These notes essentially correspond to chapter 4 of the text.

1 Consumer Choice

In this chapter we will build a model of consumer choice and discuss the conditions that need to be met for a consumer to be making optimal decisions. We will begin with an overview of the restrictions that we place on consumer preferences. Next we will discuss how these preferences are related to consumer utility. We will then develop the concept of a budget constraint. Finally, we will show how to develop the conditions that must be met for a consumer to be behaving optimally.

2 Consumer Preferences

The main presumption is that consumers get a certain benefit or satisfaction (called utility in economics) from consuming goods and services. The goal in this section is to determine the level of utility that each bundle of goods and services gives a consumer. Although the analysis extends to more than 2 goods, we will work with 2 goods for simplicity.

2.1 Properties of Consumer Preferences

There are 3 primary properties that we will deem necessary in order for our consumer preferences to be rational. The properties are defined below. There may be some notation you are unfamiliar with, so I have defined a few symbols.

$\succsim$ – "at least as good as"
$\sim$ – "indifferent to"
$\succ$ – "preferred to"

1. **Completeness** – this property says that consumers can rank their bundles such that, given 2 bundles A and B

   - $A \succsim B$
   - $B \succsim A$
   - $A \sim B$

   Thus, one of these relationships must exist for every possible bundle. Note that if a consumer is indifferent between bundles it means he receives the same level of utility for each bundle of goods.

2. **Transitivity** – given at least 3 bundles, A, B, and C, if

   - $A \succsim B$
   - $B \succsim C$

   Then it must be the case that

   - $A \succsim C$

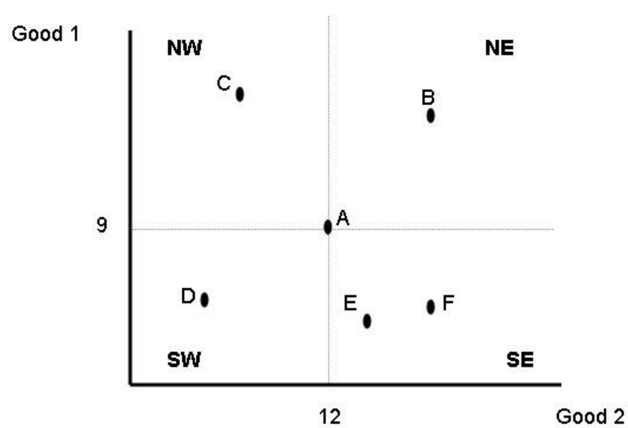
3. **Nonsatiation** (or, as it is more commonly called, more is better) – Suppose that bundle A consists of two goods, good 1 and good 2. Let q_1^A be the quantity of good 1 in bundle A and let q_2^A be the quantity of good 2 in bundle A. Suppose bundle B also consists of the same two goods, good 1 and good 2. Let q_1^B be the quantity of good 1 in bundle B and let q_2^B be the quantity of good 2 in bundle B. If $q_1^B > q_1^A$ AND $q_2^B > q_2^A$, then $B \succ A$. Notice that the relationship between the quantities of the goods is a greater than relationship (NOT greater than or equal to) and the relationship between the bundles is a preferred to relationship (NOT an at least as good relationship). If a bundle has a larger quantity of ALL goods than another bundle, then the bundle with the larger quantity is preferred to the bundle with the smaller quantity. If the case were that bundle B had a larger quantity of good 1

than bundle A but the exact same amount of good 2, then we would say that $B \succsim A$. Thus, if one bundle has more of one good but the exact same of the other goods then we say that the bundle with more of the one good can be no worse than the bundle with the lesser amounts of goods.

We will say that all consumers will have preferences that satisfy these 3 properties. You should note that our analysis still holds if we do NOT have the more is better property. The more is better property is used for two reasons. First, it seems a reasonable assumption to make that if you have more of all goods that you will be better off in the sense of having a higher utility level. Second, it makes the analysis a little more tractable.

2.2 Graphing consumer preferences

In this section we will use a graph to aid in our analysis of consumer preferences. We will focus on the positive quadrant of the Cartesian plane, as we will assume that you cannot consume negative quantities of goods. The axes of the graph will be labelled good 1 and good 2. Thus, each point (or ordered pair) on the graph will represent a bundle of goods consisting of an amount of good 1 and good 2 corresponding to that point. Below is a graph with 6 bundles distinctly labelled A–F.



Note that bundle A is given by the intersection of the 2 dotted lines, and it corresponds to a quantity of 9 of good 1 and 12 of good 2, or the ordered pair $(12, 9)$. You will also note that each section of the graph has been labelled as northeast (NE), northwest (NW), southwest (SW), or southeast (SE). These labels are in relation to point A in the graph.[1]

NE corner Now, suppose that we want to compare bundle B and bundle A based on our properties of consumer preferences. Notice that bundle B has more of both goods than bundle A. By the more is better property, it must be the case that $B \succ A$. In fact, any bundle in the NE corner of the graph is preferred to bundle A, as all of those bundles have more of both goods than bundle A.

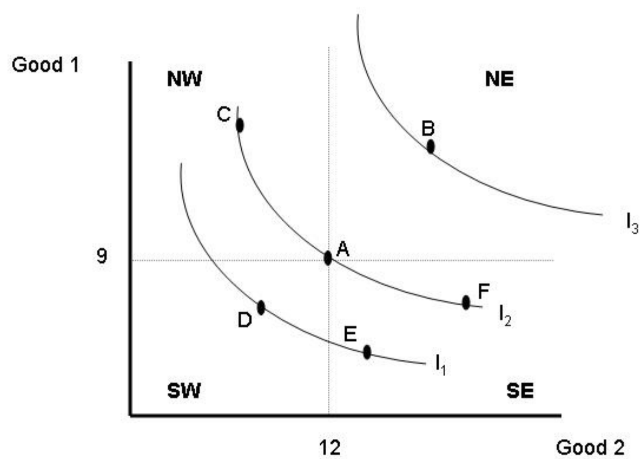
SW corner Now, let's compare bundle D and bundle A. Since bundle A has more of both goods than bundle D, by the more is better property we know $A \succ D$. Notice that bundle A has more of both goods than any bundle in the SW corner, which means that bundle A is preferred to any bundle in the SW corner.

[1] Another way to think about it is to create a new Cartesian plane with point A is the new origin. Then the NE corner is quadrant I, the NW corner is quadrant II, the SW corner is quadrant III, and the SE corner is quadrant IV.

NW and SE corners Notice that bundles in the NE corner (like bundle C) have more of good 1 than bundle A, but less of good 2. Also, bundles in the SE corner (like bundles E and F) have more of good 2 than bundle A, but less than good 1. This means we cannot use the more is better principle to determine which bundles in these corners are preferred to bundle A. Thus, the preference relation between bundles in the SE and NW corners and bundle A are determined by how much a particular consumer likes good 1 and good 2. We will use the concept of an indifference curve to determine the preference ordering of these bundles.

2.2.1 Indifference curves

An indifference curve is a plot of all the bundles that give the consumer the same level of utility (hence the name *indifference curve*, meaning that the consumer is indifferent between the bundles along the curve). Consumers have an infinite amount of indifference curves – if we were to plot all of the consumer's indifference curve we would get their indifference map. The plot below shows 3 indifference curves for this consumer. The curve through point B is labelled I_3. The curve through points C, A, and F is labelled I_2. The curve through points D and E is labelled I_1. Since C, A, and F are all on the same indifference curve, the consumer receives the same amount of utility from each bundle. Below the picture are some rules for indifference curves.



