These notes essentially correspond to chapter 14 of the text. There is a little more detail in some places.

1 Intro to game theory

Although it is called game theory, and most of the early work was an attempt at "solving" actual games (like Chess), the tools used in game theory can be applied to many economic situations (how to bid in an auction, how to bargain, how much to produce in a market setting, etc.). A game consists of the following four items:

1. Players – the agents (firms, people, countries, etc.) who actively make decisions
2. Rules – the procedures that must be followed in the game (knights must move in an L-shaped pattern in Chess, three strikes and you're out in baseball, a firm cannot produce a quantity less than 0 – these are all rules); may also include timing elements (white moves first in Chess then player's alternate moves, one firm may produce first and the other firm may observe this production before it makes a quantity decision,
3. Outcomes – what occurs once all decisions have been made (in a winner/loser game like Chess or baseball, the outcome is a win or a loss or perhaps a tie, while in a market game the outcome is more like a profit level)
4. Payoffs – the value that the player assigns to the outcome (in most of our examples outcomes and payoffs will be identical, as the outcomes will be dollars and players will just translate

1.1 Solution Concept

Our goal will be to "solve" these games. Although there are a variety of solution methods, the one we will use is the Nash Equilibrium concept (yes, named after that guy Nash in the movie). A Nash Equilibrium is a set of strategies such that no one player can change his strategy and obtain a higher payoff given the strategy the other player(s) is (are) currently using.

A strategy is a complete plan of play for the game. Suppose we were trying to solve the game of Chess (if you ever actually solve Chess you will become famous, at least within the mathematics community). There are two players, and the player with the white pieces moves first. One piece of the white pieces player's strategy might be, "move king side knight to square X to start the game". However, this is not a complete strategy – you need to write down what you will do for every possible move that you will make. By contrast, look at the beginning of the black pieces player's strategy. There are 20 possible moves that the white pieces player can use to begin Chess, and the black pieces player must have a plan of action for what he will do for EVERY possible move the white pieces player would make. That's a list of 20 moves that the black pieces player must write out just to make his FIRST move. Thus, a complete strategy of Chess is very, very, very, lengthy (even with the increases that we have seen in computing power no one has been able to program a computer to solve Chess).

1.2 Simple duopoly example

Suppose that there are two firms (Firm A and Firm B) engaged in competition. The two firms will choose quantity levels simultaneously. To keep this example simple, assume that the firms' quantity choices are restricted to be either 48 units or 64 units. If both firms choose to produce 64 units, then both firms will receive a payoff of \$4.1. If both firms choose to produce 38 units, then both firms will receive a payoff of \$4.6. If one firm chooses to produce 48 units and the other chooses to produce 64 units, the firm that produces 48 units receives a payoff of \$3.8 while the firm that produces 64 units receives a payoff of \$5.1.

When analyzing 2 firm simultaneous games (where there are a small number of strategy choices), we can use a game matrix (or the normal form or strategic form or matrix form – it has many names) as an aid in finding the NE to the game. The game matrix is similar to the table above for the monopoly, only now we have 2 firms. I will write out the matrix below and then explain the pieces as well as some terminology.

	Firm B		
		$Q_B = 48$	$Q_B = 64$
Firm A	$Q_A = 48$	\$4.6, \$4.6	\$3.8, \$5.1
	$Q_A = 64$	\$5.1, \$3.8	\$4.1, \$4.1

One player is listed on the side of the matrix (Firm A in this example) and is called the row player, as that player's strategies ($Q_A = 48$ and $Q_A = 64$ in this example)are listed along the rows of the matrix. The other player is listed at the top of the matrix (Firm B in this example) and is called the column player, as that player's strategies ($Q_B = 48$ and $Q_B = 64$ in this example)are listed along the columns of the matrix.

Each cell inside the matrix lists the payoffs to the players if they use the strategies that correspond to that cell. So the \$4.6, \$4.6 are the payoffs that correspond to the row player (Firm A) choosing to produce 48 and the column player (Firm B) also choosing to produce 48. The cell with \$5.1, \$3.8 corresponds to the row player choosing 64 and the column player choosing 48. Note that the row player's payoff is ALWAYS, ALWAYS, ALWAYS listed first (to the left) in the cell.

Now that the game is set-up, how do we find the Nash Equilibrium (NE) to the game? We could look at each cell and see if any player could make himself better off by changing his strategy.

If $Q_A = 48$ and $Q_B = 48$, then Firm A could make himself better off by choosing $Q_A = 64$ (Firm B could also have made himself better off by choosing $Q_B = 64$, but all we need is one player to want to change his strategy and we do not have a NE). Thus, $Q_A = 48$ and $Q_B = 48$ is NOT a NE.

If $Q_A = 48$ and $Q_B = 64$, then Firm A can make himself better off by choosing $Q_A = 64$, because he would receive \$4.1 rather than \$3.8. Thus, $Q_A = 48$ and $Q_B = 64$ is NOT a NE.

