N notation that I use is quite common in the economics literature on game theory, owing to the fact that the primary PhD Microeconomics text uses Γ_N as the symbol for a normal form game):

Definition 1 *Let Γ_N denote the normal form representation of a game. A normal form game $\Gamma_N = [I, \{S_i\}, \{u_i\}]$ where $\{S_i\}$ is the set of strategy spaces for all I players and $\{u_i\}$ is the set of payoff functions for all I players.*

2.1 Solving a normal form game

We still have not answered what the players should do. One way to determine what the players will do is to analyze what each player would do if the other player chose one of his strategies. For instance, what would Prisoner 1 choose if Prisoner 2 chose "Confess"? If Prisoner 1's payoff is accurately measured by the payoff in the cell, Prisoner 1 would choose "Confess" because $-6 > -10$. If Prisoner 2 chose "Not Confess", Prisoner 1 would still choose "Confess" because $0 > -10$. Thus, we (as well as Prisoner 2) can see that regardless of what Prisoner 2 chooses Prisoner 1 will choose "Confess". When a strategy for player i does at least as well against all the strategies of the other player j than another strategy for player i we say that the strategy is a **weakly dominant strategy**.

Definition 2 *A strategy $s_i \in S_i$ is a weakly dominant strategy for player i in game $\Gamma_N = [I, \{S_i\}, \{u_i\}]$ if for all $s_i' \neq s_i$ we have:*

$$u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i}) \text{ for all } s_{-i} \in S_{-i}.$$

The term s_{-i} refers to the strategies of the other players in the game besides player i. The strategy is strictly dominant if the inequality is strict (if $u_i(s_i, s_{-i}) > u_i(s_i', s_{-i})$ for all $s_{-i} \in S_{-i}$). In the Prisoner's Dilemma, "Confess" is a strictly dominant strategy for both players. So one potential way to solve the game is to look for weakly or strictly dominant strategies. If a player has a strictly dominant or weakly dominant strategy, the player should play it. Unfortunately, not many games have strictly or weakly dominant strategies.

Another possibility is to look for strategies that do strictly worse than other strategies. For instance, "Not Confess" does strictly worse than "Confess" for both prisoners (hopefully this is obvious since "Confess" is a dominant strategy). Strategies that do strictly worse than other strategies are called **strictly dominated strategies**.

Definition 3 *A strategy $s_i \in S_i$ is strictly dominated for player i in game $\Gamma_N = [I, \{S_i\}, \{u_i\}]$ if there exists another strategy $s_i' \in S_i$ such that for all $s_{-i} \in S_{-i}$*

$$u_i(s_i', s_{-i}) > u_i(s_i, s_{-i}).$$

A strategy is said to be weakly dominated if $u_i(s_i', s_{-i}) \geq u_i(s_i, s_{-i})$. Note that while strictly dominated strategies should never be played, it may be possible for weakly dominated strategies to be part of the solution to the game since they are not strictly worse than another strategy. Thus, another possible way to look for a solution to a game is to remove any strategies that are strictly dominated and then see if any of the strategies for the other player become strictly dominant. Here is an example:

		Player 1		
		Left	Center	Right
	Top	$7,4$	$6,3$	$4,11$
Player 2	Middle	$8,8$	$10,4$	$6,7$
	Bottom	$18,7$	$11,9$	$4,6$

Note that there are no dominant strategies for either player. However, "Top" is strictly dominated by "Middle" for Player 2. So we know that Player 2 will never choose top and so we can eliminate it from the game. The game is now:

		Player 1		
		Left	Center	Right
Player 2	Middle	$8,8$	$10,4$	$6,7$
	Bottom	$18,7$	$11,9$	$4,6$

Once "Top" is removed the strategy "Right" is strictly dominated by the strategy "Left" for Player 1. Note that this was not the case when "Top" was still considered by Player 2. So now we eliminate "Right" and get:

		Player 1	
		Left	Center
Player 2	Middle	$8,8$	$10,4$
	Bottom	$18,7$	$11,9$

We can now see that "Bottom" is a strictly dominant strategy for Player 2, and Player 2 will use this strategy. Also, we can say that "Middle" is strictly dominated by "Bottom" and eliminate "Middle". This leaves:

		Player 1	
		Left	Center
Player 2	Bottom	$18, 7$	$11, 9$

It is quite obvious that Player 1 will choose "Center" since $9 > 7$. Thus, the solution to this game is Player 2 chooses "Bottom" and Player 1 chooses "Center". This method of eliminating strictly dominated strategies is known as **iterated elimination of dominated strategies (IEDS)**.

Again, while IEDS is a useful tool for solving games it is not the case that all games have strictly dominated strategies. Consider the following game

> Two people wish to attend either a boxing match or an opera. Unfortunately, they have lost their cell phones and all other devices that allow for communication. If they both go to the boxing match, then Player 1 receives a payoff of 2 and Player 2 receives a payoff of 1. If they both go to the opera, then Player 1 receives a payoff of 1 and Player 2 receives a payoff of 2. However, if they show up at either event and the other person is not there they are deeply saddened and receive a payoff of 0.

