

Dynamic games of complete information*

1 Introduction

We have discussed games where both players make choices simultaneously. In many games players make choices sequentially, so that one player observes the other player's decision and then gets to act. The classic example is the game of Chess. Considering some of the "economic" examples I mentioned, it may be that one firm observes another firm's quantity choice and then gets to choose how much it will produce, that one voter observes what other voters have done and then gets to cast a vote, or that one bidder in an auction can observe a bid by another bidder (EBay). We will begin by discussing the relevant parts of a game of this type and then proceed with a discussion on how to solve games of this type. Along the way we will make a refinement of the Nash equilibrium concept. This will be our first attempt to eliminate some of the equilibria from consideration.

2 Sequential Games

We call games where players take turns moving "sequential games". Sequential games consist of the same elements as normal form games – there are players, rules, outcomes, and payoffs. However, sequential games have the added element that history of play is now important as players can make decisions conditional on what other players have done. Thus, if two people are playing a game of Chess the second mover is able to observe the first mover's initial move prior to making his initial move. While it is possible to represent sequential games using the normal (or matrix) form representation of the game (in fact, we will discuss how to do this and why we might want to do this), it is more instructive at first to represent sequential games using a game tree. In addition to the players, actions, outcomes, and payoffs, the game tree will provide a history of play or a path of play. This will be important when discussing the refinement technique.

2.1 A Game Tree

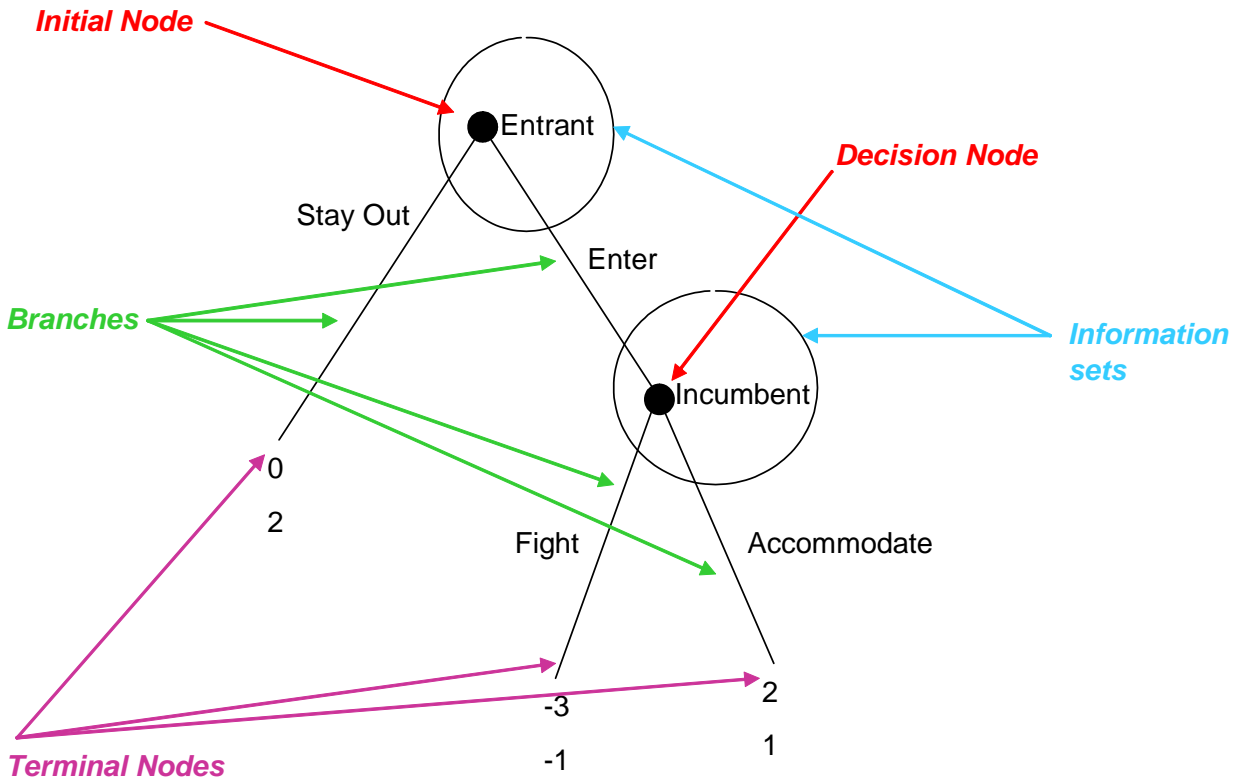
Game trees consist of the following pieces. There is an *initial node* to the game tree. This is the starting point of the game (think about the setup of the board when a game of Chess is begun – this is the initial node). From that initial node there are actions that the first mover can take. These actions are represented as *branches* to the game tree. At the end of each branch is a node. If the first mover makes a move and the game ends after that move, then we say that the game has reached a terminal node. A *terminal node* is a node at which no more actions can be taken. If the first mover makes a move and the second mover then gets to choose an action, we call this a *decision node*. The second mover's actions are then represented by branches extending from the decision node. The game tree extends until all the nodes are terminal nodes. At the terminal nodes, the payoffs to the players are listed. It is the convention to list the payoffs in the order that the players moved. One other important aspect of the game tree is the *information set*. For the games we will initially consider all decision nodes will also be information sets. However, it is possible that a game is being played and a player is uncertain as to which of a few decision nodes the player is at. In this case, the collection of decision nodes is that player's information set.

Consider a game where there is an entrant and an incumbent. The entrant moves first and the incumbent observes the entrant's decision. The entrant can choose to either enter the market

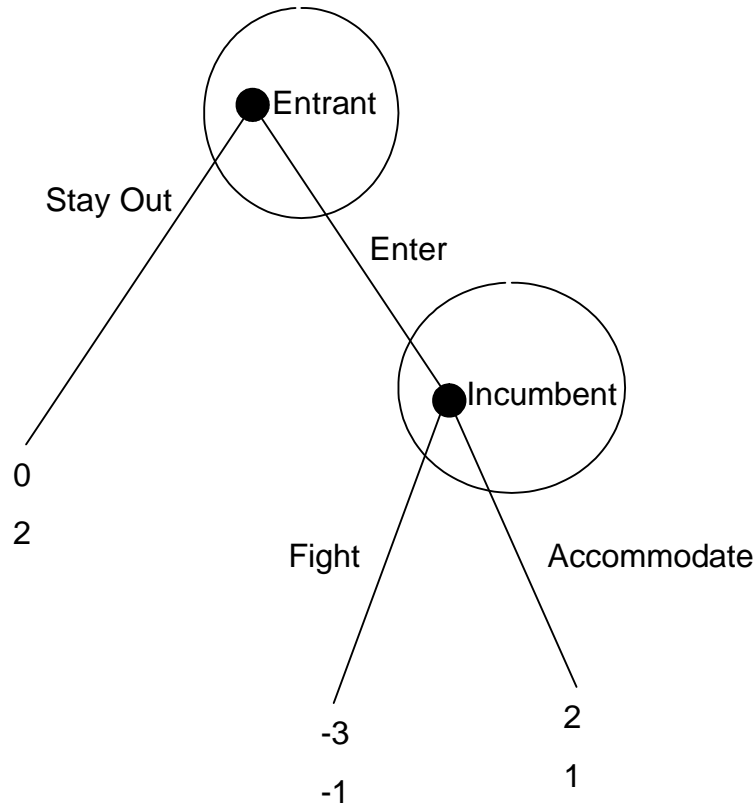
*Based on Chapter 2 of Gibbons (1992).