If $Q_A = 64$ and $Q_B = 48$, then Firm B could make himself better off by choosing $Q_B = 64$. Thus, $Q_A = 64$ and $Q_B = 48$ is NOT a NE.

If $Q_A = 64$ and $Q_B = 64$ then neither firm can make himself better off by changing his strategy (if either one of them changes then the firm that changes will receive \$3.8 rather than \$4.1). Since neither firm has any incentive to change, $Q_A = 64$ and $Q_B = 64$ is a NE to this game.

Working through each cell is a fairly intuitive, albeit time-consuming process. You can use this technique if you want, but a word of caution. You must check EVERY cell in the game as there may be multiple NE to the game – thus, even if you started by checking $Q_A = 64$ and $Q_B = 64$ and found that it was a NE you would still need to check the remaining cells to ensure that they were not NE. However, there is another method.

Another method that works to find NE of game matrices is called "circling the payoffs" (it doesn't really have a technical name). Here's the idea: hold one player's strategy constant (so suppose Firm B chooses $Q_B = 48$), then see what the other player's highest payoff is against that strategy and circle that payoff. So if Firm B chose $Q_B = 48$, then Firm A would circle the payoff of \$5.1 in the lower left-cell (the payoff of \$5.1 that corresponds to $Q_A = 64$ and $Q_B = 48$). If Firm B chose $Q_B = 64$, then Firm A would circle the payoff of \$4.1 since \$4.1 > \$3.8. So halfway through the process we have:

	Firm B		
		$Q_B = 48$	$Q_B = 64$
Firm A	$Q_A = 48$	\$4.6, \$4.6	\$3.8, \$5.1
	$Q_A = 64$	$\boxed{\$5.1}$, \$3.8	$\boxed{\$4.1}$, \$4.1

Now, we simply hold Firm A's strategy constant and figure out what Firm B would do in each situation. Firm B would circle the \$5.1 payoff if Firm A chose $Q_A = 48$ and Firm B would circle the \$4.1 payoff if Firm A chose $Q_A = 64$. Thus, the result would be:

	Firm B		
		$Q_B = 48$	$Q_B = 64$
Firm A	$Q_A = 48$	\$4.6, \$4.6	\$3.8, $\boxed{\$5.1}$
	$Q_A = 64$	$\boxed{\$5.1}$, \$3.8	$\boxed{\$4.1}$, $\boxed{\$4.1}$

Whichever cell (or cells) have both payoffs circled are NE to the game. Note that this is the same NE we found by going through each cell. Again, it is possible to have more than one NE to a game. Also, it is possible to circle more than one payoff at a time. Suppose Firm A chose $Q_A = 48$ and that Firm B received \$5.1 if it chose $Q_B = 48$ or $Q_B = 64$. In this case, since the highest payoff corresponds to two different strategies for Firm B you would need to circle both of the payoffs. The "solved" game (with the \$5.1 replacing the \$4.6 for Firm B only) would look like below:

	Firm B		
		$Q_B = 48$	$Q_B = 64$
Firm A	$Q_A = 48$	\$4.6, $\boxed{\$5.1}$	\$3.8, $\boxed{\$5.1}$
	$Q_A = 64$	$\boxed{\$5.1}$, \$3.8	$\boxed{\$4.1}$, $\boxed{\$4.1}$

2 Dynamic (or sequential) games

We had been studying simultaneous games, where each firm makes its quantity choice or price choice without observing the other firm's choice. Now, we want to extend the analysis to include sequential games, where one firm moves first, the second firm observes this decision, and then the second firm makes its decision. To analyze sequential games, a structure, called a game tree, that is slightly different than the game matrix should be used. The game tree provides a picture of who decides when, what decisions each player makes, what decisions each player has seen made prior to his decision, and which players see his decision when it is made. We can start by translating the simple quantity choice game from chapter 13 (when the firms could each only choose to produce a quantity of 64 or 48) into a sequential games framework.

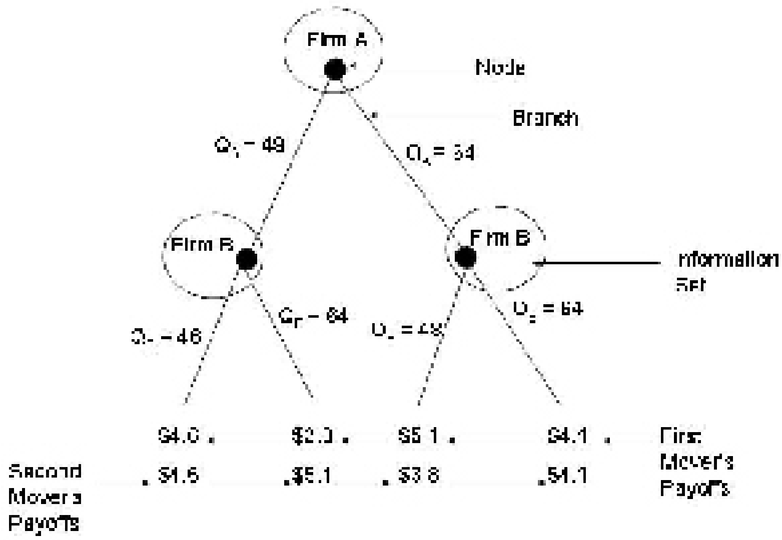
Suppose that there are two firms (Firm A and Firm B) engaged in competition. Firm A will choose its quantity level first, and then Firm B will choose its quantity level after observing Firm A's choice. To keep this example simple, assume that the firms' quantity choices are restricted to be either 48 units or 64 units. If both firms choose to produce 64 units, then both firms will receive a payoff of \$4.1. If both firms choose to produce 48 units, then both firms will receive a payoff of \$4.6. If one firm chooses to produce 48 units and the other chooses to produce 64 units, the firm that produces 48 units receives a payoff of \$3.8 while the firm that produces 64 units receives a payoff of \$5.1. This game is sequential since Firm A chooses first and Firm B observes Firm A's decision.[1]