Rules for indifference curves:

1. Bundles on indifference curves farther from the origin are preferred to those closer.

   This means that the consumer would prefer to be on I_3 rather than on I_2, and would prefer to be on I_2 rather than I_1. Using the more is better principle we can see that this makes sense. The consumer prefers bundle B to bundle A, so he must have a higher utility at bundle B than he does at bundle A. Thus, all points along the indifference curve that pass through bundle B must give a higher utility level than those that are on the indifference curve that pass through bundle A. So I_3 is preferred to I_2. A similar argument can be constructed for the relationship between I_2 and I_1.

2. There is one and only one indifference curve that passes through each point.

   If there was more than one indifference curve that passes through any point, then the consumer would be saying that a bundle gives him a utility level of 12 (from the first indifference curve passing through the point) as well as a utility level of 10 (from the second indifference curve passing through the same point). Hopefully, it is obvious that this does not make any sense.

3. Indifference curves may not cross.

   For starters, if they crossed then rule 2 above would be violated. You can also show that transitivity is violated by indifference curves that cross.

4. ***Indifference curves are downward sloping.***

   I have marked this rule because we have seen examples of indifference curves that are not exactly downward sloping. If the two goods are perfect complements, or if the consumer gets zero utility from consuming one of the goods, then the indifference curves will consist of perfectly vertical lines, perfectly horizontal lines, or a combination of the two (meaning that they are L-shaped). See the section below on special cases of indifference curves.

2.2.2 Special cases of indifference curves

We will look at 4 special cases of indifference curves. The case where the consumer receives no utility from good 1, the case where the consumer receives no utility from good 2, the case where the goods are perfect complements, and the case where the goods are perfect substitutes.

No Utility from good 1 Suppose that the consumer receives no utility from good 1. In this case, the consumer can only reach a higher level indifference curve if he receives more of good 2. Since good 2 is on the x-axis, the indifference curves for these two goods will be perfectly vertical lines. As the consumer receives more of good 2 he moves to a higher indifference curve, which is an indifference curve to the right.

No Utility from good 2 Suppose that the consumer receives no utility from good 2. In this case, the consumer can only reach a higher level indifference curve if he receives more of good 1. Since good 1 is on the y-axis, the indifference curves for these two goods will be perfectly horizontal lines. As the consumer receives more of good 1 he moves to a higher indifference curve, which is an indifference curve above the original indifference curve.

Perfect Complements If two goods must ALWAYS be consumed in the same quantities, then the two goods are perfect complements. The classic example is left shoes and right shoes. Having 26 right shoes but only 1 left shoe is not going to make you any better off than if you simply had 1 right shoe and 1 left shoe. However, the bundle of 26 right shoes and 1 left shoe has to be at least as good as the bundle of 1 right shoe and 1 left shoe. This is due to the portion of the more is better principle that says that a consumer cannot be any worse off if a bundle of goods has strictly more of one good and the exact same amount of all other goods. If we were to plot these indifference curves they would be L-shaped.

Perfect Substitutes If the consumer is indifferent between which of the 2 goods he consumes then the goods are perfect substitutes. The key is that the consumer will move to a higher indifference curve if the sum total of the 2 goods increases. It should be noted that the slope of the indifference curve for 2 goods that are perfect substitutes is (-1).

2.2.3 Slope of an indifference curve

The Marginal Rate of Substitution (MRS) is defined as the maximum amount of one good a consumer will give up to obtain one more unit of another good. Thus we want to find the amount of good A that a person will give up in order to get one more unit of good B. Writing this mathematically (assuming good B is on the x-axis and good A is on the y-axis), we have the $MRS = \frac{-\Delta Q_A}{\Delta Q_B}$. Notice that this is just a formula for a slope as we simply have a change in the quantity of good 1 divided by a change in the quantity of good 2. Also notice that the MRS is negative since we must give up some of good 1 in order to get more of good 2. On a technical note, since the indifference curve is a curve and not a straight line, the slope of the indifference curve will change depending on the point at which we evaluate the slope. We will return to this concept later in the chapter.

3 Utility

We have discussed indifference curves as running through bundles of goods that give the same level of utility. We will now make the concept of utility more formal. We suppose that every consumer has a "utility function" which allows him to take different bundles of goods and assign them levels of utility in such a manner that does not violate the properties of consumer preferences described above. For instance, let $U(Q_A, Q_B)$ be the consumer's utility function that determines the level of utility a consumer receives from consuming different quantities of goods A and B. A particular utility function might be:

$$U(Q_A, Q_B) = \sqrt{Q_A * Q_B}$$

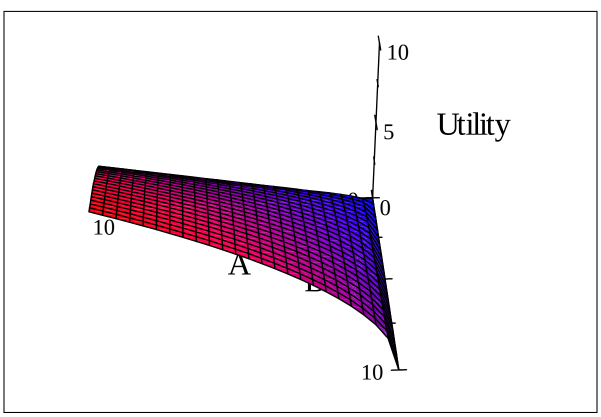
Now, for any bundle of goods A and B, we can calculate the utility level of the bundles. The table below has a few different calculations.

Q_A	Q_B	$U(Q_A, Q_B)$
9	16	12
13	13	13
12	12	12
8	18	12

Assume that the quantities in the bundles are given – then to find the utility level just plug in the quantities and calculate. You should notice that the bundles $(9, 16)$, $(12, 12)$, and $(8, 18)$ would all lie on the same indifference curve because they all have a utility level of 12. However, the bundle $(13, 13)$ would lie on a higher indifference curve because it has a utility level of 13. Note that this conforms to the more is better property because the bundle $(13, 13)$ has more of both goods than the bundle $(12, 12)$ so the consumer must prefer the bundle $(13, 13)$.

3.1 Where indifference curves come from

Indifference curves can be derived directly from utility functions. In order to do this, however, we need to use three-dimensions. Stand in the corner of a room, facing outward diagonally. Let the floor along one of the walls be the axis for the quantity of good A and let the floor along the other wall be the axis for the quantity of good B. The crease where the walls meet is the level of utility. We can now plot the utility function since we have three dimensions. It would essentially look like a cave that starts from the origin and keeps expanding outward. Alternatively, you could think about cutting a cone into two symmetric halves. If you lay one half of the cone down it (almost) looks like what we would call a utility shell. The picture below actually graphs the function $U(Q_A, Q_B) = \sqrt{Q_A * Q_B}$, although it is a little difficult to see since it is supposed to be 3-D.



Now, suppose we pick a utility level, say 2.5, and make a nice even cut through the utility shell at 2.5. If we lay the new (now smaller) utility shell directly on the ground and trace around the bottom of the shell we will have our indifference curve for utility level 2.5. If we were to do the same at every utility level, then we would have the consumer's indifference map.

