Problems on game theory and pricing practices (chapters 14 and 15)

1. Two neighbors like to listen to their favorite recording artists, Black Eyed Peas and Linkin Park. There are only three possible volumes at which they can listen – low, medium, and high. Both neighbors prefer that the other neighbor listen to his music at a lower volume, although they prefer listening to their own music at the same volume as the neighbor than listening at a lower volume than the neighbor. The matrix for the game is as follows:

		Player 2		
		High	Medium	Low
Player 1	High	$2, 2$	$8, 1$	$15, -1$
	Medium	$1, 8$	$6, 6$	$12, 5$
	Low	$-1, 15$	$5, 12$	$10, 10$

Find all Nash Equilibria of the game.

Answer:

The only PSNE to this game will be for Player 1 to choose High and Player 2 to choose High. The best responses are marked in the matrix below, and the only place where the best responses overlap is when both players choose High.

		Player 2		
		High	Medium	Low
Player 1	High	$\boxed{2},\boxed{2}$	$\boxed{8}, 1$	$\boxed{15}, -1$
	Medium	$1,\boxed{8}$	$6, 6$	$12, 5$
	Low	$-1,\boxed{15}$	$5, 12$	$10, 10$

2. There are two pigs in a pen, the Large Pig and the Little Pig. In order to receive food, one of the pigs must press a lever which is on one side of the pen. On the other side of the pen is a chute which deposits 6 pellets of food into the pen. Pressing the lever and then running across the pen to eat the food costs 0.5 pellets worth of energy. If both pigs go to press the lever then the Little Pig eats 1.5 pellets of food and the Large Pig eats 3.5 pellets of food (in total one pellet is lost in running across the pen). If the Little Pig presses the lever then the Large Pig is able to eat all 6 pellets of food prior to the Little Pig reaching the food, but it still costs the Little Pig 0.5 pellets worth of energy to run across the pen. If the Large Pig presses the lever then the Little Pig is able to eat 5 pellets of food before the Large Pig reaches the food, leaving the Large Pig with a net gain of 0.5 pellets of food. Finally, if neither pig presses the lever then no food is deposited into the pen and both pigs receive 0.

a Draw the matrix form version of this game.

Answer:

The strategic form of the game is:

		Large Pig	
		Press	Not Press
Little Pig	Press	$1.5, 3.5$	$-0.5, 6$
	Not Press	$5, 0.5$	$0, 0$

b What is the Nash equilibrium of this game? Recall that Nash equilibria are strategies.

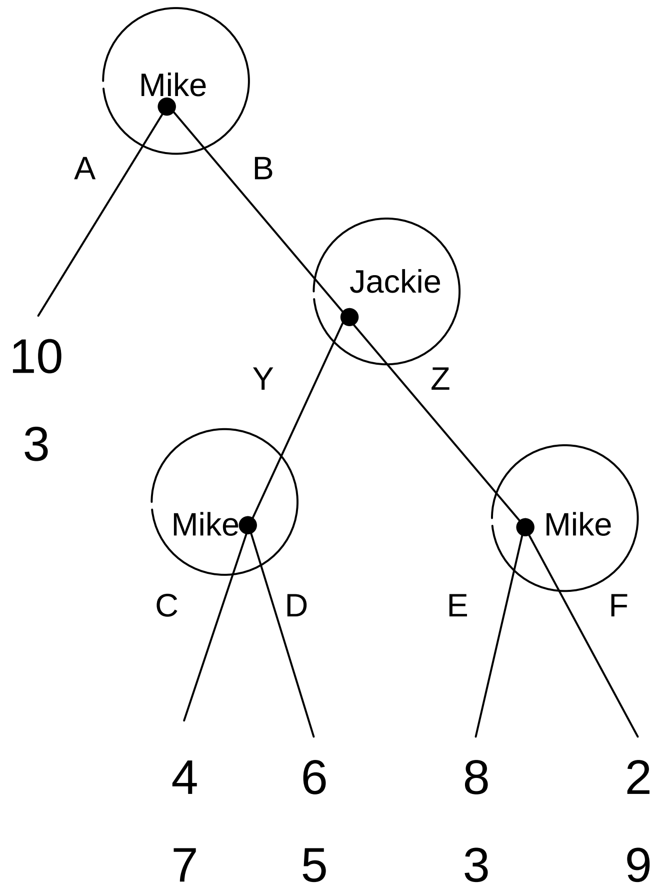
Answer:

Looking at the best responses, we see the only cell in which the best responses overlap is when the Large Pig chooses "Press" and the Little Pig chooses "Not Press".