While we could use the matrix (or box or normal) form of the game for the sequential game, there is another method for sequential games that makes the sequential nature of the decisions explicit. The method that should be used is the game tree. A game tree consists of:

1. Nodes – places where the branches of the game tree extend from
2. Branches – correspond to the strategies a player can use at each node
3. Information sets – depict how much information the player has when he moves (if the second player knows that he follows the first player but cannot observe the first player's decision then his information set is really no different than in the simultaneous move game; however, if the second player can observe the first player's decision, then his information set has changed)

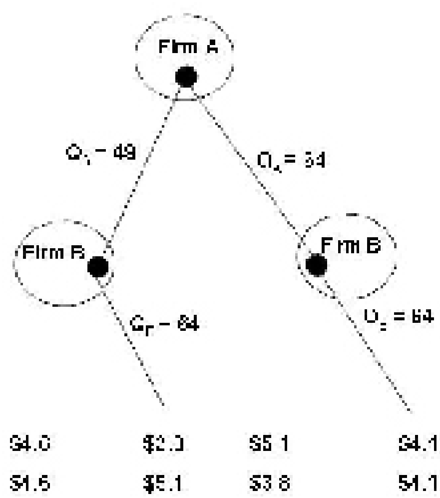
A game tree corresponding to the quantity choice game previously described is depicted below. The individual pieces of a game tree are also labelled. The label for information set is pointing to the open circle that encircles the term "Firm B". Thus, Firm B can see how much Firm A has decided to produce. If Firm B could not determine if Firm A decided to produce 48 or 64 units, then Firm B would have one information set, and there would be one open circle encircling both of Firm B's decision nodes.

[1] In the real-world Firm A may actually choose a quantity before Firm B, but if Firm B gains no additional information from Firm A's decision (such as a change in the market price), then the game is essentially one where Firm A and Firm B choose simultaneously.



To solve sequential games we start from the end of the game and work our way back towards the beginning. This is called backward induction. To find the Nash Equilibrium (NE), we first determine what Firm B would do given a quantity choice by Firm A. In this example, Firm B would choose $Q_B = 64$ as its strategy if Firm A chose $Q_A = 48$ because \$5.1 > \$4.6. Also, Firm B would choose $Q_B = 64$ if Firm A chose $Q_A = 64$ because \$4.1 > \$3.8. Thus, Firm B's strategy is: {Choose $Q_B = 64$ if Firm A chooses $Q_A = 48$; choose $Q_B = 64$ if Firm A chooses $Q_A = 48$}. We now know what Firm B will do for any given choice by Firm A, which means that we have an entire strategy for Firm B.

Firm A, knowing that Firm B will choose $Q_B = 64$ regardless of its quantity choice, can now "lop off the branches" that correspond to $Q_B = 48$. The reason that Firm A can lop off these branches is that it knows that it will never see the payoffs associated with following those branches because Firm B will never follow them. Thus, to Firm A, the game tree looks like:



I have left the payoffs there but removed the branches. Firm A has one decision to make, produce a quantity of 48 or a quantity of 64. If it produces a quantity of 48, Firm B will produce 64, and Firm A will receive a payoff of \$3.8. If it produces a quantity of 64, Firm B will produce 64, and Firm A will receive a payoff of \$4.1. Since \$4.1 > \$3.8, Firm A will choose $Q_A = 64$. Thus, the complete NE for this game is:

Firm A: Choose $Q_A = 64$

Firm B: Choose $Q_B = 64$ if Firm A chooses $Q_A = 48$; choose $Q_B = 64$ if Firm A chooses $Q_A = 48$

Now, when the game is played only one payoff is received. To find this payoff just follow the path outlined by the NE strategy. Firm A chooses $Q_A = 64$, and if Firm A chooses $Q_A = 64$ then Firm B chooses $Q_B = 64$, which leads to a payoff of \$4.1 for Firm A and \$4.1 for Firm B. Notice that we didn't use the fact that Firm B chooses $Q_B = 64$ if Firm A chooses $Q_A = 48$ because Firm A did not choose $Q_A = 48$. We still need to include that piece as part of our NE strategy even though we don't use it when we find the path that the game actually follows.