		Player 2	
		Boxing	Opera
Player 1	Boxing	$2, 1$	$0, 0$
	Opera	$0, 0$	$1, 2$

Note that in this game (called a coordination game or the Battle of the Sexes) neither player has a strictly (or even weakly) dominant strategy. Nor does either player have a strictly dominated strategy. How do we solve this game? We can check each outcome cell to see if either player would choose a different strategy. If we are in the "Boxing, Boxing" outcome cell, then Player 1 does not want to choose "Opera" because this would give Player 1 a payoff of 0, which is less than the payoff of 2 that Player 1 receives from choosing "Boxing" when Player 2 chooses "Boxing". The same is true for Player 2, since Player 2's payoff of 1 is greater than the 0 he would get by switching. So "Boxing, Boxing" is a solution to the game.

We found a solution, can we stop there? No, we must check ALL the outcome cells. It is easy to see that if the players are at "Boxing, Opera" or "Opera, Boxing" that either player would want to switch strategies. However, if both players choose "Opera" then neither player wishes to switch. Thus, "Opera, Opera" is ALSO a solution to the game. So it is possible for one game to have 2 solutions (technically there is a third solution which we will discuss later in the course).

The underlying characteristic of all these solutions is that once an outcome is reached in which neither player can unilaterally change strategies and make him or herself better off then that outcome (or set of strategies) is a solution to the game. Basically, we have achieved equilibrium since no one can choose differently to make himself better off. In game theory, we call this set of strategies a **Nash equilibrium**, after John Nash. Please note that a Nash equilibrium is a SET OF STRATEGIES.

Definition 4 *A strategy profile $s = (s_1, ..., s_I)$ constitutes a Nash Equilibrium of game $\Gamma_N = [I, \{S_i\}, \{u_i\}]$ if for every $i = 1, ..., I$,*

$$u_i (s_i, s_{-i}) \geq u_i (s'_i, s_{-i})$$

for all $s'_i \in S_i$.

Take a look at the Prisoner's Dilemma. If both players choose "Confess" (which is each player's dominant strategy), neither player can *unilaterally* choose a different strategy and make himself better off. It is certainly the case that *both* players could choose different strategies and make themselves better off (if both choose "Not Confess" both will be better off than if both choose "Confess"), but if one prisoner chooses "Not Confess" then what will the other prisoner want to do? The other prisoner will want to choose "Confess" because $0 < -2$ – this leads us right back to "Confess, Confess".

Another useful way to define a Nash equilibrium is by using the concept of a **best response correspondence**.

Definition 5 *In game $\Gamma_N = [I, \{S_i\}, \{u_i\}]$, strategy s_i is a best response for player i to his rivals' strategies s_{-i} if*

$$u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i}) \text{ for all } s_i' \in S_i$$

Strategy s_i is never a best response if there is no s_{-i} for which s_i is a best response.

So a best response correspondence only focuses on one player's best response to the other player's strategies. Essentially, we hold the strategies of the other player(s) fixed and then determine what the original player in question would do in each case. Let's consider the Battle of the Sexes game again. If Player 2 were to choose "Boxing", Player 1's best response would be to choose "Boxing". If Player 2 were to choose "Opera", then Player 1's best response would be to choose "Opera". In the normal form version of the game we can draw a geometric shape around the best response payoffs like so:

		Player 2	
		Boxing	Opera
Player 1	Boxing	$\boxed{2}, 1$	$0, 0$
	Opera	$0, 0$	$\boxed{1}, 2$

We can do the same for Player 2:

		Player 2	
		Boxing	Opera
Player 1	Boxing	$\boxed{2}, \boxed{1}$	$0, 0$
	Opera	$0, 0$	$\boxed{1}, \boxed{2}$

Any of the outcome cells where both (or all if there are more than 2 players) payoffs are enclosed is a Nash equilibrium of the game. Note that this is consistent with what we found earlier. We will denote a player's best response correspondence by $b_i(\cdot)$. A definition of Nash equilibrium using the best response concept is:

Definition 6 *A strategy profile $(s_1, ..., s_I)$ is a Nash Equilibrium of game $\Gamma_N = [I, \{S_i\}, \{u_i(\cdot)\}]$ if and only if $s_i \in b_i(s_{-i})$ for $i = 1, ..., I$.*

Thus, a set of strategies is a Nash Equilibrium if and only if the strategies are best responses to one another.

2.2 Existence and uniqueness

When forming models in economics our primary focus is on some notion of equilibrium. There are two questions we primarily ask about equilibrium:

1. Does an equilibrium exist?

2. Is that equilibrium unique?

   These questions arose primarily in response to partial and general equilibrium concepts, and not in response to game theoretic solution strategies. Nonetheless, they are still important questions to ask. Given some fairly reasonable assumptions we can show that an equilibrium exists but it is difficult to prove uniqueness. In fact, game theory suffers from "an embarrassment of riches", in that in some games (primarily those with repeated interactions, which are discussed in the next set of notes) many outcomes can be supported as an equilibrium to the game. Thus, game theory provides a third question:

3. How is one equilibrium selected over another?

This is an interesting question and one for which evolutionary game theory may provide an answer. We may discuss some of the basics of evolutionary game theory a little later in the course. Most of economic theory is concerned with static predictions (many economic models discuss the concept of being at an equilibrium and there is no discussion of how the economic agents came to be at that equilibrium), not with dynamic predictions. However, it was recognized over 100 years ago that economics may have more in common with biology and physics.[2] For now we turn to existence of equilibrium.