3.2 Marginal Utility

An important concept in consumer theory is marginal utility. Recall that marginal means additional – as in how much additional utility a person would get if he consumed one more unit of the good. We can define the marginal utility of good A as:

$$MU_A = \frac{\Delta U}{\Delta Q_A}$$

We can also define the marginal utility of good B as:

$$MU_B = \frac{\Delta U}{\Delta Q_B}$$

An interesting relationship then results if we find the ratio of marginal utilities:

$$\frac{MU_B}{MU_A} = \frac{\frac{\Delta U}{\Delta Q_B}}{\frac{\Delta U}{\Delta Q_A}}$$

Or:

$$\frac{MU_B}{MU_A} = \frac{\Delta Q_A}{\Delta Q_B}$$

Note that both of these changes in quantities are in the positive direction. Recall that:

$$MRS = \frac{-\Delta Q_A}{\Delta Q_B}$$

Now, if we multiply $\frac{MU_B}{MU_A}$ by (-1), we will get:

$$MRS = \frac{-MU_B}{MU_A}$$

Thus the Marginal Rate of Substitution is the negative of the ratio of marginal utilities of the goods. This will prove useful when showing some results later.

4 Budget Constraints

We had a few goals when developing our consumer choice problem, one of which was to discuss how consumers choose the optimal bundle in a world where they have limited income. We will now discuss this concept of limited income. First, we will make a few assumptions about consumer behavior/attitude towards this limited income.

1. We begin with a fixed budget or endowment, denoted Y. We will analyze labor-leisure decisions a little later in chapter 5, and for right now it is best to consider our consumer with a fixed income.
2. There is no borrowing allowed (thus, no credit cards).
3. There is no saving allowed. Again, a saving-spending decision could be represented with indifference curves. We will, however, assume that all of your income must be spent now or it is lost forever.
4. Only look at decisions regarding 2 goods, although the analysis extends to n goods, where $n > 2$.
5. Assume that you can purchase fractional amounts. While this may not be true at an instantaneous point in time (try to go to Outback and order $\frac{1}{4}$ of a steak), if we looked at your average purchases of Outback steaks per day then it will not likely be a nice round number (and even if it is it is still possible for it to be a fractional amount).

4.1 Deriving a budget constraint

Whenever one derives a budget constraint it must be the case that we set expenditures equal to income (technically we need expenditures to be less than or equal to income). So we would have (assuming equality – which we will show will hold for the consumer who is behaving optimally):

$$Expenditures = Income$$

We know that our consumer's income is fixed at a level of Y. Suppose we have two goods, A and B. What are our expenditures on goods A and B? They are simply the price that we pay for the goods, P_A and P_B respectively, times the amount that we consumer of those goods, Q_A and Q_B respectively (in this analysis it is implicitly assumed that the same price is paid for all units of the good).

So we can rewrite our budget constraint as:

$$P_A * Q_A + P_B * Q_B = Y$$

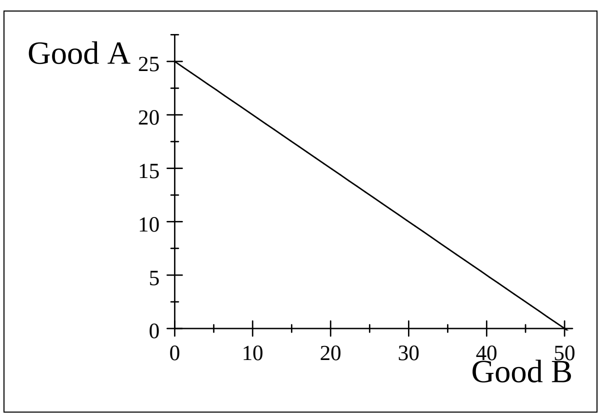
At this point you should note that the prices, P_A and P_B, as well as the income are variables whose values are known to the consumer. What the budget constraint maps out is the different quantities of goods A and B that the consumer can afford. Let's rewrite the budget constraint by solving for Q_A. We get:

$$Q_A = \frac{Y}{P_A} - \frac{P_B}{P_A} Q_B$$

Notice that the budget constraint is in the form of an equation of a line, or $y = mx + b$ form (technically it's written as $y = b + mx$ above). Note that the y-intercept of the line is $\frac{Y}{P_A}$ and the slope of the line is $\left(-\frac{P_B}{P_A}\right)$. If we were given values for Y, P_A, and P_B we could graph this line by labelling the y-axis as the quantity of good A and the x-axis as the quantity of good B. Suppose that $Y = 50$, $P_B = 1$, and $P_A = 2$. Plugging in the numbers we get:

$$Q_A = 25 - \frac{1}{2} Q_B$$

If we were to plot the budget constraint we would get:



Since plotting lines by using their equations is a little time-consuming, there is an alternative method by which we can plot the budget constraint. Recall that all you need to plot a line is 2 points, then you just connect the dots. The easiest points to find are the y-intercept and the x-intercept, and they have intuitive economic meanings. The y-intercept in this case is 25, and the bundle at this point is 0 units of good B and 25 units of good A. So the y-intercept is just the amount of good A that one could buy if one purchased 0 of good B. Since $Y = 50$ and $P_A = 2$, we can buy 25 units. It is similar for the x-intercept, which is 50 in this case. Since $Y = 50$ and $P_B = 1$, the consumer can purchase 50 units of good B if he purchases 0 units of good A.

At this point it should be noted that the consumer can purchase any bundle on the budget constraint OR inside the budget constraint. Hopefully this is intuitive. If I can afford the bundle 26 units of good

B and 12 units of good A (this is a point on the budget constraint), then I can afford 13 units of good B and 6 units of good A (this is a point inside the budget constraint). We can then define the consumer's opportunity set as the set of all the bundles that he can purchase given his income and the prices of the goods. This is the entire triangle made by the x-axis, y-axis, and budget constraint.

4.2 Income changes and the budget constraint

Suppose that the consumer's income doubled – he now has \$100. It is assumed that prices remain the same. What will happen to his budget constraint?

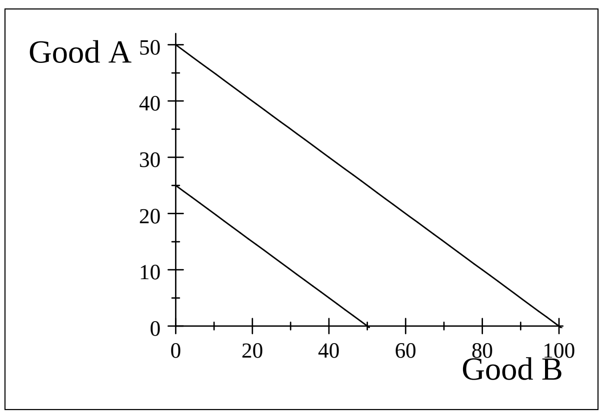
The first thing we need to do is find out how his budget constraint changes. We know that the generic formula for a budget constraint is:

$$Q_A = \frac{Y}{P_A} - \frac{P_B}{P_A} Q_B$$

If only his income changes, then only the y-intercept of the budget constraint is affected. The slope of the budget constraint remains the same since income does not enter the formula for the slope. If we plug the new income into the budget constraint formula we see that the new budget constraint is:

$$Q_A = 50 - \frac{1}{2} Q_B$$

Graphing the new budget constraint on the same graph as the old budget constraint gives us:



Since we had an increase in income the new budget constraint has made a parallel shift outward. This is reflected in the change in intercepts, the y-intercept increasing from 25 to 50 and the x-intercept increasing from 50 to 100. Notice that the consumer's opportunity set has increased as well.