3 Pricing Practices

What follows is a game theoretic analysis of some prevalent pricing practices in industry.

3.1 Limit Pricing

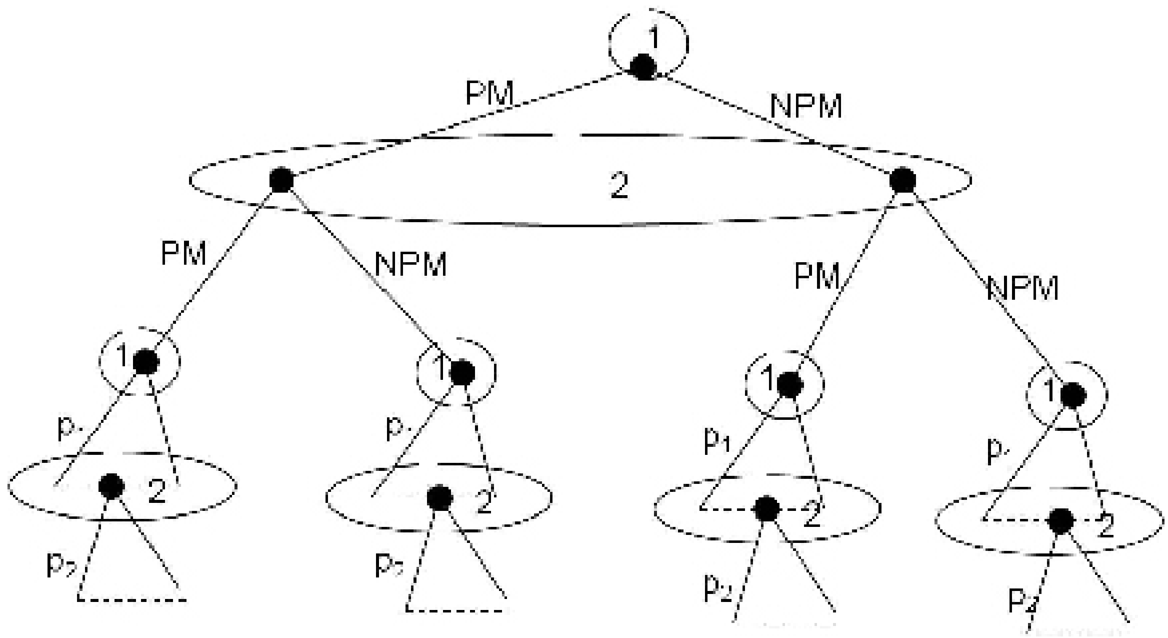
Limit pricing is an entry deterrence strategy that may be used by a monopolist. The idea is that a monopolist will underprice in one period (today) to keep potential entrants out of the market so that the monpolist can reap the rewards of the monopolist market in future periods (tomorrow, and perhaps later time periods). The monopolist's goal would be to choose the highest possible price today that would keep the entrant from entering today. Then the monopolist could choose the monopoly price in future periods.

One issue with limit pricing is why it should work – if entry is profitable, and easy enough that the monopolist would have to be concerned about it, one might think that as soon as the monopolist charges the monopoly price some firm will enter into the market. There is debate about how well limit pricing works in practice.

3.2 Price Match Guarantees

Price-matching guarantees are very popular with firms in the retail industry. Think about some advertisements you may have seen – "If you find a lower advertised price, we will match (or perhaps beat) it!" Many people view these policies as competitive policies, because, after all, firms are guaranteeing that they will sell the product at the lowest available price. But consider the following game.

In the standard simultaneous Bertrand (pricing) game with two firms there is a unique pure strategy Nash equilibrium in which both firms choose a price equal to marginal cost c (we saw this in the Chapter 13 notes). Now suppose that before the firms simultaneously choose price in the standard Bertrand game they simultaneously choose whether or not to implement a price-matching policy. If a firm implements a price-matching policy this means that firm will match the lowest price in the market. Firms observe each other's price-matching decision and then simultaneously choose their respective prices (under the assumptions of the standard Bertrand pricing game), where a price can be any real number. A basic depiction of the game is below, where PM means "price-matching", NPM means "no price-matching", and p_1 and p_2 are the price choices of firms 1 and 2 respectively:



What can happen in equilibrium if both firms choose a price-matching policy? There are many equilibria. In the other 3 pricing subgames (the ones where both firms choose NPM or one firm chooses PM and the other chooses NPM) we need $p_1 = p_2 = c$. In the price-matching subgame (the part of the tree in which both firms choose PM), any pair of prices where $p_1 = p_2$ and $p_1 \in [c, p^m]$ is a Nash equilibrium, where c is marginal cost and p^m is the monopolist's price. Consider $p_1 = p_2 = p^m$. Because both firms have a price-matching policy, any attempt to undercut the price to capture the entire market will result in the other firm having its price pulled downward (because it has a price-matching policy). Now consider $p_1 = p_2 < p^m$ (but greater than c). In this case, there is still no incentive to undercut the price as the other firm's price would be pulled downward, but there is also no incentive to increase price (because whichever firm increases price has its price pulled back to the lower price). Finally, consider $c < p_1 < p_2 < p^m$. This is not an equilibrium. While there is no incentive for the firm with the higher price to change its strategy, the firm with the lower price (p_1) would be better off charging the higher price (p_2).