or remain out of the market. If the entrant remains out of the market then the game ends and the entrant receives a payoff of 0 while the incumbent receives a payoff of 2. If the entrant chooses to enter the market then the incumbent gets to make a choice. The incumbent chooses between fighting entry or accommodating entry. If the incumbent fights the entrant receives a payoff of -3 while the incumbent receives a payoff of -1 . If the incumbent accommodates the entrant receives a payoff of 2 while the incumbent receives a payoff of 1.

The ultimate goal is to solve this game, but first we can display it as a game tree (or in *extensive form*, which is the technical name for a game tree). I have two "versions" of the extensive form of the Entry Game. The first has labels for each of the components of the game (node, branch, information set) and the second is the actual game without all the components labeled.



Game tree with its components labeled.



Game tree without the components labeled.

Now, before discussing the refinement of the Nash equilibrium concept it is instructive to find the Nash equilibria to the normal form version of this game. Any extensive form game can be represented as a normal form game (and vice versa, although it is less instructive to represent a game that is truly simultaneous as an extensive form game). All we need to do to represent the extensive form game as a normal form game is to determine the strategies available to each player (hence why the normal form is also called the strategic form). In the Entry game the Entrant has two strategies, Stay Out or Enter. The Incumbent also has 2 strategies, Fight or Accommodate. So we can represent the extensive form of the Entry game as a 2x2 normal form game:

		Incumbent	
		Fight	Accommodate
Entrant	Enter	-3, -1	2, 1
	Stay Out	0, 2	0, 2

We can now find the pure strategy Nash equilibria (PSNE) to this game. The two PSNE to this game are that the Entrant chooses Enter and the Incumbent chooses Accommodate; and that the Entrant chooses Stay out and the Incumbent chooses Fight:

		Incumbent	
		Fight	Accommodate
Entrant	Enter	-3, -1	2, 1
	Stay Out	0, 2	0, 2

Note that both of these outcomes are Nash equilibria since they both have the property that neither player would wish to unilaterally change his strategy given what the other is doing. For completeness, there is also a mixed strategy Nash equilibrium where the Entrant plays Stay Out with probability 1 (note that the Incumbent is indifferent between Fight and Accommodate if the Entrant always plays Stay Out) and the Incumbent plays Fight with probability $\frac{2}{5}$ and Accommodate with probability $\frac{3}{5}$. The actual *payoffs* with the MSNE will be the same as the payoffs where the Incumbent always chooses Fight and the Entrant always chooses Stay Out, but the *strategies* are different (because the Incumbent is using a mixed strategy).

Now, the new question is whether either of these PSNE (we will ignore the MSNE) are more reasonable than the other when looking at the game as it is played sequentially. Consider the Stay Out, Fight equilibrium from the perspective of the Entrant. The Entrant chooses Stay Out because the incumbent is threatening to Fight if the Entrant should choose to Enter. This seems somewhat logical, but would the Incumbent really choose Fight if the Entrant chose Enter? Of course not, because in this game that is only played once the Incumbent does better if it chooses to Accommodate when the Entrant chooses Enter.¹ We can see this by noting that Enter, Accommodate is an equilibrium to this game. The Entrant, moving first, knows that if it chooses Enter the Incumbent will choose Accommodate. Thus, the Entrant should always choose Enter in the *sequential game* because it knows that the Incumbent will always follow a play of Enter with a play of Accommodate. The process of eliminating one (or some) of the Nash equilibria is formalized in the next section.

2.2 Subgame Perfect Nash Equilibrium (SPNE)

The Nash equilibrium refinement that we use in dynamic games of complete and perfect information is subgame perfection.² But first we must define a *subgame*. As the name implies, a subgame is part of a game, but a subgame has some particular features. A subgame must begin at an information set that contains a single decision node. The subgame must contain all the decision nodes and terminal nodes that follow that decision node, but no decision or terminal nodes that do not follow. A subgame cannot contain part of an information set (if an information set contains 2 or more nodes the subgame must contain all of the nodes in the information set). Technically, the entire game is a subgame (consider whether all of the conditions needed for a subgame are met using the initial node) but sometimes we want to focus on those subgames that are not the entire game. Thus, a subgame that is not the entire game is called a *proper subgame*. If there are x proper subgames in a particular game, then there will be $x + 1$ subgames (including the entire game). I point out the terminology so that you know the difference in the terms – it is not a major point, so do not get bogged down in it.

In the Entry game there is one proper subgame, that being the subgame that starts with the Incumbent's decision node. The idea behind the subgame perfection refinement is to look at the smallest (or last) subgame in the game that contains terminal nodes. The smallest subgame should contain no other subgames. In the Entry game there is only the one proper subgame so that is where we begin. We know that if the decision node where the Incumbent chooses to Fight or to Accommodate is ever reached the Incumbent will choose to Accommodate because the payoff to Accommodate is greater than the payoff to Fight (i.e. $1 > -3$). In effect, this “removes” Fight as a viable strategy for the Incumbent. Since the game is one of complete and perfect information, the Entrant knows that the Incumbent will never choose Fight if the Incumbent is forced to make a decision. Thus, the Entrant can also “remove” the Fight branch from the game. Now we move to the next subgame, which happens to be the initial node. If the Entrant chooses Stay Out it will earn a payoff of 0. If the Entrant chooses Enter it will earn a payoff of 2 because the Entrant knows the Incumbent will never choose Fight. Thus, the Entrant chooses Enter at the initial node since $2 > 0$. Thus, the Enter, Accommodate equilibrium is the one that “survives” this refinement process.