4.3 Price changes and the budget constraint

Now, suppose that one of the prices change. Assume that income and the price of the other good remain constant. How does our budget constraint change?

4.3.1 Change in the price of good B

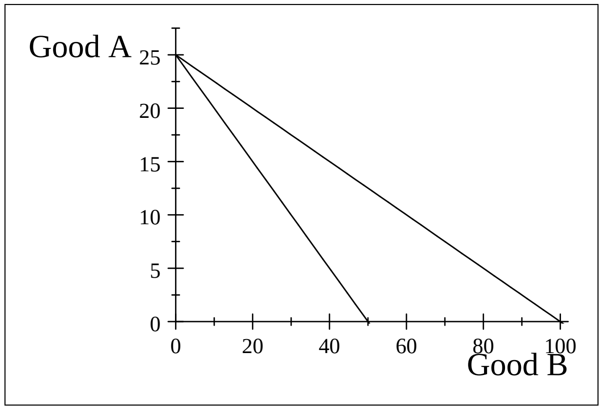
Suppose that we had a change in the price of good B. Looking at our generic formula for the budget constraint we see:

$$Q_A = \frac{Y}{P_A} - \frac{P_B}{P_A} Q_B$$

The price of good B only enters into the slope of the equation, so the y-intercept will remain the same. This should make sense, as the y-intercept tells us how much of good A we can buy if we buy 0 of good B. Since neither income nor the price of good A change we will still be able to buy exactly the same amount of good A if we buy 0 of good B. Letting the price of good B fall to 50 cents we have:

$$Q_A = 25 - \frac{1}{4}Q_B$$

Graphing this with the original budget constraint gives us:



Since the price of good B fell, we get a pivot effect on the budget constraint, as it swings out to the right. If the price of good B rose, we would still get a pivot effect, although the budget constraint would swing in to the left.

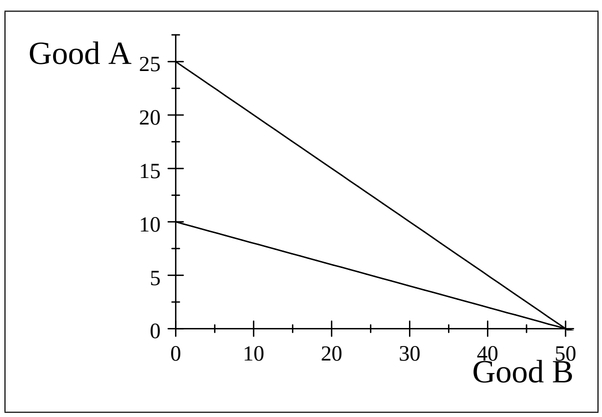
4.3.2 Change in the price of good A

Since the price of good A enters both the slope and y-intercept of our budget constraint we will see both of them change. However, the x-intercept will remain the same. What we will find is still a pivot effect on the budget constraint, only now the budget constraint pivots on the x-intercept.

Suppose the price of good A increases to \$5. Our budget constraint is now (with the price of good B being returned to it original \$1 level):

$$Q_A = 10 - \frac{1}{5}Q_B$$

Plotting this on the same graph with the original budget constraint we see:



Notice that the increase in the price of good A caused the budget constraint to swing inward. A decrease in the price of good A would have caused the budget constraint to shift outward.

The key to both changes in the price of good B and changes in the price of good A is that the slope of the budget constraint changes when either changes. As we have already seen, slopes have been important in economic analysis.

4.4 Slope of the budget constraint

The slope of the budget constraint is given a specific name in economics. We call it the Marginal Rate of Transformation (MRT). The MRT tells us the rate at which the market will allow consumers to exchange goods. If the price of good A is $2 and the price of good B is $1, then the market says that if I give up purchasing one unit of good A I can now purchase 2 additional units of good B. Mathematically then, the MRT is the $\frac{-\Delta Q_A}{\Delta Q_B}$, or how much of good A I must give up in order to get more of good B. Note that the ΔQ_A is negative, as we must give up some units of good A to receive more units of good B.

You should also notice that $\frac{-\Delta Q_A}{\Delta Q_B}$ is a formula for a slope. Specifically, the MRT is the slope of the budget constraint, which is always the same at any point along the budget constraint because the budget constraint is a line. From our generic formula for the budget constraint we know that the slope is $\frac{-P_B}{P_A}$. So we now know that:

$$MRT = \frac{-P_B}{P_A}$$

This is another useful result that we will use in the next section on optimal consumer choice.

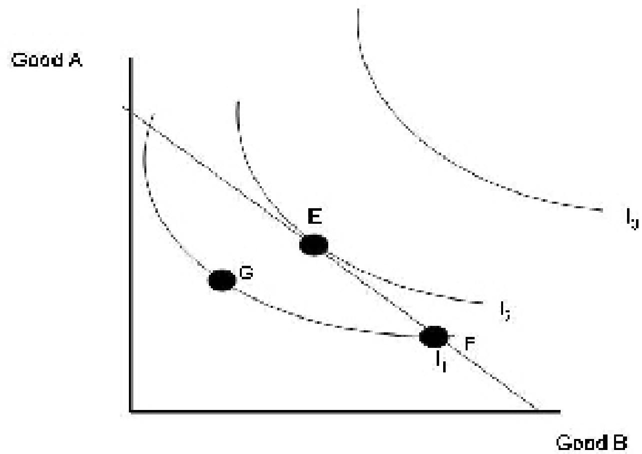
5 Optimal consumer choice – intuition and graphs

There are two types of solution we might find, an interior solution and a corner solution. It is easier to define a corner solution first. A corner solution occurs when a consumer buys either ONLY good A or ONLY good B. Thus the optimal bundle (if it is a corner solution) will look like either $(0, Q_A)$ OR $(Q_B, 0)$, where Q_A and Q_B are both assumed to be greater than zero. At an interior solution the consumer will purchase positive quantities of both goods. We will first consider the interior solution and then the corner solution.

One important point before beginning. If the consumer is acting optimally, will he purchase a bundle inside, but not on, the budget constraint? The answer is no. The easy explanation is that if the consumer chooses to purchase a bundle inside the budget constraint then he is not spending all of his money. Essentially, he is throwing money into a lake (and we are assuming he gets no utility from throwing money into a lake or a wishing well), and why would anyone throw away money when they could get goods for it? Another explanation is that for any bundle inside the budget constraint that is being considered as the optimal bundle, a different bundle ON the budget constraint can be found that has more of BOTH goods. Thus the consumer can gain utility by moving to this bundle on the budget constraint because the more is better property of consumer preferences tells us that bundles with more of both goods are preferred to bundles without as much of both goods. So if the consumer is behaving optimally he will NOT choose a point inside the budget constraint.

5.1 Interior solution

As mentioned above, an interior solution to the consumer's problem is an optimal bundle at which the consumer purchases positive quantities of both goods. Look at Figure 5.1:



A consumer's optimal choice problem – interior solution.

We know that a consumer who is optimizing will pick a point along the budget constraint, which is the downward sloping straight line in the picture. There are 2 points labelled, E and F. Suppose the consumer chooses point F. Is he behaving in an optimal manner? That is, does he maximize his utility? A consumer maximizes his utility if he chooses a bundle such that there is no other bundle that he could have chosen, given his limited income, that would place him at a higher utility level (or on a higher indifference curve). Looking at bundle F, we notice that this consumer is indifferent between bundle F and bundle G. However, bundle G lies inside the budget constraint, so there must be an affordable bundle (call it bundle X) that he prefers to bundle G. If he prefers bundle X to bundle G, then he must prefer bundle X to bundle F. Thus, F cannot be the optimal bundle.

