Algebra notes

August 23, 2022

Some background notes on algebra including graphing, solving systems of equations, and exponents.

1 Graphing

Throughout this course we will use an approach that involves graphs. While we will not necessarily be graphing specific functions, a review of graphing can be helpful.

1.1 Real Numbers

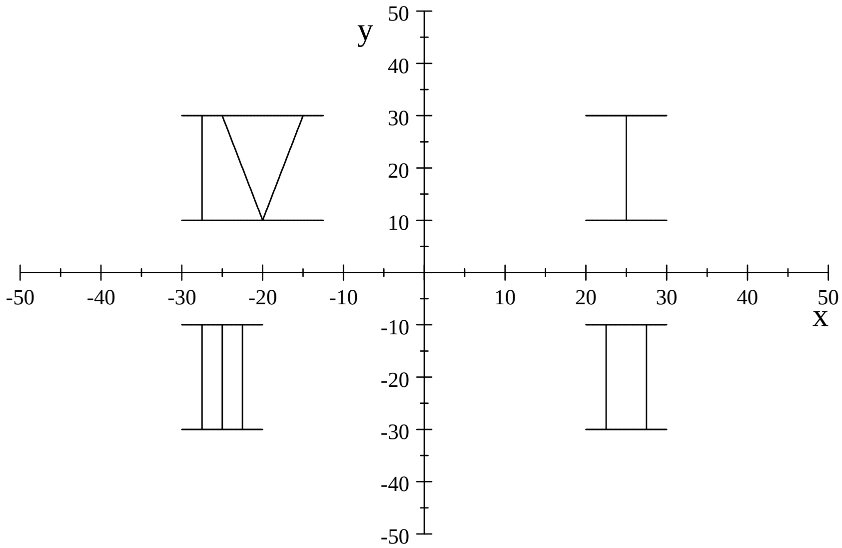
The real number line runs from negative infinity to positive infinity and include integers $(..., -3, -2, -1, 0, 1, 2, 3, ...)$, fractions (or rational numbers – which is any number that can be represented by the ratio of two integers) and irrational numbers (like $\sqrt{2}$, which cannot be represented by the ratio of two integers). The real numbers are typically denoted by $\mathbb{R}$ which you may see in more technical economics papers. The real numbers themselves are a single dimension so they can be represented by a line. At times we need more than one dimension, so if we had n dimensions we would write $\mathbb{R}^n$. We could have n dimensions when discussing a consumer who has preferences over n goods. Typically we will let $n = 2$ because it is easy to represent two dimensions on a graph.

It may seem fairly elementary to start with the set of real numbers, but there are entire courses dedicated to analyzing their properties and proving that the real number line exists. We will use the philosophy of my advisor, who, when asked to prove the existence of the real number line in one of his graduate classes, drew it on a piece of paper and said "It exists."

1.2 Cartesian plane

Our focus will be on plots in a two-dimensional space, which we will represent by the Cartesian plane. The Cartesian plane is just two real number lines that have a perpendicular intersection. In standard mathematics classes we typically label the horizontal axis the x-axis and the vertical axis the y-axis as in Figure 1.2. The intersection of the x-axis and y-axis is called the origin.

Recall that the Cartesian plane has four quadrants, typically labelled I, II, III, and IV.