		Large Pig	
		Press	Not Press
Little Pig	Press	$1.5, 3.5$	$-0.5,\boxed{6}$
	Not Press	$\boxed{5},\boxed{0.5}$	$\boxed{0}, 0$

So the Nash equilibrium to this game is: Little Pig "Not Press", Large Pig "Press".

3. Consider the following game tree:



Game 1

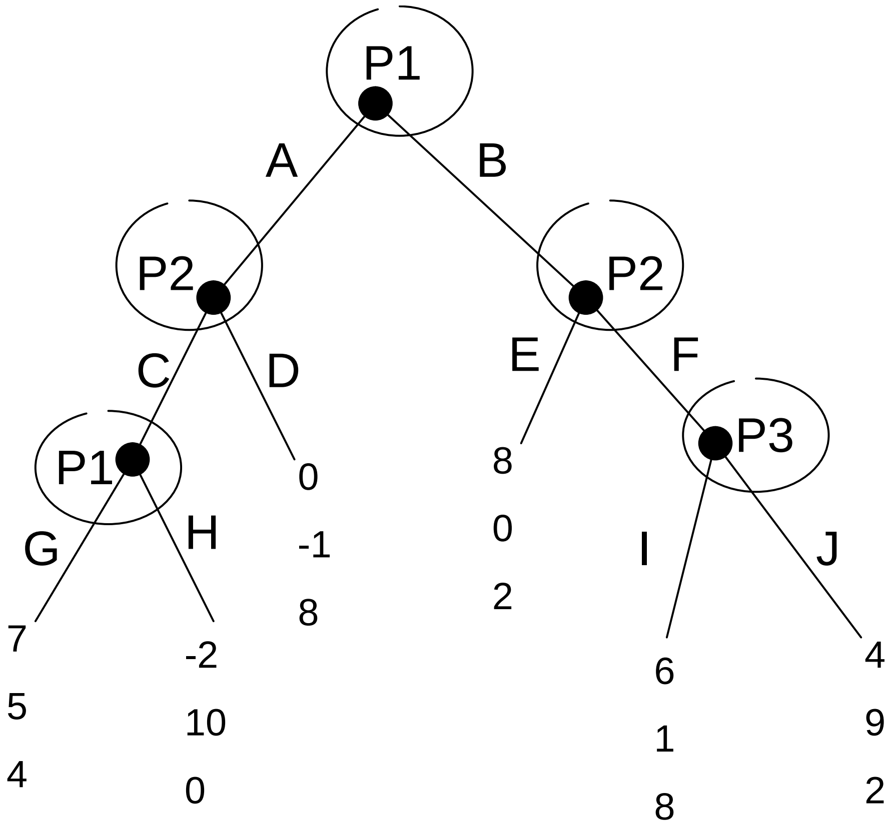
Find the subgame perfect Nash equilibrium to this game.

Answer:

If Mike gets to choose between C and D he will choose D because $6 > 4$. If Mike gets to choose between E and F he will choose E because $8 > 2$. If Jackie gets to choose between Y and Z she will receive 5 if she chooses Y (because Mike chooses D) and she will receive 3 if she chooses Z (because Mike chooses E), so she will choose Y because $5 > 3$. When Mike chooses between A and B he will get 10 if he chooses A and he will get 6 (because Jackie chooses Y and Mike then chooses D) if he chooses B, so he chooses A.

So the SPNE of this game is: Mike chooses strategy A, D, E, while Jackie chooses strategy Y.

4. Consider the following sequential game:



Find the subgame perfect Nash equilibrium to this game.

Answer:

Working from the end of the game, P3 chooses I over J (8>2). P1 chooses G over H (7>-2). P2 chooses C over D (5>-1). P2 chooses F over E (1>0). P1 chooses A over B (7>6). So the SPNE is:

P1 choose AG

P2 choose CF

P3 choose I

When the game is played using these strategies, the outcome is A, C, G with P1 receiving 7, P2 receiving 5, and P3 receiving 4.