Now, look at bundle E. The indifference curve I_2 only touches the budget constraint once (it is tangent to the budget constraint). Note that there is no other bundle that the consumer can afford that would put him on a higher indifference curve. Thus, the optimal bundle is found by finding the indifference curve that is tangent to the budget constraint.

5.1.1 A key result for interior solutions

Recall that the slope of the budget constraint is the MRT. Also recall that the slope of the indifference curve is the MRS. A result from math class (I don't remember which one) is that if a line is tangent to a curve, then the slope of the line and the slope of the curve AT THE POINT OF TANGENCY are equal. Thus, at the consumer's optimal bundle we have:

$$MRT = MRS$$

We know a few other things. We know that:

$$MRT = -\frac{P_B}{P_A}$$
$$MRS = \frac{-MU_B}{MU_A}$$

Substituting, we get:

$$-\frac{P_B}{P_A} = \frac{-MU_B}{MU_A}$$

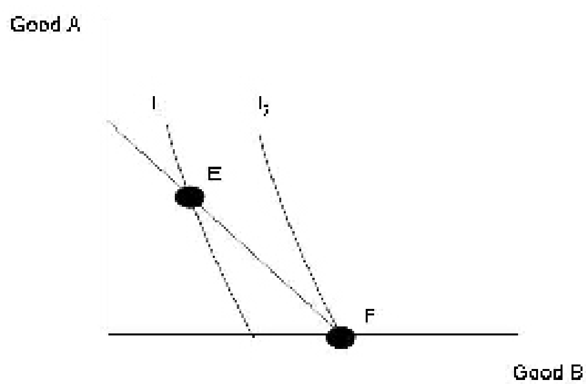
Doing some rearranging gives us:

$$\frac{MU_A}{P_A} = \frac{MU_B}{P_B}$$

Notice what this equation tells us. At the optimal bundle, the marginal utility per dollar of each good must be the same. If it is not, the consumer can do better by shifting some dollars from the good with the lower $MU/\$$ to the good with the higher $MU/\$$. As an example, suppose that the consumer has \$10 and that he goes to Rio Bravo when they sell \$1 drafts and 10-cent wings. For simplicity, assume he must buy 10 wings at a time, so that he gets 10 wings for \$1. Now suppose that you purchase 90 wings and 1 draft. You get through 10 wings and your 1 draft and think, "I would really like another draft to go with the other 80 wings that I have". Clearly you have NOT equated the marginal utilities per dollar for the two goods, otherwise you would not have thought this thought. In this case, if you could go back in time and reallocate your \$10 by making a different purchase, you would take some of the money you spent on wings (which have a low MU since you have so many of them) and you would shift those funds to drafts (which have a high MU at the bundle (90 wings, 1 draft) because you only have one draft).

5.2 Corner solution

A corner solution has slightly different implications for behavior than an interior solution. The key is that the result that we have from an interior solution, $MRS = MRT$, does NOT have to be met at a corner solution. Corner solutions typically occur when indifference curves are relatively flat or relatively steep. A relatively flat indifference curve suggests that a consumer gets more marginal utility from the good on the y-axis, while a relatively steep indifference curve suggests that the consumer heavily favors the good on the x-axis.[2] Take a look at Figure 5.2:



A picture of the consumer's choice problem – corner solution.

In this instance the consumer puts more weight on good B in the utility function than he does on good A. Thus, unless the price of good A is very small relative to the price of good B (which means that the budget constraint must be very steep), the consumer will purchase only good B. This is shown by bundle F in the picture, which is the bundle where the budget constraint and the x-axis intersect. Notice that this point is on the budget constraint, but we cannot find a bundle that would give the consumer a higher level of utility. Thus, the consumer only purchases good B to maximize his utility.

[2] Think about the cases where the consumer gets no utility from one of the two goods. They are perfectly horizontal and vertical indifference curves. The consumer will NEVER purchase a good for which he gets no utility, thus we will always end up at a corner solution in those cases.

If you look at the graphs of the interior solution and the corner solution you should see one key difference. If an interior solution occurs, there will be indifference curves that cross the budget constraint twice (such as I_1 in the interior solution picture). However, if a corner solution occurs, the indifference curves will only cross the budget constraint once.

5.2.1 Does $MRS = MRT$ at a corner solution?

The answer is, "maybe". There are cases where $MRS = MRT$ at the corner solution. These are very special cases however, and the vast majority of the time $MRS \neq MRT$ at a corner solution. If you look at the picture, notice that I_2 actually intersects the budget constraint – it is NOT tangent to the budget constraint. If there is an intersection of the budget constraint and the indifference curve at a corner solution then the result that $MRT = MRS$ will NOT hold. This is because the line is not tangent to the curve at that point. The consumer would actually be better off if he could consume negative quantities of good A in this case, as the point of tangency for his indifference curve and his budget constraint is actually in quadrant IV of the Cartesian plane in this example.

6 Optimal consumer choice

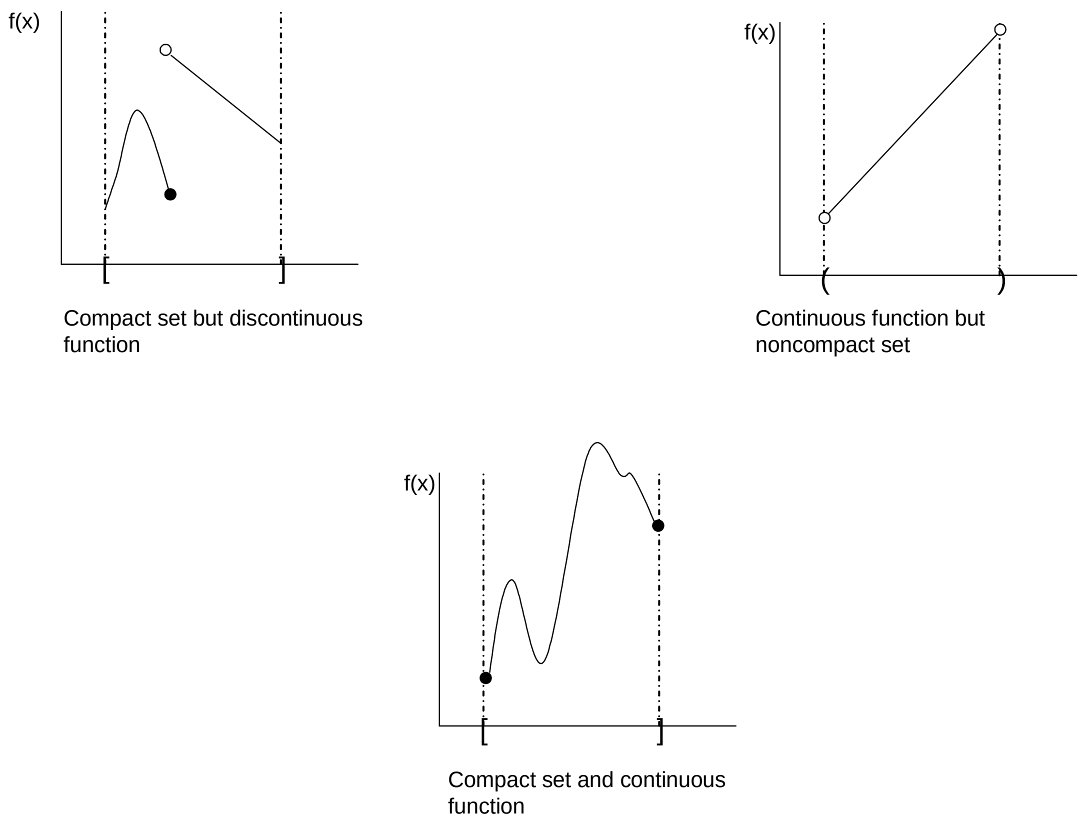
The consumer's goal is to maximize utility, given that they have a budget constraint. We can be a little more formal, saying something like the consumer's goal is to choose the most preferred consumption bundle $x(p, y)$ (note that $x(p, y)$ is a vector of quantities – there is an $x_1(p, y)$ and an $x_2(p, y)$ in our 2 good world) given prices $p >> 0$ and $y > 0$ (and hence, budget constraint $p * x \leq y$). Write this out as a maximization problem:

$$\max_{x \geq 0} u(x) \text{ subject to (or s.t.) } p * x \leq y$$

This is a nice problem, but does it have a solution? There are two questions that you will want to ask when setting up your models. The first is, Does a solution exist? The second is, Is the solution unique? So existence and uniqueness are two concepts that model builders, at least in the sense of classical demand theory, strive for.[3] We will briefly discuss the basics behind existence and uniqueness of the solution.

There is a nice result called the Weierstrass Theorem (extreme value theorem) that states that a continuous function attains a maximum (as well as a minimum) on any compact set. While we will not go through the details, it guarantees that there is actually a maximum to our problem, so that a solution does exist. Figure 6

[3] When we discuss game theory there are some game theorists who believe the fact that multiple equilibria exist in theory is useful because multiple equilibria exist in the real world. The question then becomes how one of those equilibria was selected in one case and how another was selected in a second case. So uniqueness is not necessarily that important to some game theorists, but they still strive for existence.



Examples that show a maximum may not be guaranteed without certain assumptions.

shows examples of why we need both a continuous function and a compact set to guarantee the existence of a maximum. If the set is compact but the function is discontinuous then it is possible to have the function be open where the maximum would be. Thus, the maximum would never be reached. The same is true if the function is continuous but the set is not compact. The maximum may be at the boundary of the set, but since that boundary is never reached the maximum is never reached. The bottom picture provides an example where a maximum is attained, although it is only an example and not a proof. It would be simple to construct examples for the other cases where a maximum is attained, but these counterexamples are sufficient to disprove the suggestion that a maximum would be guaranteed without a continuous function or a compact set. For uniqueness we need that the preference relation, $\succsim$, to be convex, which simply means that convex combinations of two bundles of goods are preferred to either of the two bundles.

6.1 Inequality Constrained Optimization

We now know that given our consumer's problem there is a solution and it is unique (provided the assumptions we made on $\succsim$ and $u(\cdot)$ hold). Now we will discuss the mechanics of actually solving the consumer's problem and finding $x(p, y)$.

Consider a general 2-good problem with goods x_1 and x_2. We assume that $\succsim$ is rational, continuous, monotone, and strictly convex, so that $u(x_{1,}x_2)$ is continuous, increasing, and strictly quasiconcave. The consumer faces prices $p_1 > 0$ and $p_2 > 0$ for goods x_1 and x_2 respectively, and has a level of wealth $y > 0$, and that $p_1x_1 + p_2x_2 \leq y$. We will also assume (for the current example) that $x_1^*(p, y) > 0$ and $x_2^*(p, y) > 0$, where $x_1^*(p, y)$ and $x_2^*(p, y)$ are the consumer's optimal consumption levels of x_1 and x_2. This means there is an interior solution, and not a corner solution. The consumer's problem is then:

$$\max_{x_1, x_2} u(x_1, x_2) \text{ s.t. } p_1x_1 + p_2x_2 \leq y$$

Some steps:

1. Rewrite $p_1x_1 + p_2x_2 \leq y$ as $y - p_1x_1 - p_2x_2 \geq 0$ (this is a technical aspect which we do not have time to discuss in this class – setting up the inequality in this manner lets us add the constraint instead of subtract it).

2. Form the Lagrangian,
$$\mathcal{L}(x_1, x_2, \lambda) = u(x_1, x_2) + \lambda [y - p_1x_1 - p_2x_2]$$

3. Ponder where this λ came from ... (we will discuss this shortly ... mechanics right now)

If $x_1^*(p, y) > 0$ and $x_2^*(p, y) > 0$, we get the following Kuhn-Tucker conditions:

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial x_1} &= \frac{\partial u(x_1^*, x_2^*)}{\partial x_1} - \lambda^* p_1 = 0 \\ \frac{\partial \mathcal{L}}{\partial x_2} &= \frac{\partial u(x_1^*, x_2^*)}{\partial x_2} - \lambda^* p_2 = 0 \\ y - p_1x_1 - p_2x_2 &\geq 0 \\ \lambda^* [y - p_1x_1 - p_2x_2] &= 0 \end{aligned}$$

The first 2 conditions are the first order conditions (FOCs) with respect to our consumer's 2 choice variables, x_1 and x_2 (it is a maximization problem after all). The 3^{rd} condition is our inequality constraint (it is an *inequality* constrained maximization problem after all). The last condition is called the complementary slackness condition. The consumer's goal is to maximize $u(x_1, x_2)$, NOT $\mathcal{L}(x_1, x_2, \lambda)$. This complementary slackness condition assures us that $u(x_1, x_2) = \mathcal{L}(x_1, x_2, \lambda)$. This means that either $\lambda^* = 0$ or $y - p_1x_1 - p_2x_2 = 0$. But we know from our discussion of the intuition of the consumer's optimal choice problem that $y - p_1x_1 - p_2x_2 = 0$, so we also know that condition 3 holds with equality. Now we have a system of 3 equations (the FOCs and the constraint which is now an equality) and 3 unknowns (x_1, x_2, λ).

Something we can see is that at the optimum,

$$\begin{aligned} \frac{\partial u(x_1^*, x_2^*)}{\partial x_1} &= \lambda^* p_1 \\ \frac{\partial u(x_1^*, x_2^*)}{\partial x_2} &= \lambda^* p_2. \end{aligned}$$

Note that $\frac{\partial u(x_1^*, x_2^*)}{\partial x_1}$ is the marginal utility of good x_1, or MU_{x_1} and that $\frac{\partial u(x_1^*, x_2^*)}{\partial x_2}$ is the marginal utility of good x_2, or MU_{x_2}. If we take the ratio of those 2 equations, we get: $\frac{MU_{x_1}}{MU_{x_2}} = \frac{p_1}{p_2}$, or $\frac{MU_{x_1}}{p_1} = \frac{MU_{x_2}}{p_2}$. These equations should look familiar as we discussed them as the "conditions" for an interior solution to the consumer's optimization problem when we solved the problem graphically. Note that $\frac{p_1}{p_2}$ is the slope of the budget line (the negative of the slope) and that $\frac{MU_{x_1}}{MU_{x_2}}$ is the marginal rate of substitution, or the slope of the indifference curve at x_1^* and x_2^*, so that the slope of the indifference curve is equal to the slope of the budget line at that point, or, in very technical terms, the budget line is tangent to the indifference curve at that point.