Theorem 7 *In the I-player normal form game $\Gamma_N = [I, \{S_i\}, \{u_i\}]$, if I is finite and S_i is finite for every i, then there exists at least one Nash equilibrium.*

So we basically have two conditions needed for existence of a Nash equilibrium. The number of players needs to be finite and the number of strategies each player has also needs to be finite. These seem like reasonable assumptions.

Here is a more advanced (and useful when showing a picture) version of the theorem:

Theorem 8 *A Nash Equilibrium exists in game $\Gamma_N = [I, \{S_i\}, \{u_i\}]$ if for all $i = 1, ..., I$*

1. *S_i is a nonempty, convex, and compact subset of some Euclidean space $\mathbb{R}^M$*
2. *$u_i(s_1, ..., s_I)$ is continuous in $(s_1, ..., s_I)$ and quasiconcave in s_i*

For the first part think of a strategy space that is something like the closed interval from $[0, 1]$. Alternatively, think about a firm determining how much of a good to produce. The lowest amount they can produce is 0, while the largest amount (call it $\overline{C}$) they can produce is constrained by their available technology, the amount of money they can spend, and the prices of inputs (for simplicity we assume that firms can produce any real number between 0 and $\overline{C}$). For the second part, think about the fact that there are no "large jumps" in payoffs when moving from one strategy to another that is "close" to it. The idea relies on a fixed-point theorem. An overview of a fixed-point theorem is that if we take a function $f(x)$ with domain and range of $[0, 1]$ then there exists at least one fixed-point, which is a point where $f(x^*) = x^*$. In the case of the games we are discussing, there are points in the best response correspondences of players that map back into themselves. Please note that neither version of the theorem makes any statement about uniqueness of the equilibrium.

3 Rock, Paper, Scissors

Consider the classic game of Rock, Paper, Scissors. In this game there are 2 players who simultaneously determine which object to form with their fingers. Each player has 3 strategies – form a Rock, form Paper, or form Scissors. If both players form the same object then they tie and receive 0. If one player forms a Rock and the other forms Scissors then Rock wins and receives a payoff of 1 while Scissors loses and receives a payoff of -1. If one player forms Scissors and the other forms Paper then Scissors wins and receives a payoff of 1 while Paper loses and receives a payoff of -1. If one player forms Paper and the other forms Rock then Paper wins and receives a payoff of 1 and Rock loses and receives a payoff of -1. Essentially, Rock smashes Scissors, Scissors cut Paper, and Paper covers Rock. The normal form version of the game is:

		Player 2		
		Rock	Paper	Scissors
	Rock	$0, 0$	$-1, 1$	$1, -1$
Player 1	Paper	$1, -1$	$0, 0$	$-1, 1$
	Scissors	$-1, 1$	$1, -1$	$0, 0$

Note that neither player has a dominant strategy nor a dominated strategy. We can then look at the best response correspondences for the players by enclosing the payoffs in a square.

		Player 2		
		Rock	Paper	Scissors
	Rock	$0, 0$	$-1, \boxed{1}$	$\boxed{1}, -1$
Player 1	Paper	$\boxed{1}, -1$	$0, 0$	$-1, \boxed{1}$
	Scissors	$-1, \boxed{1}$	$\boxed{1}, -1$	$0, 0$

[2] Alfred Marshall, who wrote THE principles of economics text used in the early 20^{th} century, made claims of this type. You can read about Marshall and his link to biology in two articles of the Economic Journal. A piece by Geoff Hodgson appears in the March 1993 issue (pgs. 406-415) and a piece by John Foster appears in the July 1993 issue (pgs. 975-991).

There is now a problem. Nash's theory says that an equilibrium exists if there are a finite number of players and if each of those players has a finite number of strategies. There are 2 players (fairly finite) and each player has 3 strategies (also fairly finite). But there are no outcome cells that have both payoffs enclosed. Is Nash wrong?

3.1 Mixed strategies

Think about what type of strategy you would use if you were playing this game repeatedly with a friend. Would you always use the same strategy? If you were Player 1 and you always played Rock after a few rounds your friend may just start playing Paper.[3] It might be in your best interest to mix your play of strategies by assigning some probability distribution over your available strategies. Note that games where players have dominant strategies are simply a special case of a mixed strategy. In the Prisoner's Dilemma, Prisoner 1 assigns a probability of 1 to playing "Confess" and a probability of 0 to playing "Not Confess". The same is true for Prisoner 2. When a player chooses a strategy with probability 1 we call this a pure strategy. The question then becomes how a player determines which probability to assign to each strategy.