Definition 1 *A Nash equilibrium is subgame perfect if the players' strategies constitute a Nash equilibrium at every subgame.*

Note that the definition of subgame perfection requires that every player acts in an optimal manner at each subgame. Thus, the Stay Out, Fight equilibrium is NOT a subgame perfect Nash equilibrium because for the Incumbent choosing Fight is NOT a best response if it actually gets to make a decision. Subgame perfection rules out non-credible threats, which is exactly what choosing Fight is for the Incumbent. The only way that the Incumbent can get the Entrant to Stay Out is by threatening to Fight, but if the Entrant does Enter then the Incumbent no longer has any incentive to Fight, so the threat is non-credible.

¹We will eventually discuss games that are repeated but for now our focus is on the game in question, which is played only once.

²Complete information refers to knowing the other players' payoff functions – essentially, no player has any private information. You can think of an auction as a case where players have *incomplete information* because players do not necessarily know each others' values for an item. Perfect information refers to knowing exactly which decision node one is at in the game tree.

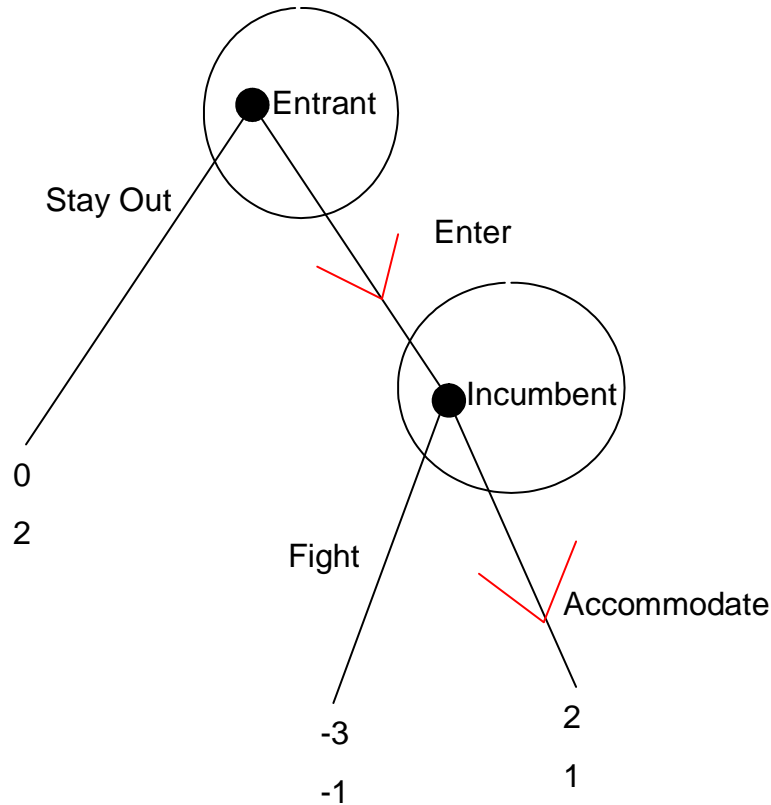


Figure 1: Illustration of SPNE of the Entry game.

The process we used, starting at the smallest (or last) subgame and working backwards by removing branches is known as backward induction. If backward induction is used in a game of complete and perfect information then the solution to the game found by using backward induction will be subgame perfect. This provides an easy method of determining the subgame perfect Nash equilibrium in games of this type. However, there may still be other Nash equilibria in the game which are NOT subgame perfect, as evidenced by the Stay Out, Fight Nash equilibria in the Entry game. One thing to remember is that all subgame perfect Nash equilibria are Nash equilibria (hence why they are called subgame perfect *Nash equilibria*), but not all Nash equilibria are subgame perfect. To denote the subgame perfect Nash equilibrium of the game we simply mark arrows down a player's action: This is just a technique used to recall which actions are chosen at the decision nodes.

As for existence of NE and SPNE in sequential games, we have two propositions.

Proposition 2 (*Zermelo's Theorem*) *Every finite game of perfect information has a pure strategy Nash Equilibrium that can be derived through backward induction. Moreover, if no player has the same payoffs at any two terminal nodes, then there is a unique Nash Equilibrium that can be derived in this manner.*

Proposition 3 *Every finite game of perfect information has a pure strategy subgame perfect Nash Equilibrium that can be derived through backward induction. Moreover, if no player has the same payoffs at any two terminal nodes, then there is a unique subgame perfect Nash Equilibrium that can be derived in this manner.*

Note that we need perfect information in order to have these propositions hold, not just common knowledge. Perfect information requires that all information sets contain a single decision node. Thus, the simultaneous move game of Rock, Paper, Scissors and the simultaneous move Prisoner's Dilemma game do not have perfect information, so there is no guarantee that there is a PSNE to either of those games (there might be though).

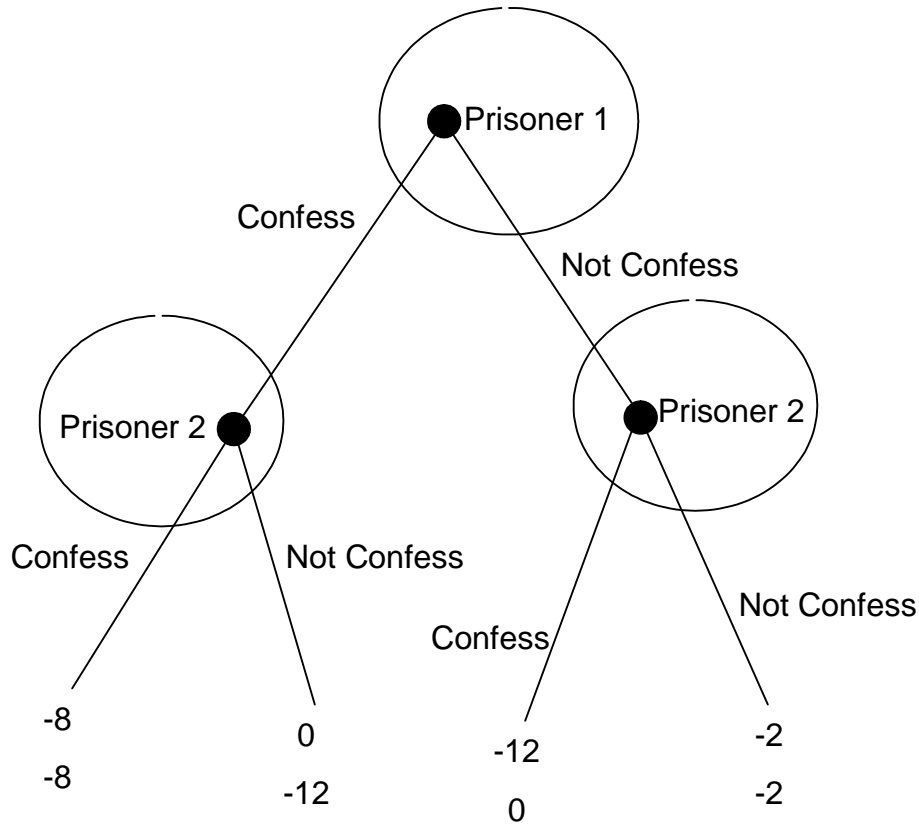


Figure 2: Sequential move Prisoner's Dilemma game.