5. Consider two firms engaged in price (Bertrand) competition, such that the firm that charges the lowest price produces the entire market quantity. If the 2 firms charge the same price they split the market quantity evenly. This means that each firm has the following demand function:

	q_1	q_2
if $p_1 > p_2$	0	$\frac{500-p_2}{2}$
if $p_1 = p_2$	$\frac{1}{2} * \frac{500-p_1}{2}$	$\frac{1}{2} * \frac{500-p_2}{2}$
if $p_1 < p_2$	$\frac{500-p_1}{2}$	0

Each firm has total cost equal to $TC(q_i) = 20 * q_i$, so that each firm has a constant marginal and average cost of production of 20.

a Suppose that the pricing choices are made simultaneously. Find the pure strategy Nash equilibrium to this game.

Answer:

This is the same Bertrand pricing game from class, only now each firm's marginal cost is equal to 20, so the PSNE to this game is $p_1 = p_2 = 20$. Note that both firms receive a profit of zero if these strategies are used. If either firm lowers its price it will capture the entire market but it will be charging a price less than its cost so it will be making a loss. So neither firm will lower its price. If either firm raises its price then the other firm captures the entire market and the firm that raised its price would still receive a profit of zero. So neither firm has an incentive to raise its price. Because no firm can unilaterally deviate from the set of strategies, we have that $p_1 = p_2 = 20$ is a PSNE to the game.

Now suppose that Firm 1 announces the following policy: We are going to charge \$260 for our product. If any customer finds a lower price for this product than \$260 then tell us and we will not only match that price but offer a refund equal to the difference in the two prices. For instance, if another firm charges \$240, we will only charge \$220 (take \$260 − \$240 = \$20 and then deduct another \$20 so that the total amount deducted from our price of \$260 is \$40). This is known as a price-beating policy.

b Find Firm 2's best response to Firm 1's policy announcement. *Reminder: This is only a one period game.*

Answer:

What happens if Firm 2 chooses a price below \$260? Then it will receive a profit of zero, because, according to Firm 1's policy, Firm 1 will end up charging a price below Firm 2 and Firm 2 will receive no sales. What happens if Firm 2 prices above \$260? Then Firm 1 receives all the sales. But if Firm 2 prices just at \$260 then Firm 1 will end up splitting the market with Firm 1 and both will make a positive profit. So Firm 2's best response is to choose $p_2 = \$260$.

c Given the best response you found for Firm 2 in part **b**, is that best response and Firm 1's strategy (announced policy and price choice of \$260) a pure strategy Nash equilibrium to the game? Explain.

Answer:

We know that Firm 2 is best responding to Firm 1's strategy, but technically Firm 1 could choose a different price than \$260 (say \$259) and be better off. This would then lead to Firm 2 charging the same price as Firm 1, Firm 1 undercutting again, and so on and so forth until both end up pricing at marginal cost again as in part **a**.

6. It is typical for the government to allocate construction contracts, such as repaving a highway, by holding an auction for the contract. The auction rules are as follows. Each bidder is to submit a sealed bid. The lowest bidder will win the contract, and the winning bidder will be paid an amount equal to the second lowest bid. Suppose that each bidder draws a cost, c_i, of completing the job from the uniform distribution over the interval $[0, 100]$. The cost draws are made independently of each other. All bidders are aware of the common distribution of costs as well as the fact that cost draws are made independently of one another.

a Suppose that the bidders are risk-neutral. Find a Nash equilibrium for this auction.

Answer:

Note that this is a 2^{nd}-price sealed bid procurement auction. Because it is a 2^{nd}-price auction, the weakly dominant strategy will be to submit a bid equal to cost. Thus, $b_i(c_i) = c_i$. One can use the same arguments from class to show that a bidder in this auction would not choose to submit a bid greater than or less than cost.

b Suppose that the bidders are risk-neutral. Suppose we changed the format so that the winning bidder, who is still the lowest bidder, now receives a payment equal to his bid if he wins (instead of a payment equal to the 2^{nd} lowest bid as before). Would this bidder bid more than or less than the bidder's cost draw c_i? Explain.[1]

[1] Finding the actual equilibrium bid function for this type of auction involves solving a differential equation, which I am not going to make you all do.