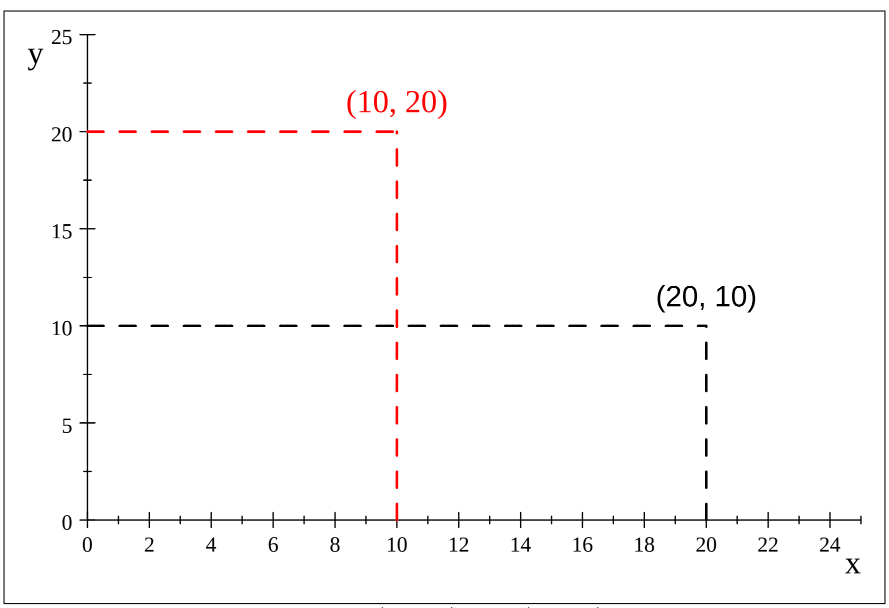


Cartesian plane, which is a representation of a two-dimensional space of real numbers.

Quadrant I is the space where both of those sets of real numbers are positive; Quadrant III is the space where both of those sets of real numbers are negative; Quadrants II and IV are spaces with one positive set of real numbers and one negative set. Quadrant I will be our primary focus as we will be concerned with nonnegative amounts of goods and services.

1.2.1 Coordinates and plotting a point

The most basic element to plot is a point. In our one-dimensional space, the real number line, a point is just a number. In our two-dimensional space, the Cartesian plane, a point is a pair of numbers (x, y) or coordinates or an ordered pair. The convention is to list the x coordinate first and the y coordinate second in the pair.[1] The origin is assigned the coordinates $(0, 0)$. To plot $(10, 20)$ we move 10 positive points away from the origin along the x-axis and then 20 positive points along the y-axis; to plot $(20, 10)$ we move 20 positive points along the x-axis and then 10 positive points along the y-axis.



Plots of $(10, 20)$ and $(20, 10)$.

[1] That we list the x coordinate and the y coordinate in a specific order is what we mean by an ordered pair.

Typically we will relabel the x- and y-axes with economic variables (quantity, price, cost, \$, etc.) that will depend upon the question we are answering.

1.3 Functions

A function defines a very specific relationship between x and y coordinates. In essence, a function is just a collection of points or a set, though it is a set that has particular properties. We can think of Quadrant I of the Cartesian plane as a set – it is the collection of all points such that both x and y are greater than zero. However, in that set (Quadrant I) there are multiple values of y for each x and vice versa.

With a function, for each value of x we only want a single value of y. We can have functions so that each x is associated with a unique value of y. The equation $y = x$ is just the collection of points where x and y are the same: $(1, 1), (-12, -12), \left(\frac{1}{9}, \frac{1}{9}\right)$, etc. However, we can also have functions where two different x values might have the same y value but still there is only one y value associated with each x. The equation $y = x^2$ is just the collection of points where y is the square of x. In this case, whether $x = 5$ or $x = -5$ we have $y = 25$. So $(5, 25)$, $(-5, 25)$, $(12, 144)$, $(-12, 144)$, $\left(\frac{1}{3}, \frac{1}{9}\right)$, $\left(-\frac{1}{3}, \frac{1}{9}\right)$ are all examples of points in this set.

At times you may see the term "polynomial function." A polynomial function is a function has non-negative integer exponents of the variable in the equation. The functions $y = x$ and $y = x^2$ are both polynomial functions. The exponent on the variable (x) in the latter is 2 and recall that any variable that does not have an explicitly stated exponent has an exponent of 1, so $y = x$ is the same as $y = x^1$ and is a polynomial function. The equations $y = x^3 - 10x^2 + 4$ and $y = -x^5 + 2x^2 - 3x - 4$ are also examples of higher order (larger exponents) polynomials. Some examples of equations that are not polynomials are $y = x^2 + x^{-4} + 7$ (because the exponent on the second term is -4) and $y = x^9 - x^{1/2}$ (because the exponent on the second term is a fraction, not an integer).