2.2.1 The sequential Prisoner's Dilemma

The Entry game was a fairly simple game. Now consider the *sequential* Prisoner's Dilemma, which is only slightly more complicated but which will illustrate the difference between an *action* and a *strategy* (finally). The sequential Prisoner's Dilemma has the same format as the simultaneous Prisoner's Dilemma except that Prisoner 1 makes an observable decision to Confess or Not Confess prior to Prisoner 2 making a choice. The extensive form of this game is: The payoffs have been changed slightly from the initial game that was discussed but that is unimportant. Note that in this game there are 2 proper subgames and 3 total subgames (including the entire game). We can start at either of the two proper subgames. If Prisoner 2 is at the decision node where he has seen Prisoner 1 already Confess, what should Prisoner 2 do? Spending 8 months in prison is better than spending 12 months in prison, so Prisoner 2 should Confess if Prisoner 1 chooses Confess. We can eliminate the Not Confess branch extending from that node. Now, the next smallest subgame is the one where Prisoner 2 observes that Prisoner 1 chose Not Confess. If Prisoner 2 is at this node what should Prisoner 2 do? Well, spending 0 months in prison is better than spending 2 months in prison, so Prisoner 2 should Confess if Prisoner 1 chooses Not Confess. Now we move to the next subgame, which is the entire game. Both of Prisoner 2's Not Confess branches have been removed, which leaves Prisoner 1 with the choice of spending 8 months in Prison if he chooses Confess or 12 months in prison if he chooses Not Confess. Since spending 8 months in prison is better than spending 12 months in prison Prisoner 1 chooses Confess. Thus, the subgame perfect Nash equilibrium to this game is that Prisoner 1 chooses Confess and Prisoner 2 chooses Confess if Prisoner 1 chooses Confess and Confess if Prisoner 1 chooses Not Confess.

But wait a second, why do we need to state what Prisoner 2 will do if he sees Prisoner 1 choose Not Confess? We know that Prisoner 1 is choosing Confess, so why can't we just say that Prisoner 2 chooses Confess when Prisoner 1 chooses Confess? That is an excellent question, and it illustrates the difference

between a strategy and an action. At a decision node (technically, at an information set, but since all of our decision nodes are information sets right now they are the same thing – for now) each player must choose an action that he would take at that decision node. The collection of actions that a player takes at ALL of his decision nodes (technically, information sets) is the player’s strategy. A strategy is a *complete contingent* plan of action, so actions that are not along the equilibrium path must be specified. The *equilibrium path* shows the outcome reached using the chosen strategies by the players, but the actions taken off the equilibrium path must be specified. Why do they need to be specified? When Prisoner 1 is choosing his optimal strategy at the initial node he needs to know what Prisoner 2 will do at both nodes. If Prisoner 2 were to choose Not Confess for some reason when Prisoner 1 chose Not Confess then Prisoner 1 would want to choose Not Confess because spending 2 months in prison would be better than spending 8 months in prison. Thus the off equilibrium path actions are an integral part of the game.

Now, Prisoner 2 in this game has 4 strategies:

1. Confess if Prisoner 1 chooses Confess (C if P1 C) and Confess if Prisoner 1 chooses Not Confess (C if P1 NC)
2. Confess if Prisoner 1 chooses Confess (C if P1 C) and Not Confess if Prisoner 1 chooses Not Confess (NC if P1 NC)
3. Not Confess if Prisoner 1 chooses Confess (NC if P1 C) and Confess if Prisoner 1 chooses Not Confess (C if P1 NC)
4. Not Confess if Prisoner 1 chooses Confess (NC if P1 C) and Not Confess if Prisoner 1 chooses Not Confess (NC if P1 NC)

Prisoner 1 still only has 2 strategies, Confess or Not Confess. Writing this game in the normal (or strategic) form (note that because Prisoner 2 has 4 strategies it is easier to make Prisoner 2 the row player when typing the notes) and solving it:

		Prisoner 1	
		Confess	Not Confess
Prisoner 2	C if P1 C, C if P1 NC	8, 8	0, -12
	C if P1 C, NC if P1 NC	8, -8	-2, -2
	NC if P1 C, C if P1 NC	-12, 0	0, -12
	NC if P1 C, NC if P1 NC	-12, 0	-2, -2

There are now 8 outcome cells although each payoff is repeated twice. This is what happens in most extensive form games – there are more outcome cells than there are terminal nodes so payoffs end up being repeated. Note that while the outcome in this sequential Prisoner’s Dilemma is the same as the outcome in the simultaneous Prisoner’s Dilemma, neither player has a dominant strategy. This is because in the case where Prisoner 2 uses the 2nd strategy in the list Prisoner 1 would do better by choosing Not Confess. But Prisoner 2 is unlikely to use that strategy since it is weakly dominated by the 1st strategy in the list. You should also note that in this game there is only one pure strategy Nash equilibrium (PSNE) and that specific PSNE is also the subgame perfect Nash equilibrium (SPNE). Don’t be confused by the acronyms PSNE and SPNE.

3 Repeated Games

Another type of sequential game is a repeated game. Consider the *simultaneous* Prisoner’s Dilemma that is played multiple times by the same two players. One might think that if the players are playing this game repeatedly that the Not Confess, Not Confess outcome, which has strictly better payoffs for both prisoners than the Confess, Confess outcome, might be a more viable strategy than it is when the game is played only once.³ We will look at repeated games of two types – those that are repeated a finite amount of times and those that are repeated an infinite amount of times.

³One might also think that if the two players are repeatedly being picked up by the police and charged with crimes that the two players find new occupations.

3.1 Finitely Repeated Games

Some games may be played a finite amount of times. Consider the two prisoners in the Prisoner's Dilemma playing the game a known finite amount of times. Let's start with two times (the figure is kind of large and is probably on the next page): When solving this twice-repeated *simultaneous* game we find a subgame perfect Nash equilibrium that is very similar to the single shot simultaneous game.⁴ Regardless of which player is at which node that player should choose Confess.⁵ Well, that is of no help because the point of repeating the game was to try to find a different SPNE. Of course, we could try repeating the game 3 times, or 4 times, or 100 times, or 1062 times, but the fact of the matter is that no matter how many times the game is finitely repeated the subgame perfect Nash equilibrium to this game is still the same – both prisoners Confess whenever they get a chance to take an action.