Answer:

The bidder would bid more than the cost draw c_i because the bidder needs to be able to pay the cost draw. Thus, $b_i(c_i) > c_i$ in this type of auction. Note that this auction is similar to the first-price sealed bid auction we discussed in class, only now we are looking for the lowest bid.

7. Coach Industries, Inc., is a leading manufacturer of recreational vehicle products. Its products include travel trailers, fifth-wheel trailers (towed behind pick-up trucks), and van campers, as well as parts and accessories. Coach offers its fifth-wheel trailers to both dealers (wholesale) and retail customers. Ernie Pantusso, Coach's controller, estimates that each fifth-wheel trailer costs the company \$10,000 in variable labor and material expenses. Demand relations for fifth-wheel trailers are

$$\begin{aligned} P_W &= \$15,000 - \$5Q_W \text{ (wholesale)} \\ P_R &= \$50,000 - \$20Q_R \text{ (retail)} \end{aligned}$$

a Assuming that the company can price discriminate between its two types of customers, calculate the profit-maximizing price, output, and profit contribution levels.

Answer:

With price discrimination, profits are maximized by setting $MR = MC$ in each market, where $MC = \$10,000$.

For the wholesale market:

$$\begin{aligned} MR_W &= MC_W \\ \$15,000 - \$10Q_W &= \$10,000 \\ \$5,000 &= \$10Q_W \\ 500 &= Q_W \end{aligned}$$

When $Q_W = 500$, we have $P_W = \$12,500$.

For the retail market we have:

$$\begin{aligned} MR_R &= MC_R \\ \$50,000 - \$40Q_R &= \$10,000 \\ \$40,000 &= \$40Q_R \\ 1000 &= Q_R \end{aligned}$$

When $Q_R = 1000$, we have $P_R = \$30,000$.

The profit for this firm is given by:

$$\begin{aligned} \Pi &= P_W Q_W + P_R Q_R - AVC(Q_W + Q_R) \\ \Pi &= 12,500 * 500 + 30,000 * 1,000 - 10,000 * (500 + 1,000) \\ \Pi &= \$21,250,000 \end{aligned}$$

The reason that we calculate cost as $AVC * (Q_W + Q_R)$ is because we do not know the total cost function in this particular question (we do not know the fixed cost component). But we do know that each additional unit costs $\$10,000$ in variable costs, so we can at least determine how profitable production is excluding the fixed cost component.

b Calculate point price elasticity for each customer type at the activity levels identified in part **a**. Are the differences in these elasticities consistent with your recommended price differences in part **a**? Why or why not?

Answer:

The point price elasticity of demand for the wholesale market is given by:

$$\varepsilon_P = \frac{\partial Q_W}{\partial P_W} * \frac{P_W}{Q_W}$$

Now, be careful. The demand function as it is written is $P_W = 15,000 - 5Q_W$. However, the partial derivative in the elasticity formula is $\frac{\partial Q_W}{\partial P_W}$, so we would need the reciprocal of the coefficient on Q_W. So our point elasticity for the wholesale market would be:

$$\begin{aligned} \varepsilon_P &= -\frac{1}{5} * \frac{12,500}{500} \\ \varepsilon_P &= -5 \end{aligned}$$

The point elasticity for the retail market is given by:

$$\begin{aligned} \varepsilon_P &= \frac{\partial Q_R}{\partial P_R} * \frac{P_R}{Q_R} \\ \varepsilon_P &= -\frac{1}{20} * \frac{30,000}{1,000} \\ \varepsilon_P &= -1.5 \end{aligned}$$

A higher price for retail customers is consistent with the lower degree of price elasticity observed in that market.

8. The Heritage Club at Harbor Town offers elegant accommodations for discriminating vacationers on Hilton Head Island, South Carolina. Like many vacation resorts, Heritage Club has discovered the advantages of offering its services on an annual membership or "time-sharing" basis. To illustrate, assume that an individual vacationer's weekly demand curve can be written as:

$$P = \$6,500 - \$1,250Q$$

where P is the price of a single week of vacation time, and Q is the number of weeks of vacation time purchased during a given year. For simplicity, assume that the resort's marginal cost for a week of vacation time is \$1,500, and that fixed costs are zero. This gives the following total cost function:

$$TC = \$1,500Q$$

a Calculate the profit-maximizing price, output, profit, and consumer surplus assuming a uniform per unit price is charged each customer.