To determine which probability to assign to each strategy we need a criterion (or set of criteria) that tells us what the players are attempting to do. When determining the probabilities for the mixed strategy Nash equilibrium the goal is to make the other player indifferent over ANY of his pure strategies (this will also make the other player indifferent over his mixed strategies). For instance, we know that Player 1 should not always choose Rock. However, suppose Player 1 chooses Rock 50% of the time, Paper 25% of the time, and Scissors 25% of the time. What should Player 2 do? Well, if Player 2 always chooses Paper (or chooses paper with a probability of 1 or 100%) then Player 2 will win 50% of the time and tie 25% of the time. That's better than choosing Rock all the time for Player 2 (25% wins, 50% ties, and 25% losses) and better than choosing Scissors all the time (25% wins, 25% ties, and 50% losses). So Player 1 choosing Rock 50% of the time, Paper 25% of the time, and Scissors 25% of the time CANNOT be a mixed strategy Nash equilibrium (MSNE). How then do we determine the probabilities?

Let $p1_{rock}$ be the probability that Player 1 plays Rock, $p1_{paper}$ be the probability that Player 1 plays Paper, and $p1_{scissors}$ be the probability that Player 1 plays Scissors. Let $p2_{rock}$, $p2_{paper}$, and $p2_{scissors}$ be the respective probabilities for Player 2. Player 1's goal is to make Player 2 indifferent among his pure strategies. We know that if Player 1 uses his mixed strategy and Player 2 ALWAYS chooses Rock, then Player 2 will receive 0 with probability $p1_{rock}$, -1 with probability $p1_{paper}$, and 1 with probability $p1_{scissors}$. If Player 2 ALWAYS chooses Paper, then Player 2 will receive 1 with probability $p1_{rock}$, 0 with probability $p1_{paper}$, and -1 with probability $p1_{scissors}$. If Player 2 ALWAYS chooses Scissors, then Player 2 will receive -1 with probability $p1_{rock}$, 1 with probability $p1_{paper}$, and 0 with probability $p1_{scissors}$. We can say that the expected value for Player 2 of playing each of these strategies is then:

$$\begin{aligned} E_2[Rock] &= 0 * p1_{rock} + (-1) * p1_{paper} + 1 * p1_{scissors} \\ E_2[Paper] &= 1 * p1_{rock} + 0 * p1_{paper} + (-1) * p1_{scissors} \\ E_2[Scissors] &= (-1) * p1_{rock} + 1 * p1_{paper} + 0 * p1_{scissors} \end{aligned}$$

We now have 3 unknowns – $p1_{rock}$, $p1_{paper}$, and $p1_{scissors}$. It must be that Player 2 has:

$$\begin{aligned} E_2[Rock] &= E_2[Paper] \\ E_2[Paper] &= E_2[Scissors] \end{aligned}$$

By transitivity, this gives that $E_2[Rock] = E_2[Scissors]$. However, there are only 2 equations and 3 unknowns. The third equation is that probabilities must sum to 1, so that our 3 equations are now:

$$\begin{aligned} 0 * p1_{rock} + (-1) * p1_{paper} + 1 * p1_{scissors} &= 1 * p1_{rock} + 0 * p1_{paper} + (-1) * p1_{scissors} \\ 1 * p1_{rock} + 0 * p1_{paper} + (-1) * p1_{scissors} &= (-1) * p1_{rock} + 1 * p1_{paper} + 0 * p1_{scissors}. \\ p1_{rock} + p1_{paper} + p1_{scissors} &= 1 \end{aligned}$$

[3] There is a scene in a Simpson's episode where Bart and Lisa play Rock, paper, scissors. Bart is thinking "Good old Rock never loses". Lisa is thinking, "Poor predictable Bart. He always picks Rock". There is a similar scene in an episode of That 70's Show where Fez is a little confused about the circular nature of the game. In essence, Fez is looking for a dominant strategy.

We can now solve the 3 equations for $p1_{rock}$, $p1_{paper}$, and $p1_{scissors}$. Rewrite $p1_{scissors} = 1 - p1_{rock} - p1_{paper}$ and substitute into the first two equations. We get:

$$\begin{aligned}(-1) * p1_{paper} + 1 * (1 - p1_{rock} - p1_{paper}) &= 1 * p1_{rock} + (-1) * (1 - p1_{rock} - p1_{paper}) \\ 1 * p1_{rock} + (-1) * (1 - p1_{rock} - p1_{paper}) &= (-1) * p1_{rock} + 1 * p1_{paper}\end{aligned}$$

Now it is just a simple matter of solving the system.

$$\begin{aligned}-p1_{paper} + 1 - p1_{rock} - p1_{paper} &= p1_{rock} - 1 + p1_{rock} + p1_{paper} \\ p1_{rock} - 1 + p1_{rock} + p1_{paper} &= -p1_{rock} + p1_{paper}\end{aligned}$$

Simplifying:

$$\begin{aligned}-3 * p1_{paper} + 2 &= 3 * p1_{rock} \\ 3 * p1_{rock} - 1 &= 0\end{aligned}$$