Why is cooperation (meaning Not Confess in this game) not a viable strategy in this game? The problem is that once the end of the game is reached both players have an incentive to choose Confess since there are no more repetitions of the game. If there is no future to the game then there is no way to punish a noncooperator (someone who chooses Confess). Without the ability to punish (or be punished), there is no incentive to continue to cooperate in the last period of the game.

Why don't the prisoners cooperate until the last period? Consider it is the next to last period of the game. The prisoners know that in the last period both will choose Confess since there is no incentive to choose Not Confess in the last period. But now neither prisoner can punish the other in the last period by choosing Confess since they are both already choosing Confess (the only way to punish someone is by choosing Confess when he or she chooses Not Confess). So now there is no incentive to cooperate in the next to last period as there is no possibility of punishment in the last period. This process continues to the beginning of the game, and the game unravels. Now, it is not all games that unravel in this manner, but ones with strictly dominant strategies do. Thus, there is no way that rational players can escape playing the dominant strategy in a finitely repeated Prisoner's Dilemma.

Since this course will eventually deal with laboratory experiments, you should be aware that there is experimental evidence that strongly suggests that human subjects do NOT behave in this manner in finitely repeated games, particularly when there are more than 5-10 periods remaining in the game. Only when the game nears the end (say within the last 10 periods) does the unraveling typically occur. This leads to our next section.

3.2 Infinitely Repeated Games

One may ask why we bother to study infinitely repeated games since most players of games outside of Duncan MacLeod tend to expire at some point in time. There are a few good reasons to study infinitely repeated games. One is that not all players need to expire – consider a corporation. Corporations can be infinitely lived and they play many economic games. A second reason is that although we will all eventually expire the endpoint of the game is (hopefully) uncertain. There are results that show that games that are repeated finitely with an uncertain endpoint are consistent with games that are infinitely repeated. The experimental evidence on play in finitely repeated games suggests a third reason. The subgame perfect Nash equilibrium is a poor predictor of behavior in some of these repeated games. Studying infinitely repeated games allows for a different set of SPNE to be chosen. There is one major drawback to infinitely repeated games, and that is that multiple equilibria (even multiple SPNE) are bound to exist. Some view this as a problem, but it just shifts the focus from trying to show that an equilibrium exists to trying to show why one of the equilibria should be selected over another.

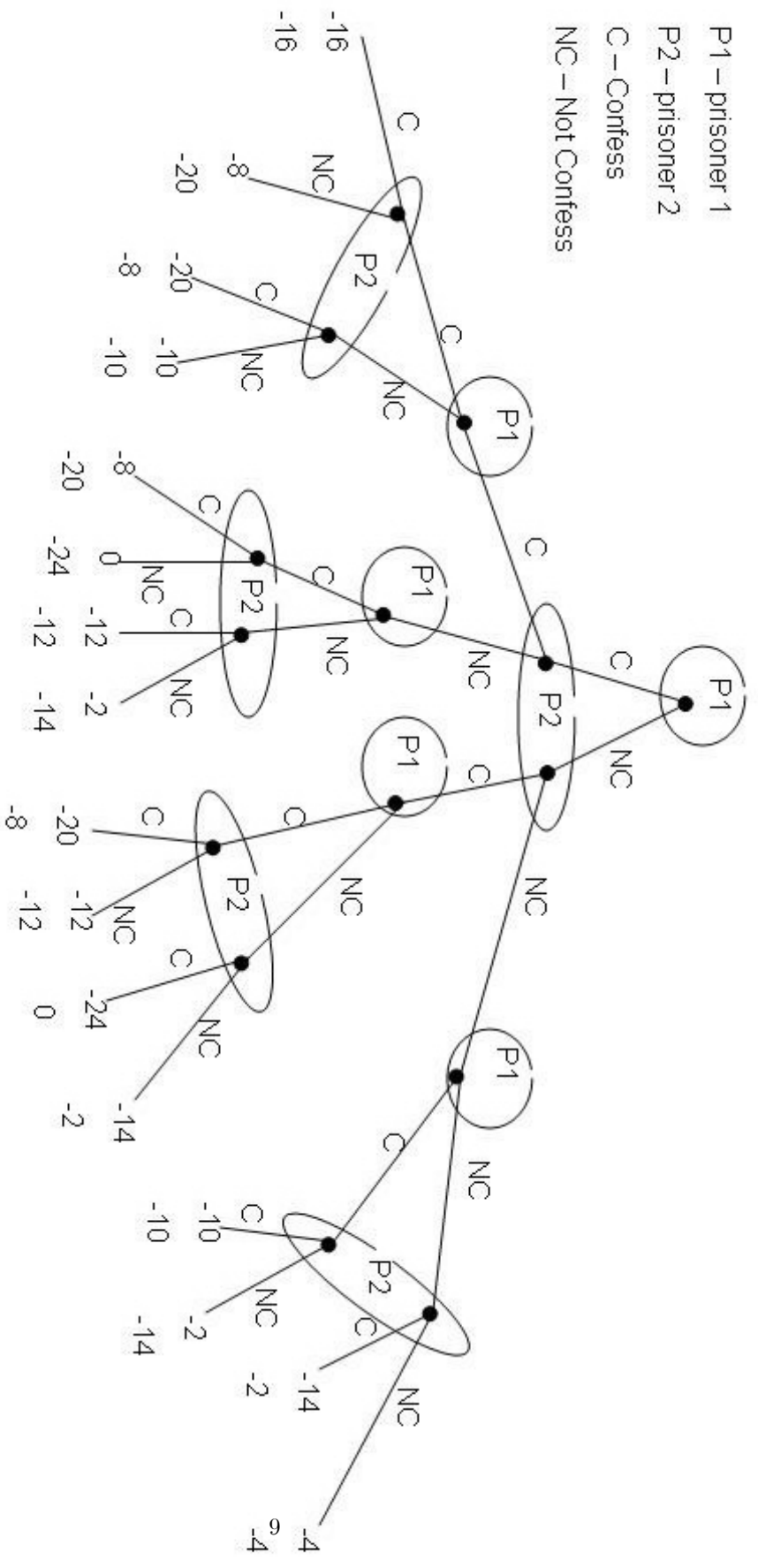
3.2.1 Evaluating strategies in infinite games

We will use a slight modification of the Prisoner's Dilemma, mainly so that we can get rid of the negative payoffs. Consider the following game:

⁴The normal form of this game is a 32x32 matrix – you can check that to find the other NE to the game. I have a hunch that there is more than just the one NE to the game.