Answer:

To maximize profits by charging a single per unit price just find the quantity that maximizes profit:

$$\begin{aligned} \Pi &= (6500 - 1250Q)\,Q - 1500Q \\ \Pi &= 6500Q - 1250Q^2 - 1500Q \\ \Pi &= 5000Q - 1250Q^2 \\ \frac{\partial \Pi}{\partial Q} &= 5000 - 2500Q \\ Q &= 2 \end{aligned}$$

When $Q = 2$ we have $P = \$4,000$ and $\Pi = \$5,000$.

The consumer surplus is the area under the demand curve but above the price being charged. We can think of this as a triangle with a base equal to the quantity sold, 2, and the height equal to the difference between the intercept of the demand curve (6,500) and the price being charge (4,000). Then use the formula for the area of a triangle:

$$\begin{aligned} CS &= \frac{1}{2} * 2 * (6500 - 4000) \\ CS &= 2500 \end{aligned}$$

b Calculate the profit-maximizing price, output and profit assuming the optimal two-part pricing strategy is adopted for each customer.

Answer:

For the optimal two-part pricing strategy the firm needs to charge a price equal to marginal cost, and then calculate consumer surplus under that price and charge a lump sum fee equal to the consumer surplus. Marginal cost is constant at $\$1,500$, so $P = \$1,500$. When $P = \$1,500$, we have that $Q = 4$. Now, we use the same method to determine consumer surplus (the area of the triangle) as in part **a**, only now the base is 4 (because 4 units are being sold) and the height of the triangle is 6,500-1,500 (because the price is now 1,500). The consumer surplus is:

$$
\begin{aligned}
CS &= \frac{1}{2} * 4 * 5000 \\
CS &= 10,000
\end{aligned}
$$

So the optimal lump sum fee is $\$10,000$, and the firm charges a per-unit price of $\$1,500$. The firm makes zero profit from each customer for its per-unit sales, but it makes a profit of $\$10,000$ from each consumer from the lump sum fee. Thus the profit is twice as high (at least in this example) from using the optimal two-part pricing strategy when compared to the optimal single per-unit price.

9. Each ton of ore mined from the Baby Doe Mine in Leadville, Colorado, produces one ounce of silver and one pound of lead in a fixed 1:1 ratio. Marginal costs are \$10 per ton of ore mined. The demand curve for silver is:

$$P_S = \$11 - \$0.00003 Q_S$$

and the demand curve for lead is:

$$P_L = \$0.4 - \$0.000005 Q_L$$

where Q_S is ounces of silver and Q_L is pounds of lead.

a Calculate profit-maximizing sales quantities and prices for silver and lead.

Answer:

The firm's profit function is:

$$
\begin{aligned}
\Pi &= P_S Q_S + P_L Q_L - \$10Q \\
\Pi &= (\$11 - \$0.00003 Q_S)\, Q_S + (\$0.4 - \$0.000005 Q_L)\, Q_L - \$10Q
\end{aligned}
$$

Because silver and lead are produced in a 1:1 ratio, we can set $Q_L = Q_S = Q$ and solve for the optimal quantity. Now we have:

$$
\begin{aligned}
\Pi &= (\$11 - \$0.00003Q)\, Q + (\$0.4 - \$0.000005Q)\, Q - \$10Q \\
\frac{\partial \Pi}{\partial Q} &= 11 - 0.00006Q + 0.4 - 0.00001Q - 10 \\
0 &= 1.4 - 0.00007Q \\
Q &= 20,000
\end{aligned}
$$

When $Q = Q_S = Q_L = 20,000$, we have $P_S = \$10.40$ and $P_L = \$0.30$. However, because we are examining joint production we need to make sure that each product is making a positive contribution towards covering cost (we need MR_S and MR_L to be positive).