We have $p1_{rock} = \frac{1}{3}$ from the last equation. Substituting that into the first equation we get:

$$-3 * p1_{paper} + 2 = 3 * \frac{1}{3}$$

Solving for $p1_{paper}$ gives $p1_{paper} = \frac{1}{3}$. Now, using $p1_{scissors} = 1 - p1_{rock} - p1_{paper}$ we find that $p1_{scissors} = \frac{1}{3}$. So if Player 1 plays Rock $\frac{1}{3}$ of the time, Paper $\frac{1}{3}$ of the time, and Scissors $\frac{1}{3}$ of the time this will make Player 2 indifferent over his pure strategies. The expected value for Player 2 of playing Rock is 0, of playing Paper is 0, and of playing Scissors is 0. We can also check some mixed strategies for Player 2. If Player 2 plays Rock 50% of the time and Paper 50% of the time his expected value of playing that strategy is 0. If Player 2 plays Rock 50% of the time, Paper 25% of the time, and Scissors 25% of the time his expected value is 0. This is what is required when finding a MSNE.

Now, we have only found the probabilities for Player 1. We know that all strategies (pure or mixed) by Player 2 provide the same expected value when Player 1 chooses Rock, Paper, and Scissors $\frac{1}{3}$ of the time each. Can Player 2 then just choose any strategy? No, because not any strategy will make Player 1 indifferent over his strategies. We then have to find $p2_{rock}$, $p2_{paper}$, and $p2_{scissors}$ using the same methodology that we just used to find $p1_{rock}$, $p1_{paper}$, and $p2_{scissors}$. Luckily, in the Rock, Paper, Scissors game the two players have symmetric payoffs and strategies, so that the probabilities for Player 2 that make Player 1 indifferent between his strategies are $p2_{rock} = \frac{1}{3}$, $p2_{paper} = \frac{1}{3}$ and $p2_{scissors} = \frac{1}{3}$. The mixed strategy Nash equilibrium for this game (and the only Nash equilibrium of this game) is that Player 1 chooses Rock, Paper, and Scissors each with $\frac{1}{3}$ probability, and Player 2 chooses Rock, Paper, and Scissors each with $\frac{1}{3}$ probability. Note that the expected value for both players of playing this game is 0.

Now, suppose that Player 2 uses a different strategy, like $p2_{rock} = \frac{1}{3}$, $p2_{paper} = \frac{1}{2}$ and $p2_{scissors} = \frac{1}{6}$. Should Player 1 respond with the exact same strategy? NO. If Player 1 uses the exact same strategy as Player 2 then his expected value of using that strategy is also 0. Player 1 can do BETTER than 0 if he uses a strategy like "Always choose Scissors". If Player 1 uses this strategy against $p2_{rock} = \frac{1}{3}$, $p2_{paper} = \frac{1}{2}$ and $p2_{scissors} = \frac{1}{6}$ then Player 1 will have an expected value of $\frac{1}{6}$ because he will earn (-1) with $\frac{1}{3}$ probability (when Player 2 picks Rock), 1 with $\frac{1}{2}$ probability (when Player 2 picks Paper) and 0 with $\frac{1}{6}$ probability (when Player 2 chooses Scissors). Of course, if Player 1 always chooses Rock then Player 2 would always choose Paper. But then Player 1 would always choose Scissors. And the cycle would go on. The only time it stops is when $p1_{rock} = \frac{1}{3}$, $p1_{paper} = \frac{1}{3}$ and $p1_{scissors} = \frac{1}{3}$ and $p2_{rock} = \frac{1}{3}$, $p2_{paper} = \frac{1}{3}$ and $p2_{scissors} = \frac{1}{3}$.