⁵Technically, Prisoner 1's strategy is Confess at the initial node, Confess if he sees $\{Confess, Confess\}$ after the first repetition, Confess if he sees $\{Confess, Not Confess\}$ after the first repetition, Confess if he sees $\{Not Confess, Confess\}$ after the first repetition, and Confess if he sees $\{Not Confess, Not Confess\}$ after the first repetition.

P1 – prisoner 1
 P2 – prisoner 2
 C – Confess
 NC – Not Confess



		Player 2	
		Defect	Cooperate
Player 1	Defect	8, 8	32, 4
	Cooperate	4, 32	25, 25

Note that this game is essentially a Prisoner's Dilemma. Both players would be better off if they could both choose Cooperate instead of both choosing Defect, but Defect is a dominant strategy. In order to evaluate strategies in infinite games it will be necessary to add a particular parameter to the discussion. The parameter added will be the player's discount rate, δ . It is assumed that $\delta \in [0, 1)$, and that players have exponential discounting. All that exponential discounting means is that a payoff one time period from today is discounted at δ and a payoff two time periods from today is discounted at δ^2 , etc. Thus, a player's payoff stream from the infinite game would look like:

$$\delta^0 \Pi_0 + \delta^1 \Pi_1 + \delta^2 \Pi_2 + \delta^3 \Pi_3 + \dots$$

where Π_k denotes the player's payoff in each period k . The $\delta \in [0, 1)$ assumption will be justified shortly.⁶ It is typically assumed that players (and people in general) prefer \$1 today to \$1 tomorrow, and \$1 tomorrow to \$1 two days from now. Thus, the sooner a player receives a payoff the less he discounts it. Why add this discount rate? Well, if we do not have a discount rate then the players' payoffs from following ANY strategy (assuming that there are no negative payoffs that the player could incur) of an infinite game would be infinite. Well, that's not very interesting. This is also why we assume that $\delta < 1$ rather than $\delta \leq 1$. If $\delta = 1$, then a player weights all payoffs equally regardless of the time period, and this leads to an infinite payoff. If $\delta = 0$, then the player will only care about the current period. As δ moves closer to 1, the player places more weight on future periods. It is possible to motivate this discount rate from a present value context, which I believe would make $\delta = \frac{1}{1+r}$, where r is "the interest rate". Thus, if $r = 0.05$, then $\delta \approx 0.95$. All this says is that getting \$1 one period from today is like getting 95 cents today, and getting \$1 two periods from today is like getting 90.7 cents today. While this interpretation of the discount rate is the most closely linked to economic behavior, we will not assume that the discount rate is directly related to the interest rate, but that it is simply a parameter that states how players value payoffs over time.

Now, suppose that players 1 and 2 use the following strategies:

Player 1 chooses Cooperate in the initial period (at time $t = 0$) and continues to choose Cooperate at every decision node unless he observes that player 2 has chosen Defect. If Player 1 ever observes Player 2 choosing Defect then Player 1 will choose Defect at every decision node after that defection. Player 2's strategy is the same. These strategies call for Cooperation at every decision node until a Defection is observed and then Defection at every decision node after Defection is observed. Note that this is a *potential* SPNE because it is a set of strategies that specifies an action at every decision node of the game. The question then becomes whether or not this is a SPNE of the game. Recall that a strategy profile is an SPNE if and only if it specifies a NE at every subgame. Although each subgame of this game has a distinct history of play, all subgames have an identical structure. Each subgame is an infinite Prisoner's Dilemma exactly like the game as a whole. To show that these strategies are SPNE, we must show that after any previous history of play the strategies specified for the remainder of the game are NE.

Consider the following two possibilities:

1. A subgame that contains a deviation from the Cooperate, Cooperate outcome somewhere prior to the play of the subgame
2. A subgame that does not contain a deviation from the Cooperate, Cooperate outcome

If a subgame contains a deviation then the players will both choose Defect, Defect for the remainder of the game. Since this is the NE to the one-shot version (or stage game) of the Prisoner's Dilemma, it induces a NE at every subgame. Thus, the "Defect if defection has been observed" portion of the suggested strategy induces NE at every subgame.

Now, for the more difficult part. Suppose that the players are at a subgame where no previous defection has occurred. Consider the potential of deviation from the proposed strategy in period $\tau \geq t$, where t

⁶The exponential discounting assumption is used because it allows for time consistent preferences. Hyperbolic discounting is another type of discounting that has been suggested as consistent with choices made by individuals in experiments, although hyperbolic discounting does not necessarily lead to time consistent preferences.

is the current period. If player 2 chooses Defect in period τ he will earn $\delta^\tau \Pi^{Deviate} + \delta^\tau \sum_{i=1}^{\infty} \delta^i \Pi^D$ for the remainder of the game, where $\Pi^{Deviate}$ is player 2's payoff from deviating and Π^D is his payoff each period from the (Defect, Defect) outcome. If player 2 chooses to follow the proposed strategy, then he will earn $\delta^\tau \sum_{i=0}^{\infty} \delta^i \Pi^C$, where Π^C is his payoff from the (Cooperate, Cooperate) outcome. The question then becomes under what conditions will the payoff from deviating be greater than that from the payoff of following the proposed strategy. To find the condition simply set up the inequality:

$$\delta^\tau \Pi^{Deviate} + \delta^\tau \sum_{i=1}^{\infty} \delta^i \Pi^D \geq \delta^\tau \sum_{i=0}^{\infty} \delta^i \Pi^C$$

We can cancel out the δ^τ terms to obtain:⁷

$$\Pi^{Deviate} + \sum_{i=1}^{\infty} \delta^i \Pi^D \geq \sum_{i=0}^{\infty} \delta^i \Pi^C$$

Now, using results on series from Calculus, we have:

$$\Pi^{Deviate} + \frac{\delta}{1-\delta} \Pi^D \geq \frac{1}{1-\delta} \Pi^C$$

Now, we can substitute in for $\Pi^{Deviate}$, Π^D , and Π^C from our game to find:

$$32 + 8 \frac{\delta}{1-\delta} \geq 25 \frac{1}{1-\delta}$$

Or:

$$\begin{aligned} 32 - 32\delta + 8\delta &\geq 25 \\ 7 - 24\delta &\geq 0 \\ 7 &\geq 24\delta \\ \frac{7}{24} &\geq \delta \end{aligned}$$

Thus, choosing to deviate from the proposed strategy only provides a higher payoff if $\delta \leq \frac{7}{24}$, so that continuing to cooperate is a best response if $\delta \geq \frac{7}{24}$. The discount rate will be a key factor in determining whether or not a proposed equilibrium is a SPNE. In fact, when looking at infinitely repeated games, it is best to have a particular strategy in mind and then check to see what the necessary conditions are for it to be a SPNE, given the multiplicity of equilibria.