$$
\begin{aligned}
MR_S &= 11 - 0.00006Q \\
MR_S &= 11 - 0.00006 * 20,000 \\
MR_S &= 9.8 \\
& \text{and} \\
MR_L &= 0.4 - 0.00001Q \\
MR_L &= 0.4 - 0.00001 * 20,000 \\
MR_L &= 0.20
\end{aligned}
$$

Because both are positive $Q = Q_S = Q_L = 20,000$ is the profit-maximizing quantity in both markets.

b Now assume that wild speculation in the silver market has created a fivefold (or 500%) increase in silver demand. A 500% increase in silver demand means that our new demand curve is $P_S = 5\,(\$11 - \$0.00003Q_S)$. Calculate optimal sales quantities and prices for both silver and lead under these conditions.

Answer:

Assuming that all quantity will be sold (this assumption is the starting point for multiple product markets), we have the following profit function:

$$\begin{aligned}
\Pi &= P_S Q_S + P_L Q_L - \$10Q \\
\Pi &= (5\,(\$11 - \$0.00003Q_S))\,Q_S + (\$0.4 - \$0.000005Q_L)\,Q_L - \$10Q
\end{aligned}$$

Setting $Q_S = Q_L = Q$ we have:

$$\begin{aligned}
\Pi &= 55Q - 0.00015Q^2 + 0.4Q - 0.000005Q^2 - 10Q \\
\Pi &= 45.4Q - 0.000155Q^2
\end{aligned}$$

Finding the optimal quantity we have:

$$\begin{aligned}
\frac{\partial \Pi}{\partial Q} &= 45.4 - 0.00031Q \\
0 &= 45.4 - 0.00031Q \\
Q &\approx 146,452
\end{aligned}$$

Thus, profit maximization with equal sales of each product requires that the firm mine $Q = 146,452$ tons of ore. The firm's profit is $\$3,324,500$.

However, under the assumption that the firm sells all units of both products, marginal revenues for the two products are:

$$\begin{aligned}
MR_S &= \$55 - \$0.0003Q_S \\
MR_S &= \$55 - \$0.0003 * 146,452 \\
MR_S &= \$11.06 \\
&\text{and} \\
MR_L &= \$0.4 - \$0.00001Q_L \\
MR_L &= \$0.4 - \$0.00001 * 146,452 \\
MR_L &= -\$1.06
\end{aligned}$$

Because $MR_L < 0$, the firm would be better off selling less lead on the market. In this case, the firm should maximize profit for the silver market, and then assume the marginal cost of lead is zero and maximize its profit. Finding the optimal quantity of silver first we have:

$$\begin{aligned}
\Pi_S &= P_S Q_S - \$10Q_S \\
\Pi_S &= (5\,(\$11 - \$0.00003Q_S))\,Q_S - \$10Q_S \\
\Pi_S &= \$55Q_S - \$0.00015Q_S^2 - \$10Q_S \\
\Pi_S &= \$45Q_S - \$0.00015Q_S^2
\end{aligned}$$

Finding the optimal quantity we have:

$$\begin{aligned}
\frac{\partial \Pi_S}{\partial Q_S} &= 45 - 0.0003Q_S \\
0 &= 45 - 0.0003Q_S \\
Q_S &= 150,000
\end{aligned}$$

The optimal price of silver is then $P_S = \$32.50$, and we have $\Pi_S = \$3,375,000$. Already the firm is earning more profit just by selling silver even if it decides to scrap all of its lead production. However, the firm does not need to do this. It will sell lead assuming a marginal cost of zero:

$$\begin{aligned} MR_L &= MC_L \\ \$0.4 - \$0.00001Q_L &= 0 \\ Q_L &= 40,000 \end{aligned}$$

When $Q_L = 40,000$, $P_L = \$0.20$. The firm's profit from lead is:

$$\begin{aligned} \Pi_L &= P_LQ_L - \$0Q_L \\ \Pi_L &= 0.2 * 40,000 \\ \Pi_L &= \$8,000 \end{aligned}$$

The reason that cost is zero for lead is because the firm already paid the cost of producing lead when calculating the optimal price, quantity, and profit for silver. Thus, the firm keeps 110,000 pounds of lead off of the market.