Far from being a competitive pricing policy, price-matching guarantees could be anti-competitive. While this model is really simple, the end result that price-matching policies can lead to anti-competitive outcomes, holds up under more realistic assumptions.

4 Auctions

In this section I will describe the four basic auction formats that we will discuss. The description will include the process by which bids are submitted and the assignment rule for the winner. For now, consider only the cases where we have a single, indivisible unit for sale. Keep in mind that the auctions we are discussing are for those of many buyers (bidders) and a single seller; many results still hold if there is a single buyer (such as a government awarding a contract) and multiple sellers (firms attempting to obtain the government contract), though the bidding functions will change. The latter type of auction (single buyer, multiple sellers) are known as reverse auctions (or procurement auctions) and are used by a variety of businesses.

4.1 1^{st}-price sealed bid auction

Process All bidders submit a bid on a piece of paper to the auctioneer.

Assignment rule The highest bidder is awarded the object. The price that the high bidder pays is equal to his bid.

Examples Many procurement auctions are 1^{st}-price sealed bid. Procurement auctions are typically run by the government to auction off a construction job (such as paving a stretch of highway).

4.2 Dutch Auction

Process There is a countdown clock that starts at the top of the value distribution and counts backwards. Thus, the price comes down as seconds tick off the clock. When a bidder wishes to stop the auction he or she yells, "stop".

Assignment rule The bidder who called out stop wins the auction, and the bidder pays the last price announced by the auctioneer.

Examples The Aalsmeer flower auction, in the Netherlands, is an example of this type of auction. Hmmm, wonder where the phrase "Dutch" auction comes from...

By the way, the Ebay dutch auctions are NOT, NOT, NOT Dutch auctions. They are multi-unit ascending $k + 1$ price auctions. Perhaps we will discuss those later.

4.3 2^{nd}-price sealed bid auction

Process Bidders submit their bids on a piece of paper to the auctioneer.

Assignment rule The highest bidder wins, but the price that the highest bidder pays is equal to the 2^{nd} highest bid. Hence the term 2^{nd}-price auction.

Examples Ebay is kind of a warped 2^{nd}-price auction. If you think about the very last seconds of an Ebay auction (or if you consider that every person only submits one bid), think about what happens. You are sending in a bid. If you have the highest bid you will win. You will pay an amount equal to the 2^{nd} highest bid plus some small increment. Thus if you submit a bid of \$10 and the second highest bid is \$4, you pay \$4 plus whatever the minimum is (I think it's a quarter). So you would pay \$4.25.

There are other reasons to think that ebay is not actually a 2^{nd}-price auction – perhaps we will discuss them in class or on a homework.

4.4 Ascending clock auction

Process A clock starts at the bottom of the value distribution. As the clock ticks upward, the price of the item rises with the clock. This is truly supposed to be a continuous process, but it is very difficult to count continuously, so we will focus on one tick of the clock moving the price up one unit. If you like we can consider the unit to be a penny, or we can consider a unit to be the smallest denomination of the most worthless currency on the planet (those of you who know me know that I have very little background in international economics – therefore, I leave it up to you to decide the most worthless currency). The idea is that this is the smallest amount that anyone could possibly bid – that is how the ticks on the clock move the price up. All bidders are considered in the auction (either they are all standing or they all have their hands on a button – some mechanism to show that they are in). When the price reaches a level at which the bidder no longer wishes to purchase the object, the bidder drops out of the auction (sits down or releases the button). Bidders cannot reenter the auction. Eventually only two bidders will remain. When the next to last bidder drops out, the last bidder wins.

Assignment rule The winning bidder is the last bidder left in the auction. The bidder pays a price equal to the last price on the clock.

Examples The typical example given is Japanese fish markets. I'm trying to find a specific reference, but I have recently been told that the Japanese fish market story may be an urban legend. Thus, the English clock auction may only be a theoretical construct.

4.5 Bidding strategies

The previous section is meant to introduce you to the auction formats. In this section we will discuss the NE bidding strategies. In some cases we will "derive" the NE strategies, while in others I will discuss the intuition behind the NE and leave the gory details to those interested.[2]

4.5.1 General Environment

Before discussing the bidding strategies we need to set up the general environment. This suggests that if the environment (or pieces of it) change, the NE bidding strategies will change.

The general name for the environment is the Symmetric Independent Private Values environment (SIPV) with Risk-neutral bidders. We will also assume that we are auctioning off a single, indivisible unit of the good.