Are there other SPNE to the game? Well, consider a modified version of the game:

		Player 2	
		Defect	Cooperate
Player 1	Defect	8, 8	80, 4
	Cooperate	4, 80	25, 25

The only change in this game is that the payoff of 32 that the player received from Defecting when the other player Cooperates has been changed to 80. We can show that both players using a strategy of cooperating until a defection occurs (the same proposed strategy from before) is a SPNE if:

$$80 + 8 \frac{\delta}{1-\delta} \geq 25 \frac{1}{1-\delta}$$

or $\delta \geq \frac{55}{72}$. Thus, if both players are sufficiently patient then the proposed strategy is still a SPNE. Note that the discount rate increased in this example since the payoff to deviating increased. But, is there a strategy that yields higher payoffs? What if the following strategies were used by players 1 and 2:

If no deviation has occurred, Player 1 chooses Defect in all even time periods and chooses Cooperate in all odd time periods. If a deviation occurs Player 1 always chooses Defect.

If no deviation has occurred, Player 2 chooses Cooperate in all even time periods and chooses Defect in all odd time periods. If a deviation occurs Player 2 always chooses Defect.

⁷This canceling out of the δ^τ terms typically leads to the assumption that if deviation is going to occur in an infinitely repeated game it will occur in the first time period. I proceed under this assumption in later examples.

A deviation (from player 1's perspective) occurs when Player 2 chooses Defect in an even time period. A deviation (from player 2's perspective) occurs when Player 1 chooses Defect in an odd time period. Note that we start the game at time $t = 0$, so that Player 1 receives 80 first.

Look at what this strategy would do. It would cause the outcome of the game to alternate between the $(Defect, Cooperate)$ and $(Cooperate, Defect)$ outcomes, giving the players alternating periods of payoffs of 80 and 4, as opposed to 25 each period using the "cooperate until defect is observed, then always defect" strategy. On average (and ignoring discounting for a moment), each player would receive 42 per period under this new strategy and only 25 per period under the old. Is the new strategy a SPNE? We should check for both players now that they are receiving different amounts of payoffs in different periods.

For Player 1:

$$\begin{aligned}\Pi^{Deviate} &= 80 + \sum_{i=1}^{\infty} \delta^i 8 \\ \Pi^C &= \sum_{i=0}^{\infty} \delta^{2i} 80 + \sum_{i=0}^{\infty} \delta^{2i+1} 4\end{aligned}$$

If $\Pi^C \geq \Pi^{Deviate}$ then Player 1 will choose NOT to deviate:

$$\begin{aligned}80 \frac{1}{1-\delta^2} + 4 \frac{\delta}{1-\delta^2} &\geq 80 + 8 \frac{\delta}{1-\delta} \\ 80 + 4\delta &\geq 80(1-\delta^2) + 8\delta(1+\delta) \\ 4\delta &\geq -80\delta^2 + 8\delta + 8\delta^2 \\ 72\delta^2 - 4\delta &\geq 0 \\ 18\delta - 1 &\geq 0 \\ \delta &\geq \frac{1}{18}\end{aligned}$$

This is true, for any $\delta \geq \frac{1}{18}$.

For Player 2:

$$\begin{aligned}\Pi^{Deviate} &= \sum_{i=0}^{\infty} \delta^i 8 \\ \Pi^C &= \sum_{i=0}^{\infty} \delta^{2i} 4 + \sum_{i=0}^{\infty} \delta^{2i+1} 80\end{aligned}$$

If $\Pi^C \geq \Pi^{Deviate}$ then Player 2 will choose NOT to deviate:

$$\begin{aligned}4 \frac{1}{1-\delta^2} + 80 \frac{\delta}{1-\delta^2} &\geq 8 \frac{1}{1-\delta} \\ 4 + 80\delta &\geq 8 + 8\delta \\ 72\delta &\geq 4 \\ \delta &\geq \frac{1}{18}\end{aligned}$$

Thus, both players need to have a discount rate greater than or equal to $\frac{1}{18}$ to support this strategy. Note that this discount rate is much lower than the one needed to support the "cooperate until defect is observed, then always defect" strategy. However, it also illustrates the "embarrassment of riches" of infinitely repeated games because for any $\delta \geq \frac{55}{72}$ either of these strategies could be played. And those are NOT the only two strategies.

3.2.2 Some results

There are a number of formal results for SPNE that one can show concerning infinite games. Most of these results hinge upon a discount rate δ being sufficiently close to 1. In the Prisoner's Dilemma type games we have considered the punishment for deviating from the "cooperation" strategy is for the other player to play the stage game (or single shot) Nash equilibrium for the remainder of the game (choose Defect forever). Since there are only two actions a player can take at any decision node (Cooperate or Defect) the only method of punishment is to play Defect. Equilibria where the punishment takes the form of playing the stage game Nash equilibrium are known as Nash reversion since the game reverts back to the Nash equilibrium once a defection is observed.

Supporting average payoffs greater than stage game Nash As we showed in the second example (when the players alternated choosing the Cooperate and Defect strategies) it need not be the case that the players always “agree”⁸ to choose the same strategy in each period. It is possible to show that ANY payoff stream that yields average (undiscounted) payoffs above the Nash equilibrium level can be supported by the threat of Nash reversion IF the discount rate is sufficiently close to 1. Again, consider the Cooperate, Defect game:

		Player 2	
		Defect	Cooperate
Player 1	Defect	8, 8	80, 4
	Cooperate	4, 80	25, 25

Suppose that you believe that Player 1 is a much tougher player than Player 2, whatever being “tougher” means. A potential SPNE strategy is as follows: Player 2 always Cooperates unless a deviation is observed and then chooses Defect forever once deviation is observed. Exactly what a deviation is will be made clear momentarily, but consider Player 1 who plays Cooperate in the first period, then Defect for 4 periods, then Cooperate in the next period, then Defect for 4 periods, etc., unless a deviation is observed. A deviation by Player 2 is any play of Defect. A deviation by Player 1 is a choice of Defect in any period in which he should be choosing Cooperate. If Player 1 were to choose Cooperate in periods 2, 3, 4, 5, 7, 8, 9, 10, etc. Player 2 would NOT view this as a defection. Note that the payoff stream for Player 2 is 25, 4, 4, 4, 4, 25, 4, 4, 4, 4, etc. Every 5 periods Player 2 receives an undiscounted payoff of 41, with an average payoff of 8.2 per period. Since this average is greater than the payoff from playing the stage game Nash equilibrium (8), Player 2 will “agree” to play this equilibrium IF his discount rate is close enough to 1. Given that we have actual payoffs we can see that Player 2 will choose to always Cooperate if:

$$\sum_{t=0}^{\infty} (\delta^5)^t 25 + \delta \sum_{t=0}^{\infty} (\delta^5)^t 4 + \delta^2 \sum_{t=0}^{\infty} (\delta^5)^t 4 + \delta^3 \sum_{t=0}^{\infty} (\delta^5)^t 4 + \delta^4 \sum_{t=0}^{\infty} (\delta^5)^t 4 \geq 80 + \delta \sum_{t=0}^{\infty} \delta^t 8$$