10. The Bristol, Inc. is an elegant dining establishment that features French cuisine at dinner six nights per week, and brunch on weekends. In an effort to boost traffic from shoppers during the Christmas season, the Bristol offered Saturday customers \$4 off its \$16 regular price for brunch. The promotion proved successful, with brunch sales rising from 250 to 750 units per day.

a Calculate the arc price elasticity of demand for brunch at the Bristol.

Answer:

Arc price elasticity is given by:

$$\begin{aligned} E_P &= \frac{\frac{Q_1-Q_0}{Q_1+Q_0}}{\frac{P_1-P_0}{P_1+P_0}} \\ E_P &= \frac{\frac{250-750}{250+750}}{\frac{16-12}{16+12}} \\ E_P &= \frac{\frac{-500}{1000}}{\frac{4}{28}} \\ E_P &= \frac{\frac{-1}{2}}{\frac{1}{7}} \\ E_P &= -\frac{1}{2} * \frac{7}{1} \\ E_P &= -3.5 \end{aligned}$$

b Assume that the arc price elasticity (from part A) is the best available estimate of the point price elasticity of demand. If marginal cost is \$8.56 per unit for labor and materials, calculate the Bristol's optimal markup on cost and its optimal price.

Answer:

By assumption, $\varepsilon_P = E_P = -3.5$. The optimal markup on cost is:

$$\begin{aligned} \frac{-1}{1+\varepsilon_P} &= \frac{-1}{1-3.5} \\ \frac{-1}{1+\varepsilon_P} &= 0.4 \end{aligned}$$

So the optimal markup on cost is 40%.

The optimal price, given $MC = \$8.56$, is:

$$
\begin{aligned}
0.4 &= \frac{P - 8.56}{8.56} \\
3.424 &= P - 8.56 \\
11.984 &= P
\end{aligned}
$$

11. Brake-Checkup, Inc., offers automobile brake analysis and repair at a number of outlets in the Philadelphia area. The company recently initiated a policy of matching the lowest advertised competitor price. As a result, Brake-Checkup has been forced to reduce the average price for brake jobs by 3%, but it has enjoyed a 15% increase in customer traffic. Meanwhile, marginal costs have held steady at \$120 per brake job.

    a Calculate the point price elasticity of demand for brake jobs.

    Answer:

    Typically we would need an initial quantity and price as well as a new quantity and price, but the definition of elasticity is the percentage change in quantity divided by the percentage change in price, and we know both of those numbers.

$$
\begin{aligned}
\varepsilon_P &= -\frac{15}{3} \\
\varepsilon_P &= -5
\end{aligned}
$$

    b Calculate Brake-Checkup's optimal markup on cost as well as its optimal price.

    Answer:

    Given $\varepsilon_P = -5$, the optimal markup on cost is:

$$
\begin{aligned}
\frac{-1}{1+\varepsilon_P} &= \frac{-1}{1-5} \\
markup &= \frac{1}{4}
\end{aligned}
$$

    Given that $MC = \$120$, the optimal price is:

$$
\begin{aligned}
0.25 &= \frac{P - 120}{120} \\
30 &= P - 120 \\
150 &= P
\end{aligned}
$$

12. Dr. John Dorian, chief of staff at the Northern Medical Center, has asked you to propose an appropriate markup pricing policy for various medical procedures performed in the hospital's emergency room. To help in this regard, you consult a trade industry publication that provides data about the price elasticity of demand for medical procedures. Unfortunately, the abrasive Dr. Dorian failed to mention whether he wanted you to calculate the optimal markup as a percentage of price or as a percentage of cost. To be safe, calculate the optimal markup on price and optimal markup on cost for each of the following procedures:

Procedure	Price Elasticity	Optimal Markup on Cost	Optimal Markup on Price
A	−1	–	100%
B	−2	100%	50%
C	−3	50%	33.3%
D	−4	33.3%	25%
E	−5	25%	20%

Answer:

I have filled in the table with the optimal markups. For the optimal markup on cost, we use:

$$Optimal\ Markup\ on\ Cost = \frac{-1}{1 + \varepsilon_P}$$

For the optimal markup on price we use:

$$Optimal\ Markup\ on\ Price = \frac{-1}{\varepsilon_P}$$

Note that the optimal markup on cost for Procedure A is undefined as we have zero in the denominator.