1. There needs to be a probability distribution for player values, denoted v_i. We will assume that all player values are drawn from the uniform distribution on the unit interval. This means that all values are drawn from the interval $[0, 1]$ with equal probability. More importantly, if you draw a value of 0.7, then the probability that someone else drew a value less than you is also 0.7. Since probabilities must add up to 1, and since the other player's value draw must either be greater than your value or less than your value. We will not allow for the fact that someone else could draw the exact same value (theoretically, ties cannot occur with positive probability in a continuous probability distribution). This means that the probability that the other player has a value greater than yours is $1 - 0.7 = 0.3$.

2. The setting is symmetric in the sense that all players know that the other player's value(s) is drawn from the same probability distribution.

3. The setting is independent in the sense that your value draw has NO impact on the value draw of the other player(s).

4. The setting is private in the sense that only you know your value – thus, it is private information.

5. We add the fact that our bidders are risk-neutral, as risk aversion will alter some results. Thus, our utility function will be:

$$u(x) = \begin{cases} x \; if \; win \; the \; auction \\ 0 \; if \; don't \; win \end{cases}$$

The term x in the utility function can typically that of as $v_i - b_i$, where v_i is the player i's value and b_i is player i's bid.

This should lead you to our first bidding rule (in this particular environment).

1. Do NOT overbid, where overbidding is defined as the act of placing a bid greater than your value. Overbidding is weakly dominated by not bidding.

Ascending clock auction – bidding strategy Consider the following example. You have a value of 10. The clock begins at 0 and ticks upward: 0, 1, 2, 3, ..., 9, 10, 11, 12, 13, ... The question is, when should you sit down (or drop out of the auction)? Consider three possible cases:

1. The clock reaches 11:

   In this case you should drop out. While you increase your chances of winning the item by staying in, note that you will end up paying more than the item is worth to you. You can do better than this by dropping out of the auction and receiving a surplus of zero. So, as soon as the price on the clock exceeds your value you should drop out.

[2] Wolfstettar, Elmar (1999) *Topics in Microeconomics: Industrial Organization, Auctions, and Incentives* is an excellent reference for such gory details.

2. The clock is at some price less than 10:

   In this case you should remain in the auction. If you drop out you will receive 0 surplus. However, if you remain in the auction then you could win a positive surplus. If you drop out before your value is reached you are essentially giving up the chance to earn a positive surplus. Since this positive surplus is greater than the 0 surplus you would receive if you dropped out, you should stay in the auction.

3. The clock is at 10:

   What happens when the price on the clock reaches your value? Well, if you win the auction you get 0 surplus and if you drop out you get 0 surplus, so regardless of what you do you get 0 surplus. We will say that you stay in at 10, and drop out at 11. For one thing, it makes the NE bidding strategy simple – stay in until your value is reached, then drop out. Another way to motivate this is to consider that peoples values are drawn from the range of numbers $[0.01, 1.01, 2.01, 3.01, ...]$ instead of $[0, 1, 2, 3, ...]$. However, assume the prices increase as $[0, 1, 2, 3, ...]$. It is clear that if you have a value of 3.01 you should be in at 3, while if you have a value of 3.01 you should be out at 4. This is the "add a small amount to your value" approach that I mentioned in class.

So what is the NE strategy? Stay in until your value is reached and drop out as soon as it is passed by the clock.

2^{nd}-price sealed bid auction – bidding strategy In this auction you submit a bid and pay a price equal to that of the 2^{nd} highest bid. How should you bid?

One method of finding a NE (or a solution in general) is to propose that a strategy is a NE and then verify it. Naturally, it is a good idea to propose the right strategy the first time. So, consider the strategy: submit your value. Is this a good strategy?

What else could we do? We could submit a bid greater than the value or less than the value. Let's examine each of these.

Bid above your value Suppose we submit a bid above our value. What could this possibly change? Well, if we were to win when submitting our value then absolutely nothing changes – we still pay the same price since the price (if we win) is not tied to our bid. What happens if we submit a bid greater than our value and this causes us to switch from losing the auction to winning the auction? Suppose our value is 10 and the other player's value is 12. The other player submits 12 and we submit 10. We lose and earn 0 surplus. Now suppose we were to bid 13. We win, which is good, but we have to pay 12 for something that is only worth 10 to us. So we earn a surplus of (-2). This is bad. We could have done better by placing a bid of 10 (our value) and earning 0. So placing a bid equal to our value is better than placing a bid above the value in this case.

Bid below your value Suppose we submit a bid below our value. What could this possibly change? Well, if we were going to lose by submitting our value, then we still lose when submitting a bid below the value. So this changes nothing (at least not for us – it would help the highest bidder if we were the 2^{nd} highest bid!) as we still receive 0 surplus. Suppose we lower our bid and still win – again nothing changes because the 2^{nd} highest bidder has still submitted the same bid. It is possible though that we lower our bid and lose – here's where the problem occurs. Suppose our value is 15 and the other value is 9. We submit a bid of 15, we win, and we get a surplus of $(15 - 9) = 6$. We submit a bid of 14, we still get a surplus of 6. Now suppose we submit a bid of 8 – we go from getting a surplus of 6 to getting a surplus of 0. It would be much better to submit a bid equal to your value and get a surplus of 6. So placing a bid equal to our value is at least as good in most cases and strictly better in some cases.