You should check that the payoff stream on the left-hand side is the actual payoff stream. Simplifying this expression gives:

$$\frac{25}{1-\delta^5} + \frac{4\delta}{1-\delta^5} + \frac{4\delta^2}{1-\delta^5} + \frac{4\delta^3}{1-\delta^5} + \frac{4\delta^4}{1-\delta^5} \geq 80 + \frac{8\delta}{1-\delta}$$

This is not an easy equation to solve for δ , so we can evaluate it numerically. The goal (for the example) is not to find the actual discount rate but to show that for some discount rate close to 1 that this set of strategies constitutes a SPNE. Suppose that $\delta = 0.99$. The left-hand side (Cooperate) is:

$$\left[\frac{25}{1-\delta^5} + \frac{4\delta}{1-\delta^5} + \frac{4\delta^2}{1-\delta^5} + \frac{4\delta^3}{1-\delta^5} + \frac{4\delta^4}{1-\delta^5} \right]_{\delta=.99} = 828.48$$

The right-hand side (Defect) is:

$$\left[80 + \frac{8\delta}{1-\delta} \right]_{\delta=.99} = 872.0$$

So even for a $\delta = .99$ Player 2 would not play this equilibrium. But 0.99 is not as close as we can get to

1. What if $\delta = 0.999$? The left-hand side is:

$$\left[\frac{25}{1-\delta^5} + \frac{4\delta}{1-\delta^5} + \frac{4\delta^2}{1-\delta^5} + \frac{4\delta^3}{1-\delta^5} + \frac{4\delta^4}{1-\delta^5} \right]_{\delta=.999} = 8208.4$$

While the right-hand side is:

$$\left[80 + \frac{8\delta}{1-\delta} \right]_{\delta=.999} = 8072.0$$

So, for some discount rate between 0.99 and 0.999 this set of strategies becomes a potential solution.

Another way to think about this is to consider the case where $\delta = 1$. Now, since the game is played infinitely any set of strategies will lead to an infinite payoff, but it may be that one set of strategies gets to infinity “faster”. Consider the first 355 periods of the game. Using the “defection strategy”, Player 2 will have received 80 in the first period and 8 for the next 354 periods. This leads to an undiscounted payoff of 2912 for the 355 periods. Using the “cooperation strategy”, Player 2 will have received an average payoff of 8.2 each period for the 355 periods for an undiscounted payoff of 2911. In the 356th period Player 2 would receive 8 using the “defection strategy”, bringing the total payoff up to 2920, and using the “cooperation strategy” will receive 25, bringing the total payoff up to 2936. Up until the 356th period the total payoff

⁸This is another slight problem with infinite games – there are so many SPNE that it is difficult to say how one particular one arose.

from “defection” is less than the total payoff from “cooperation”, but from the 356th period onward the total payoff from “cooperation” is ALWAYS greater than or equal to that from “defection”. From period 361 onward the total payoff from “cooperation” is ALWAYS strictly greater than that from “defection”. In a sense, the “cooperation strategy” overtakes the “defection strategy” at some point in time and from that point in time onward is NEVER overtaken by the “defection” strategy.

Supporting average payoffs less than stage game Nash It is also possible to support payoffs LESS than the stage game Nash equilibrium payoffs. However, this cannot happen in our Cooperate, Defect game because the minimum payoff a player can guarantee himself in that game is 8, which is the stage game Nash payoff (if someone plays Defect this guarantees that person will receive at least 8). It might be the case that punishment can be WORSE than the Nash equilibrium to the stage game (again, to be clear, this is not the case in the Cooperate, Defect game, but it could be in some other game with more strategies). Thus it is possible to support average payoffs that are less than the stage game Nash equilibrium payoffs as an SPNE as long as the discount rate is close to 1. For example, if the Nash equilibrium payoff is 10, but the highest amount a player can guarantee himself is 6, then it is possible to find an SPNE where that player receives an average of 6.1 each period.

Carrot-and-stick approach So far all of our SPNE have used what is known as a “grim trigger” strategy. When using a grim trigger strategy, once a defection is observed play reverts to the Nash equilibrium (or worse) – forever. This is an extremely harsh punishment as it allows no room for error. An alternative is to use a carrot-and-stick approach. The punishment portion of the strategy specifies that the punisher will only punish for x periods rather than every ensuing period. That is the stick. The carrot is the cooperation payoff that the player receives x periods in the future once a defection is observed, provided the defecting player returns to cooperating. This approach is more forgiving than the grim trigger strategy, and in games where (1) mistakes may be made (2) actions may be misinterpreted or (3) there is some uncertainty that influences the players’ payoffs in addition to the players’ chosen actions this more forgiving approach may yield higher payoffs than the grim trigger strategy. A good example of this approach can be found in Green and Porter (1984),⁹ *Econometrica*, Noncooperative Collusion Under Imperfect Price Information, 87-100. In that model there is a group of firms who wish to collude. The market price is influenced by the total quantity produced by each firm. In addition, the market price is also influenced by a random shock. Thus, the market price may be low due to either (1) overproduction on the part of the firms or (2) bad luck. However, since individual firm quantity choices are unobservable to all firms, it is impossible to verify the true cause of the low market price. This typically means that the firms would be unable to sustain a collusive agreement. However, using a punishment system where the firms punish for x periods if the market price ever drops below some level \underline{p} regardless of the reason (either bad luck or overproduction) the firms are able to sustain a noncooperative collusive agreement.

How do we know which equilibrium will be played? We don’t really know which equilibrium will be played – it typically depends on the type of game. There is an interesting paper by Todd Kaplan and Bradley Ruffle title “Which Way to Cooperate” which discusses a particular type of game and the conditions under which different equilibria may be expected to arise.

⁹Green and Porter (1984), *Econometrica*, Noncooperative Collusion Under Imperfect Price Information, vol. 52:1, 87-100.