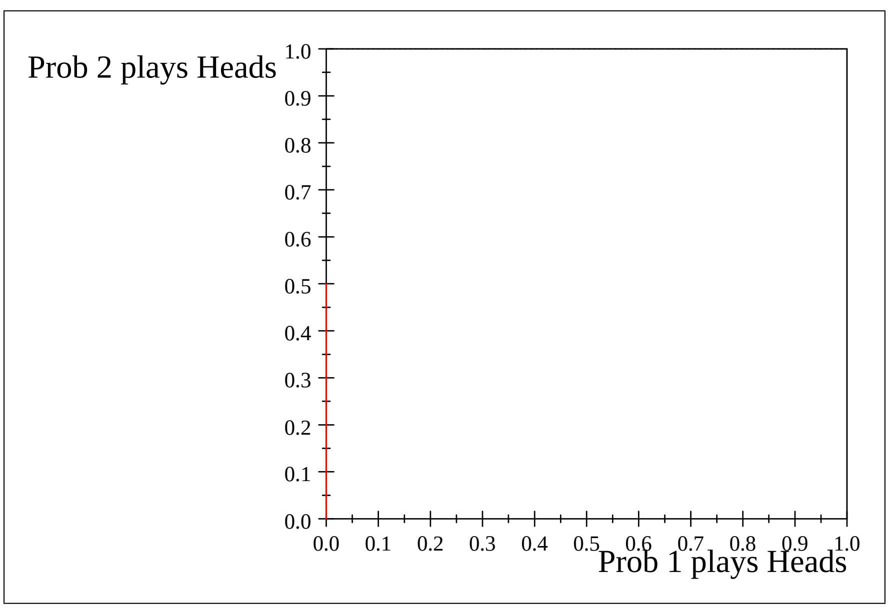
3.2 Matching Pennies

In this game there are two players who each have a penny. They simultaneously place the pennies on the counter. If the pennies match (meaning both pennies show heads OR both pennies show tails), then Player 1 wins $1 and Player 2 loses $1. If the pennies do not match (meaning one shows a head and the other shows a tail), then Player 2 wins $1 and Player 1 loses $1. The normal form of this game is:

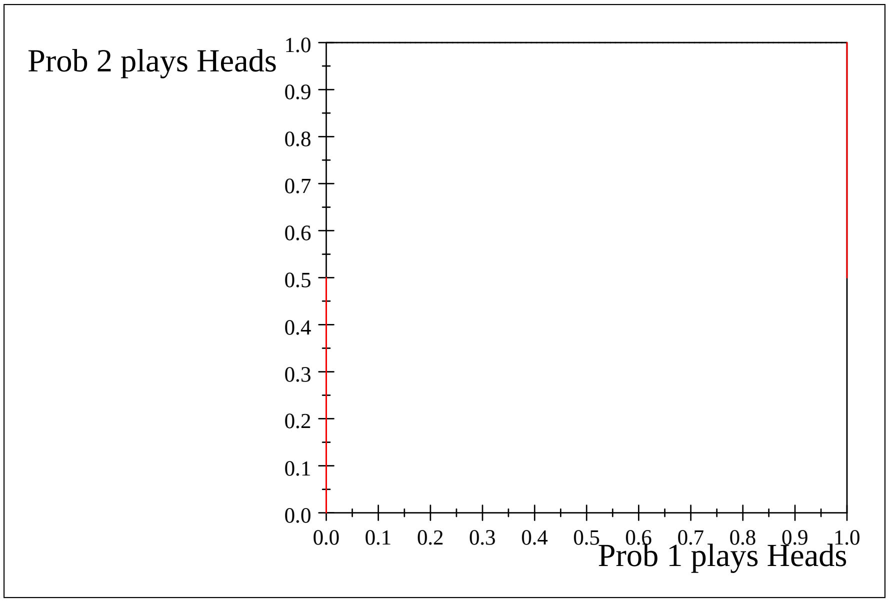
		Player 2	
		Heads	Tails
Player 1	Heads	$1, -1$	$-1, 1$
	Tails	$-1, 1$	$1, -1$

It is easy to show that there are no pure strategy Nash equilibria to this game. It is also easy to show that the mixed strategy Nash equilibrium to the game is for Player 1 to play Heads 50% of the time and play Tails 50% of the time, while Player 2 also plays Heads 50% of the time and Tails 50% of the time. These are simple enough to show, but the reason that I mention this game is to show the best response correspondences for this game.

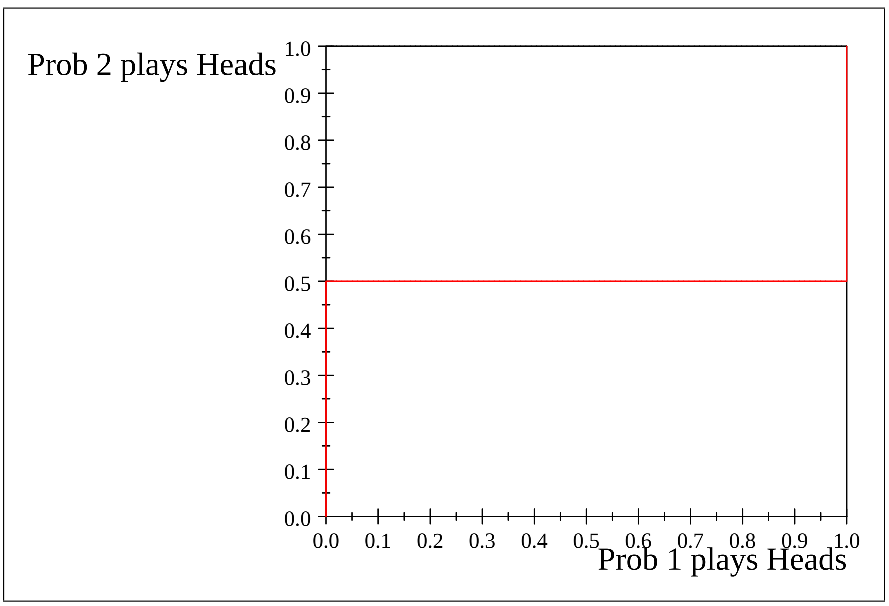
Let the x-axis represent the probability that Player 1 chooses Heads, so that the origin represents a choice (for Player 1) of playing Tails 100% of the time while the point $(1, 0)$ represents a choice (again for Player 1) of playing Heads 100% of the time. Let the unit interval $[0, 1]$ on the y-axis represent Player 2's choice of choosing Heads, so that the origin represents Player 2 choosing Tails 100% of the time and the point $(0, 1)$ represents Player 2 choosing Heads 100% of the time. Every point in the unit square now represents some outcome over the possible mixed (and pure) strategies that may be used by both players. What you should do now is think about the best responses (start for Player 1) to any mixed strategy choice of Player 2 and then plot them. Start with Player 2 choosing Tails 100% of the time. If Player 2 did this then Player 1 would choose Tails 100% of the time as well and win every single game (earning \$100 every 100 games). What if Player 2 chooses Tails 99% of the time and Heads 1% of the time? Player 1 should still choose Tails 100% of the time. In essence, Player 1 wins 99 games and loses 1, so he would earn \$98 per every \$100 games. It is hopefully clear that Player 1 should not choose Heads 100% of the time, because if he did this then he would win 1 game and lose 99 games, losing \$98 every \$100 games (on average). But what if Player 1 chose a mixed strategy like 50% Heads, 50% Tails while Player 2 chose 1% Heads, 99% Tails. Player 1 would win 50% of the games and lose the other 50%, so that he would earn (on average) \$0 over 100 games. Note that this is worse than playing Tails 100% of the time. Even if Player 1 used a strategy like 1% Heads and 99% Tails, he only wins 98.02% of the games, which is less than the 99% of the games he won playing Tails 100% of the time. So if Player 2 uses ANY mixed strategy that is weighted more heavily on Tails than Heads (even something like 49.5% Heads and 50.5% Tails), Player 1's best response is to play Tails 100% of the time. So we can fill this in as follows (red line is Player 1's best response):