1.3.1 Plotting linear functions

A commonly used function is a line or linear function. A line can be defined by two parameters: the y-intercept, which is where it intersects the y-axis, and the slope, which defines the direction of the line as well as how steep it is. A standard equation for a line is $y = mx + b$, where m is the variable for the slope and b is the variable for the y-intercept. Both m and b can be positive or negative. It is important to note that both y and x have an (unstated) exponent of 1. If we have y^2 or y^3 or x^2 or x^{18}, etc. then we may have some other function but it will not be a line.

The y-intercept, b, is fairly straightforward to see. Recall that the y-intercept is the value of y when $x = 0$. If we let $x = 0$, we have:

$$\begin{aligned} y &= mx + b \\ y &= 0 * x + b \\ y &= b \end{aligned}$$

So when $x = 0$ we have $y = b$. We can think about $(0, b)$ as a place to start the line, though it will extend in both directions. Let $b = 6$ so that the coordinate for our y-intercept is $(0, 6)$. The slope, m, then tells us how many steps up or down and left or right to take. A slope of 3 is different from a slope of -3 and both are different from $\frac{1}{3}$ and all are different from $-\frac{1}{3}$. The numerator of the slope tells us the rise or decline of the line and the denominator tells us how far left or right.

When the slope is a positive number we move up and to the right (or down and to the left). You can think about it as moving in the direction of the quadrants where the x and y coordinates have the same sign (Quadrants I and III). With a slope of 3, we move three units up (down) for every unit to the right (left). So if we start from $(0, 6)$ then $(1, 9)$, $(2, 12)$, $(3, 15)$, etc. will all be on the line. So will $(-1, 3)$, $(-2, 0)$, $(-3, -3)$, etc. With a slope of $\frac{1}{3}$, we move one unit up (down) for every three units to the right (left). So if we start from $(0, 6)$ then $(3, 7)$, $(6, 8)$, $(9, 9)$, etc. will all be on the line, as will $(-3, 5)$, $(-6, 4)$, $(-9, 3)$, etc.

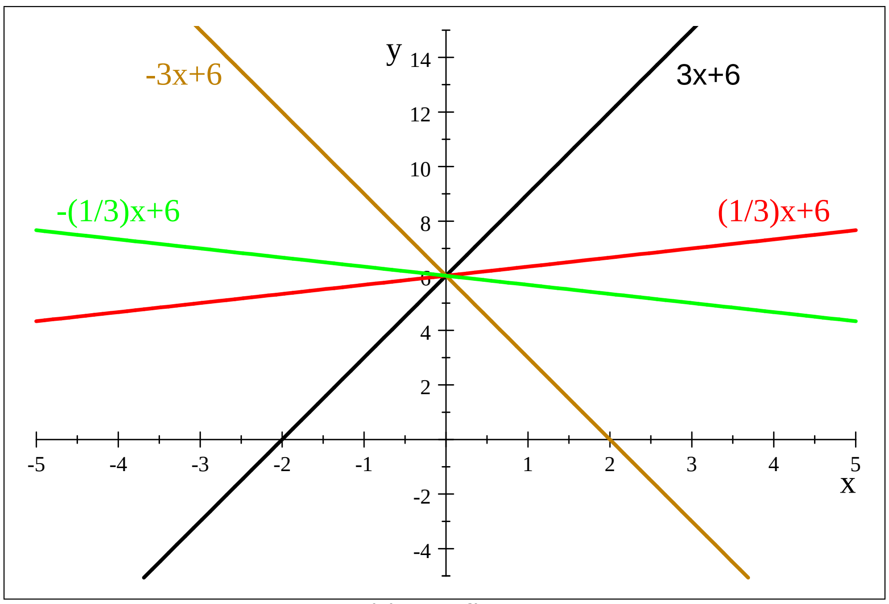
When the slope is a negative number we move up and to the left (or down and to the right). You can think about it as moving in the direction of the quadrants where the x and y coordinates have the opposite sign (Quadrants II and IV). With a slope of -3, we move three units up (down) for every unit to the left (right). So if we start from $(0, 6)$ then $(-1, 9)$, $(-2, 12)$, $(-3, 15)$, etc. will all be on the line. So will $(1, 3)$,