We have now determined that submitting a bid equal to our value is at least as good as submitting a bid greater than or lower than the value in some cases, and strictly better in other cases. Therefore, submitting a bid equal to your value is a weakly dominant strategy.

NE for 2^{nd}-price auction: Submit a bid equal to your value.

You should note that the 2^{nd}-price sealed bid auction and the ascending clock auction are strategically equivalent. This means that all players have the same bidding strategies in either auction, even though the mechanism that produces the winner of the auction is slightly different.

1^{st}-price sealed bid auction – bidding strategy In this auction you pay an amount equal to your bid if you win. The first question is, should you submit a bid equal to your value?

Bid equal to your value If you submit a bid equal to your value then you will expect to earn 0 surplus. If you win, then you will have to pay an amount equal to your value and if you lose you receive nothing. It stands to reason that you may be able to do better than this by submitting a bid below your value. The question is how far below your value?

Bid equal to the lowest possible value If you submit a bid equal to the lowest possible value that could be drawn then you will also receive 0 surplus. The reason is that you will never win because your bid was so low. Taken together with the fact that you will bid below your value, this means your actual bid should fall between the lowest possible value and your value draw.

Actual problem The actual problem facing someone bidding in a 1^{st}-price sealed bid auction is to maximize their expected utility. Their expected utility can be written as:

$$E\left[1^{st} - price\ auction\right] = \Pr(win) * (v_i - b_i) + \Pr(lose) * 0$$

Since the term $\Pr(lose) * 0 = 0$, we can drop that from the equation to get:

$$E\left[1^{st} - price\ auction\right] = \Pr(win) * (v_i - b_i)$$

If you have taken a course like math econ, then this is a maximization problem, where the choice variable is b_i. You should note that the larger b_i is the greater the probability of winning will be, but the larger b_i is then the lower the surplus will be if you win.

The idea is to pick the bid that maximizes this function. The general bidding strategy, for N bidders IN THE SIPV-RN environment,[3] is to bid $\frac{N-1}{N} v_i$. Thus you are shaving your bid depending on how many other bidders there are. The more bidders, the less you shave your bid.

Dutch auction – bidding strategy Recall that with a Dutch auction the bidder watches as the clock descends, and then calls out when he sees a price that he wishes to pay. The problem facing the bidder is to maximize their expected utility. Their expected utility can be written as:

$$E\left[1^{st} - price\ auction\right] = \Pr(win) * (v_i - b_i) + \Pr(lose) * 0$$

Notice that this is the same problem faced in the first price auction. This suggests that the 1^{st}-price and the Dutch auctions are strategically equivalent. Thus, the bidding strategy in the Dutch auction is to yell out stop when the clock reaches $\frac{N-1}{N}$ of your value.

4.6 Which format is "better"?

Now that we have seen the different formats, the question turns to which one is better. Better can mean 2 things. From the standpoint of a benevolent social planner, better could mean more efficient. We will say that an auction is efficient if the item goes to the person with the highest value. Of course, an individual seller does not necessarily care about social goals such as efficiency, but about the revenue that the auction will generate for himself. The relevant question for the individual seller is then which format generates more revenue. We will look at both of these notions of "better".

[3] Note that if the assumptions about the environment are changed then the NE bidding strategy may (likely will) change.

4.6.1 Efficiency

We will define the level of efficiency in an auction as $\frac{V_w}{V_H}$, where V_w is the value of the winning bidder and V_H is the value of the high bidder. Note that if the winner is the high bidder, then efficiency is 1 or 100%. The question is, in all of our auction formats will the bidder with the highest value bid more than, less than, or an amount equal to bidders with lower values. It is easy to see that in an ascending clock or 2^{nd}-price sealed bid auction that higher values lead to higher bids because bidders simply submit their values as bids. In the Dutch and 1^{st}-price auctions, the bid function is $b_i = \frac{N-1}{N} v_i$. The question is, who will submit the highest bid? It should be fairly easy to see that higher values will submit higher bids. Technically, we can say that the bid function is increasing in the value draw – as the value draw increases, the bid increases. Thus, bidders with higher values will submit higher bids, and the bidder with the highest value will submit the highest bid. These auctions will also be 100% efficient, assuming that all of our conditions hold and bidders use the NE bidding strategies.

4.6.2 Revenue

As far as revenue goes we know that the 1^{st}-price and Dutch auctions are strategically equivalent and that the ascending clock auction and the 2^{nd}-price are strategically equivalent. Thus we know that the revenue from the 1^{st}-price and Dutch will be equal and the revenue from the ascending clock auction and the 2^{nd}-price will be equal. The question is, does the 1^{st}-price generate more revenue than the 2^{nd}-price?