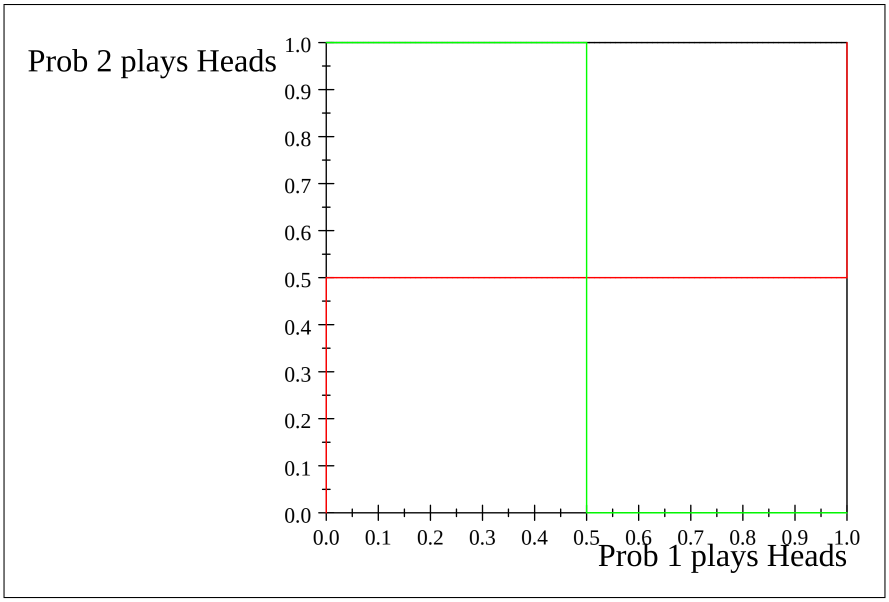
Now what if Player 2 chooses a mixed strategy that is more heavily weighted towards Heads (meaning some strategy in which he plays Heads more than 50% of the time)? Then Player 1's best response is to choose Heads 100% of the time. Here is this new portion of the best response for Player 1 added to the picture.



Now what if Player 2 uses 50% Heads and 50% Tails? Then it does not matter what strategy Player 1 uses – if Player 1 uses 100% Heads he wins 50% of the time; if Player 1 uses 100% Tails he wins 50% of the time; if Player 1 uses 50% Heads and 50% Tails he wins 50% of the time. In short, Player 1 is indifferent among all strategies if Player 2 uses a 50/50 mix of Heads/Tails. So we finish up Player 1's best response correspondence as:



Now we can find Player 2's best responses using the same logic. If Player 1 uses a mixed strategy that is weighted more towards Tails then Player 2 would choose 100% Heads. If Player 1 uses a mixed strategy that is weighted more towards Heads then Player 2 would use the 100% Tails strategy. If Player 1 chooses a 50/50 strategy, then Player 2 is indifferent among all of his strategies. Adding Player 2's best response correspondence we have:



These are the best response correspondences for Players 1 (in red) and 2 (in green) for the Matching Pennies game. Note that the intersect only one time, when both players use the mixed strategy 50% Heads and 50% Tails.