Now, what is λ? This variable λ tells us the marginal or shadow value of relaxing the constraint in the consumer's problem. When applied to the budget constraint, it is the marginal value of wealth. Think about when $\lambda > 0$ and $\lambda = 0$. If $\lambda > 0$, then wealth has a positive marginal value, and more y will increase $u(\cdot)$ or $\mathcal{L}(\cdot)$, if we hold the other variables constant. If $\lambda = 0$ (and this is only hypothetically speaking here), then additional y is worthless to the consumer, even holding the other variables constant. That is because the constraint would be non-binding, and the consumer would have chosen x_1^* and x_2^* such that $y - p_1x_1^* - p_2x_2^* > 0$. Our concern as of right now is not with a specific value of λ, but whether or not $\lambda > 0$ or $\lambda = 0$.

Here is an actual example with a utility function. Let $u(x_1, x_2) = \alpha \ln x_1 + (1 - \alpha) \ln x_2$, $\alpha \in (0, 1)$. The consumer's problem is:
$$\max_{x_1 > 0, x_2 > 0} u(x_1, x_2) \text{ s.t. } p_1x_1 + p_2x_2 \leq y$$

We know that we will need to formulate the Lagrangian,

$$\mathcal{L}(x_1, x_2, \lambda) = u(x_1, x_2) + \lambda [y - p_1 x_1 - p_2 x_2]$$

and to obtain the Kuhn-Tucker conditions:

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial x_1} &= \frac{\partial u(x_1^*, x_2^*)}{\partial x_1} - \lambda^* p_1 = 0 \\
\frac{\partial \mathcal{L}}{\partial x_2} &= \frac{\partial u(x_1^*, x_2^*)}{\partial x_2} - \lambda^* p_2 = 0 \\
y - p_1 x_1 - p_2 x_2 &\geq 0 \\
\lambda^* [y - p_1 x_1 - p_2 x_2] &= 0
\end{aligned}$$

We know that if $u(x_1, x_2) = \alpha \ln x_1 + (1 - \alpha) \ln x_2$, then

$$\begin{aligned}
\frac{\partial u(x_1^*, x_2^*)}{\partial x_1} &= \frac{\alpha}{x_1} \\
\frac{\partial u(x_1^*, x_2^*)}{\partial x_2} &= \frac{1-\alpha}{x_2}
\end{aligned}$$

We also know that $y - p_1 x_1 - p_2 x_2 = 0$, so we have 3 equations with 3 unknowns. I won't type out all the rearranging of terms for this term to find the solution, but you should be able to verify that

$$\begin{aligned}
x_1^*(p, y) &= \frac{\alpha y}{p_1} \\
x_2^*(p, y) &= \frac{(1-\alpha) y}{p_2} \\
\lambda^* &= \frac{1}{y} > 0
\end{aligned}$$

Now suppose that $p_1 = 10$, $p_2 = 5$, $\alpha = \frac{1}{3}$, and $y = 100$. We can actually find our consumer's optimal bundle in terms of a number. Plugging in those values we get that $x_1^*(p, y) = \frac{10}{3}$, $x_2^*(p, y) = \frac{40}{3}$, and $\lambda^* = \frac{1}{100} > 0$. Moreover, $\frac{MU_{x_1}}{MU_{x_2}} = \frac{1/3}{10/3} / \frac{2/3}{40/3} = 2$. Also, $\frac{p_1}{p_2} = 2$. So, $\frac{MU_{x_1}}{MU_{x_2}} = \frac{p_1}{p_2}$.

6.1.1 Slightly more general notation

We may not have a guarantee of an interior solution, but we still want to restrict $x_1 \geq 0$ and $x_2 \geq 0$. So, our consumer's problem is still to maximize utility subject to his budget constraint, but now we have the additional constraints that $x_1 \geq 0$ and $x_2 \geq 0$. Writing this out for a two good problem we have:

$$\max_{x_1, x_2} u(x_1, x_2) \text{ s.t. } p_1 x_1 + p_2 x_2 \leq y,\ x_1 \geq 0,\ x_2 \geq 0.$$

We can still follow the same steps as before, making sure that all our constraints are written as $\geq$ constraints. Since $x_1 \geq 0$ and $x_2 \geq 0$ are already written in this manner, that just leaves rewriting the budget constraint as $y - p_1 x_1 - p_2 x_2 \geq 0$. Now we can form the Lagrangian:

$$\mathcal{L}(x_1, x_2, \lambda_1, \lambda_2, \lambda_3) = u(x_1, x_2) + \lambda_1 [y - p_1 x_1 - p_2 x_2] + \lambda_2 [x_1] + \lambda_3 [x_2]$$

We will now have a full set of Kuhn-Tucker conditions for both our choice variables and our Lagrange multipliers:

$$\begin{array}{lll}
\frac{\partial \mathcal{L}}{\partial x_1} \leq 0, & x_1 \geq 0, & x_1 * \frac{\partial \mathcal{L}}{\partial x_1} = 0 \\
\frac{\partial \mathcal{L}}{\partial x_2} \leq 0, & x_2 \geq 0, & x_2 * \frac{\partial \mathcal{L}}{\partial x_2} = 0 \\
\frac{\partial \mathcal{L}}{\partial \lambda_1} \geq 0, & \lambda_1 \geq 0, & \lambda_1 * \frac{\partial \mathcal{L}}{\partial \lambda_1} = 0 \\
\frac{\partial \mathcal{L}}{\partial \lambda_2} \geq 0, & \lambda_2 \geq 0, & \lambda_2 * \frac{\partial \mathcal{L}}{\partial \lambda_2} = 0 \\
\frac{\partial \mathcal{L}}{\partial \lambda_3} \geq 0, & \lambda_3 \geq 0, & \lambda_3 * \frac{\partial \mathcal{L}}{\partial \lambda_3} = 0
\end{array}.$$

Note that in this case we have complementary slackness conditions for the choice variables because we are uncertain as to whether or not the constraint is binding. If we end up at a corner solution, then either $x_1 = 0$ or $x_2 = 0$, so one of the constraints will be binding. Notice why when we assumed that we had an interior solution that we did not have $\frac{\partial \mathcal{L}}{\partial x_1} \leq 0$ and $\frac{\partial \mathcal{L}}{\partial x_2} \leq 0$, but $\frac{\partial \mathcal{L}}{\partial x_1} = 0$ and $\frac{\partial \mathcal{L}}{\partial x_2} = 0$. If $x_1 > 0$ and $x_2 > 0$, then those partial derivatives must be zero. Technically, these Kuhn-Tucker conditions are one piece of the necessary and sufficient conditions for a general maximization problem with no guarantee of an interior solution. The theorem is called the Arrow-Enthoven Theorem, which we will not discuss in detail here but which you can look up on your own.