Let V_1 be the highest value and V_2 be the second highest value. Then the expected revenue of the 1^{st}-price auction is:

$$Revenue\left(1^{st} - price\right) = \frac{N-1}{N} E\left[V_1\right]$$

The expected revenue of the 2^{nd}-price auction is:

$$Revenue\left(2^{nd} - price\right) = E\left[V_2\right]$$

We will assume that there are the same number of bidders in each auction. We now need to know what $E\left[V_1\right]$ and $E\left[V_2\right]$ are in order to answer which of the auctions will generate more revenue. To do this we use the concept of an order statistic – basically, an order statistic tells us what the expected value of the k^{th} highest draw from a distribution will be given that we make N draws from the distribution. In our case, we are using the uniform distribution over the range 0 to 1. We find that the k^{th} highest value will be equal to:

$$\frac{N-k-1}{N+1}$$

So:

$$E\left[V_1\right] = \frac{N}{N+1}$$

$$E\left[V_2\right] = \frac{N-1}{N+1}$$

This means that the expected revenue from the 1^{st}-price auction is equal to $\frac{N-1}{N+1}$ and the expected revenue from the 2^{nd}-price auction is also equal to $\frac{N-1}{N+1}$. Thus, both auction formats are expected to generate the same revenue.

While that may surprise some of you, we have a more powerful result called the revenue equivalence theorem. Essentially, if the conditions of the theorem (laid out below) are met, then any mechanism designed will lead to the same expected revenue.

Revenue Equivalence Theorem Assume our set-up – SIPV with N risk-neutral agents. Values are drawn from a distribution $F(v)$ that is strictly increasing and atomless on $[\underline{v}, \overline{v}]$. Suppose no buyer wants more than 1 of k identical, indivisible objects for sale.

Any mechanism in which:

1. Objects always go to the k buyers with the highest values
2. any buyer with value $v = \underline{v}$ expects 0 surplus

yields the same expected revenue and results in a buyer with value v making the same expected payment.

This is a very powerful result that extends beyond the scope of auctions, as you should see on the homework.

4.7 Breaking revenue equivalence and efficiency

If all formats are perfectly efficient and generate the same revenue in expectation, why do auctioneers prefer one type or the other? In the sections below we will look at how to "break" the results from above.

4.7.1 Breaking revenue equivalence

Suppose that instead of risk-neutral agents we had risk-averse agents. They still have the exact same problem as before – they want to maximize their expected surplus. In the 2^{nd}-price and ascending clock auctions, there was no "maximization" problem – bidders simply submitted their bids or dropped out when the clock reached their value. Thus, the strategy should not change in these types of auctions if bidders are risk averse since they can do no better following another strategy. Since the strategy does not change the expected revenue from the 2^{nd}-price auction is still the same.

Consider the 1^{st}-price auction. Bidders wanted to maximize their expected surplus, given by:

$$E\left[1^{st} - price\right] = \Pr(win) * (v_i - b_i)$$

However, in the risk averse case bidders want to maximize something like:

$$E\left[1^{st} - price\right] = \Pr(win) * \sqrt{(v_i - b_i)}$$

Recall that in the risk-neutral version of the 1^{st}-price auction the bidder bid $\frac{1}{2}$ of his value. In this risk averse case, the bidder will bid $\frac{2}{3}$ of his value. Thus, we can see that the bidder is going to bid more in the risk averse case. Intuitively, if the bidder were to bid $\frac{1}{2}$ of his value in the risk averse case the marginal benefit from increasing the bid (the increase in the probability of winning) would be greater than the marginal cost (the amount of surplus lost). So we increase the bid until the marginal benefit of increasing the bid equals the marginal cost, just like we do with many other applications in economics.

4.7.2 Breaking efficiency

Suppose we want to break efficiency. The true version of the ascending clock auction has the price moving up continuously with the tick of the clock. However, we know that people do not have continuous values, or, even if they do, there is some rational minimum amount by which their values must increase. In the US the smallest value one can have for a good is a penny, so it is not a stretch to think that the smallest unit in which values can be denominated is a penny. If this is a case, then a clock which moves at the rate of 1 penny per second (or 1 penny per hour or 1 penny per half-second – the rate is not important, but the units that it counts are) will still be perfectly efficient in the sense that the highest valued bidder will get the object. However, consider a clock that increases the price at a rate of 1 penny per second. Now consider the following prices and the corresponding amount of time it will take to auction off objects of these values:

$10 – 16.67 minutes

$1 *million* – 3.17 years

$1 *billion* – 3170 years

It doesn't really seem "efficient" to take 3170 years to auction off an item. In fact, it seems quite inefficient. So what auctioneers will typically do is impose a minimum bid increment. This minimum bid increment is the minimum amount by which the clock will increase (or the minimum amount by which bidders must increase the bid if they wish to place a new bid). While this speeds up the process, the introduction of the minimum increment can also destroy the efficiency results of auctions. For instance, suppose 2 players have values of \$14.08 and \$14.92 respectively. If the clock ticks up at \$1 per second, then both bidders will drop out at \$14. In this case, a tie is declared and we must use the tie-breaking mechanism. The tie-breaking mechanism is usually a coin flip or some other equal probability game. Thus, on average, the bidder with a value of \$14.08 will get the item half of the time. As you can see, the minimum increment introduces the possibility of inefficiency into the auction process.