3.3 Boxing-Opera and MSNE

When we initially discussed the Boxing-Opera game in class I mentioned that there was a third NE. The first two are that Player 1 and Player 2 both choose Boxing, and that Player 1 and Player 2 both choose Opera. Here is the game again:

		Player 2	
		Boxing	Opera
Player 1	Boxing	$2,1$	$0,0$
	Opera	$0,0$	$1,2$

There is also a MSNE to this game. Let $p1_{boxing}$ and $p1_{opera}$ be the probabilities with which Player 1 chooses Boxing and Opera respectively. Let $p2_{boxing}$ and $p2_{opera}$ be the probabilities with which Player 2 chooses Boxing and Opera respectively. Player 1 must make Player 2 indifferent over his 2 pure strategies, so:

$$E_2\left[Boxing\right] = E_2\left[Opera\right]$$

Or:

$$1 * p1_{boxing} + 0 * p1_{opera} = 0 * p1_{boxing} + 2 * p1_{opera}$$

We also have that $p1_{boxing} + p1_{opera} = 1$. Using these two equations we find that:

$$p1_{boxing} = 2\left(1 - p1_{boxing}\right)$$

Or:

$$p1_{boxing} = \frac{2}{3}$$

This means that $p1_{opera} = \frac{1}{3}$. Now we need to find Player 2's mixed strategy. Note that while this game looks symmetric it is not. We need:

$$E_1\left[Boxing\right] = E_1\left[Opera\right]$$

Or:

$$2 * p2_{boxing} + 0 * p2_{opera} = 0 * p2_{boxing} + 1 * p2_{opera}.$$

Using $p2_{boxing} + p2_{opera} = 1$ we have:

$$2p2_{boxing} = 1 - p2_{boxing}.$$

Or:

$$p2_{boxing} = \frac{1}{3}.$$

This means that $p2_{opera} = \frac{2}{3}$. Thus, the MSNE for the Boxing-Opera game is that Player 1 chooses Boxing with $\frac{2}{3}$ probability and Opera with $\frac{1}{3}$ probability and Player 2 chooses Boxing with $\frac{1}{3}$ probability and Opera with $\frac{2}{3}$ probability. Note that the expected value of either player from playing this set of strategies is $\frac{2}{3}$, which is lower (for both players) then following either one of the two pure strategy Nash equilibria. But with these mixed strategies neither player can do any better by changing his strategy.

3.3.1 It's your opponents payoffs that matter

In the Boxing-Opera game it looks fairly reasonable – each player chooses the venue that he prefers with a higher probability than the other venue. But what if the game looked like:

		Player 2	
		Boxing	Opera
Player 1	Boxing	100, 1	0, 0
	Opera	0, 0	1, 2

In this game, Player 1 REALLY likes going to the Boxing match with Player 2. Would the probabilities that Player 1 used for his mixed strategy in the earlier version (where the 100 was a 2) change?

No.

Player 1's probabilities depend on Player 2's payoffs. They have nothing whatsoever to do with his own payoffs. In fact, we could turn the 100 into a $\frac{1}{2}$ and Player 1's probabilities would not change. But Player 2's probabilities would change because Player 1's payoffs had changed. In this new game (with the 100 payoff), Player 2's Nash equilibrium mixed strategy would be $p2_{boxing} = \frac{1}{101}$ and $p2_{opera} = \frac{100}{101}$. Because Player 1 has a larger payoff of going to the Boxing match this actually reduces the amount of times the 2 players end up at the boxing match.

3.3.2 Randomize

One last note on mixed strategies. If you only play a game that requires mixed strategies once in your life then it can never be shown that you did not correctly calculate the mixed strategy (there is only observation after all). However, consider playing Rock, Paper, Scissors repeatedly. It would be easy to devise a statistical test that determines whether or not your play is consistent with the probabilities of the MSNE. However, the key to using a MSNE is to *RANDOMIZE*. If you played 99 games of Rock, Paper, Scissors and you followed the strategy of "Choose Rock in the first game", "Choose Paper in the second game", and "Choose Scissors in the third game", then repeat (Rock in 4^{th}, 7^{th}, 10^{th} etc. games, Paper in 5^{th}, 8^{th}, 11^{th} etc. games, Scissors in 6^{th}, 9^{th}, 12^{th} etc. games) it would not be very difficult to beat you every time because the pattern is predictable. Even though you would be playing the strategies in the correct proportions you would not end up earning 0 on average if you were playing someone with any shred of intelligence. The person would beat you practically every time, and certainly every time after the 10^{th} round or so. Sometimes when ESPN shows the World Series of Poker they will cut away to the game of Rochambeau (which is the fancy way of saying Rock, Paper, Scissors). Some people when playing the game will use a randomizing device to determine their strategy for the game (one person in particular uses the first character of a dollar bill to determine what to do – I am still not quite certain what the mechanics of the process are). The key is to have the randomizing device yield the probabilities of $\frac{1}{3}$ for each strategy. Thus, throwing a single die and assigning the rolls of 1 and 2 to choosing Rock, 3 and 4 to choosing Paper, and 5 and 6 to choosing Scissors would be one method of randomizing with $\frac{1}{3}$ probability for each strategy.