6.1.2 An example with a binding constraint

Given our original problem, with $u(x_1, x_2) = \alpha \ln x_1 + (1-\alpha)\ln x_2$, budget constraint $y - p_1x_1 - p_2x_2 \geq 0$, and an interior solution (**note**: think $x_1 > 0$ and $x_2 > 0$), we know that $x_1^*(p, y) = \frac{\alpha y}{p_1}$ and $x_2^*(p, y) = \frac{(1-\alpha)y}{p_2}$. Furthermore, when $\alpha = \frac{1}{3}$, $p_1 = 10$, $p_2 = 5$, and $y = 100$, $x_1^*(p_1 = 10, p_2 = 5, y = 100) = \frac{10}{3}$ and $x_2^*(p_1 = 10, p_2 = 5, y = 100) = \frac{40}{3}$. Now we will add the constraint that $x_1 \geq 4$, which forces the consumer to consume 4 units of good x_1. Additionally, we will make one more assumption, that $y > 4p_1$ rather than $y > 0$. This ensures that our consumer can actually afford 4 units of x_1. Now, before we even start, will this new constraint be binding using the parameters of $\alpha = \frac{1}{3}$, $p_1 = 10$, $p_2 = 5$, and $w = 100$? Of course it will, since the consumer only chose to consume $\frac{10}{3} < 4$ units of x_1 when the constraint was not imposed. Now, let's set up the Lagrangian:

$$\mathcal{L}(x_1, x_2, \lambda_1, \lambda_2) = \alpha \ln x_1 + (1-\alpha)\ln x_2 + \lambda_1[y - p_1x_1 - p_2x_2] + \lambda_2[x_1 - 4].$$

We get:

$$\begin{array}{cc} \frac{\partial \mathcal{L}}{\partial x_1} = \frac{\alpha}{x_1} - \lambda_1 p_1 + \lambda_2 = 0 & \\ \frac{\partial \mathcal{L}}{\partial x_2} = \frac{(1-\alpha)}{x_2} - \lambda_1 p_2 = 0 & \\ y - p_1x_1 - p_2x_2 \geq 0 & \lambda_1 * [w - p_1x_1 - p_2x_2] = 0 \\ x_1 - 4 \geq 0 & \lambda_2 * [x_1 - 4] = 0 \end{array}.$$

We know that the budget constraint will hold with equality, so that $y - p_1x_1 - p_2x_2 = 0$. Now focus on our last equation, $x_1 - 4 \geq 0$. Either $x_1 - 4 = 0$ or $\lambda_2 = 0$ (there is the remote possibility that both occur, which would happen using our numbers if we changed w from 100 to 120. However, $\lambda_2 = 0$ in this case because the constraint does not bind, meaning that the consumer would choose LESS than 4 units at his optimal consumption bundles). Now, if $\lambda_2 = 0$ then the constraint is not binding and we are right back to where we started, with $x_1^*(p, y) = \frac{\alpha y}{p_1}$ and $x_2^*(p, y) = \frac{(1-\alpha)y}{p_2}$ (just impose $\lambda_2 = 0$ in $\frac{\partial \mathcal{L}}{\partial x_1}$ to see this). If the constraint is binding, then $x_1^*(p, y) = 4$. If we know that $x_1^*(p, y) = 4$, then from the budget constraint we know that $x_2^*(p, y) = \frac{y - 4p_1}{p_2}$. So, our Walrasian demand function would be:

$$\begin{array}{c} x_1^*(p, y) = \begin{cases} 4 \text{ if } \lambda_2 > 0 \\ \frac{\alpha y}{p_1} \text{ if } \lambda_2 = 0 \end{cases} \\ x_2^*(p, y) = \begin{cases} \frac{y - 4p_1}{p_2} \text{ if } \lambda_2 > 0 \\ \frac{(1-\alpha)y}{p_2} \text{ if } \lambda_2 = 0 \end{cases} \end{array}.$$

Now, how do we check for our specific problem? From $\frac{\partial \mathcal{L}}{\partial x_2}$ we know that $\lambda_1 = \frac{(1-\alpha)}{x_2 p_2}$. Substituting this back into $\frac{\partial \mathcal{L}}{\partial x_1}$ we can see that $\lambda_2 = \frac{(1-\alpha)p_1}{x_2 p_2} - \frac{\alpha}{x_1}$ (**Note**: if $w = 120$, and the remaining parameters are kept as before, then $x_1 = 4$ and $x_2 = 16$. Plug those values into the equation for λ_2 and this illustrates that $\lambda_2 = 0$ despite the fact that $x_1 = 4$. Since the consumer would have chosen $x_1 = 4$ without the constraint, the constraint is not binding.). Now, if $\lambda_2 = 0$, then we are right back to the original utility maximization problem because $x_1 = \frac{x_2 \alpha p_2}{(1-\alpha)p_1}$. When we plug this back into the budget constraint and solve for x_2 we find that $x_2^* = \frac{(1-\alpha)y}{p_2}$ and $x_1^* = \frac{\alpha y}{p_1}$ if $\lambda_2 = 0$. However, once we substitute in our original parameters of $\alpha = \frac{1}{3}$, $p_1 = 10$, $p_2 = 5$, and $y = 100$, we see that $x_1^* = \frac{10}{3}$, which violates the constraint that $x_1 \geq 4$, and so we know that $\lambda_2 > 0$.

In the general problem, which is to maximize $u(x_1, x_2)$ subject to the budget constraint, you would typically have to check 8 different cases.

x_1	x_2	λ
$+$	$+$	$+$
$+$	$+$	0
$+$	0	$+$
0	$+$	$+$
0	0	$+$
0	$+$	0
$+$	0	0
0	0	0

You would have to check the cases that all of the choice variables are positive, all are equal to 0, or some variables are positive and some are equal to zero. With 3 choice variables we would have $2^3 = 8$ possibilities, with 4 we would have 16, with 5 we would have 32, etc. However, in our general two good problem with a budget constraint we know that $\lambda > 0$, and we know that since $y > 0$ and $p_1 x_1 + p_2 x_2 = y$, and x_1 and x_2 are both greater than or equal to zero that we cannot have λ positive with both $x_1 = x_2 = 0$, so we are now down to 3 cases. Either $\lambda, x_1, x_2 > 0$, or $\lambda, x_1 > 0$ and $x_2 = 0$, or $\lambda, x_2 > 0$ and $x_1 = 0$. So basically all you would have to do is check to see if either $x_1 = 0$ or $x_2 = 0$. You can do this by checking whether utility is higher at either $x_1 = \frac{y}{p_1}$ and $x_2 = 0$, or $x_1 = 0$ and $x_2 = \frac{y}{p_2}$, or at the interior solution you found. For our parameters of $\alpha = \frac{1}{3}$, $p_1 = 10$, $p_2 = 5$, and $y = 100$, and our optimal bundles $x_1 = \frac{10}{3}$ and $x_2 = \frac{40}{3}$, we had $u\left(\frac{10}{3}, \frac{40}{3}\right) = \frac{1}{3}\ln\frac{10}{3} + \frac{2}{3}\ln\frac{40}{3} = 2.1282$. For $u(10, 0) = \frac{1}{3}\ln 10 + \frac{2}{3}\ln 0$ we get something undefined since $\ln 0$ is undefined, and the same for $u(0, 20) = \frac{1}{3}\ln 0 + \frac{2}{3}\ln 20$. So we "know" that we have an interior solution, at least with the original problem without the constraint that $x_1 \geq 4$. It is easier to see this if we use $u(x_1, x_2) = x_1^{\alpha} x_2^{1-\alpha}$. For our parameters the optimal consumption bundle is still $x_1 = \frac{10}{3}$ and $x_2 = \frac{40}{3}$, so that $u\left(\frac{10}{3}, \frac{40}{3}\right) = \frac{10}{3}^{1/3} \frac{40}{3}^{2/3} = 8.3995$. If x_1 or x_2 equal 0, then our utility will be 0, which is less than 8.3995.