$(2, 0)$, $(3, -3)$, etc. With a slope of $-\frac{1}{3}$, we move one unit up (down) for every three units to the left (right). So if we start from $(0, 6)$ then $(-3, 7)$, $(-6, 8)$, $(-9, 9)$, etc. will all be on the line, as will $(3, 5)$, $(6, 4)$, $(9, 3)$, etc.

In general, the slope of a line can be found from any two points $(x_{1,}y_1)$ and (x_2, y_2) on a function as:

$$slope = \frac{y_2 - y_1}{x_2 - x_1}$$

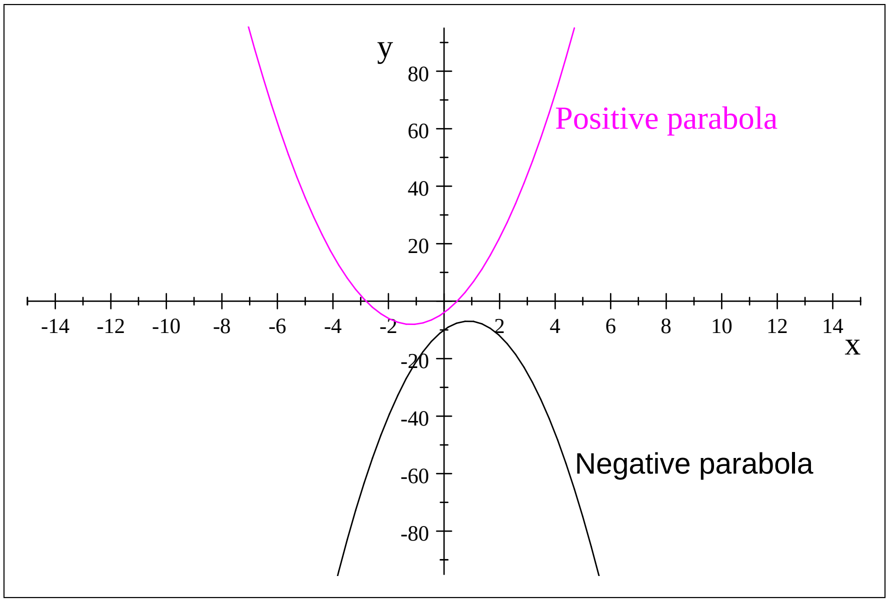
To plot a line all you need is a Cartesian plane (i.e. some graph paper), a ruler, and two points on the line. The y-intercept is one point so all you need is one more. Plot the two points, lay the ruler down so that you can connect those two points in a straight line, and draw the line. This method with the ruler works for a linear function because the slope of the line is constant; it never changes regardless of where you are on the line. However, the reality is that very few people plot functions by hand. In class I will plot very general functions by hand; I am not trying to be precise. In the notes I (mostly) use software to plot equations because I want to be precise.



Plots of four different lines.

1.3.2 Plotting nonlinear functions

There are many useful functions other than lines. The standard at this time is to use software to plot these functions but it is helpful to understand some general properties of particular functions. We oftentimes use equations with a squared term in economics (technically a polynomial function of degree 2 because the largest exponent is two), the result of which is a graph of a parabola. The parabola is U-shaped and can be either positive (looks like a U) or negative (looks like an upside down U). The sign on the variable with the squared term determines whether the parabola is positive or negative. For instance, $y = 3x^2 + 7x - 4$ is positive, while $y = -4x^2 + 7x - 10$ is negative. The $3x^2$ term has a positive 3 so the parabola is positive while the $-4x^2$ term has a negative 4 so that parabola is negative. The graph below shows the two equations plotted.

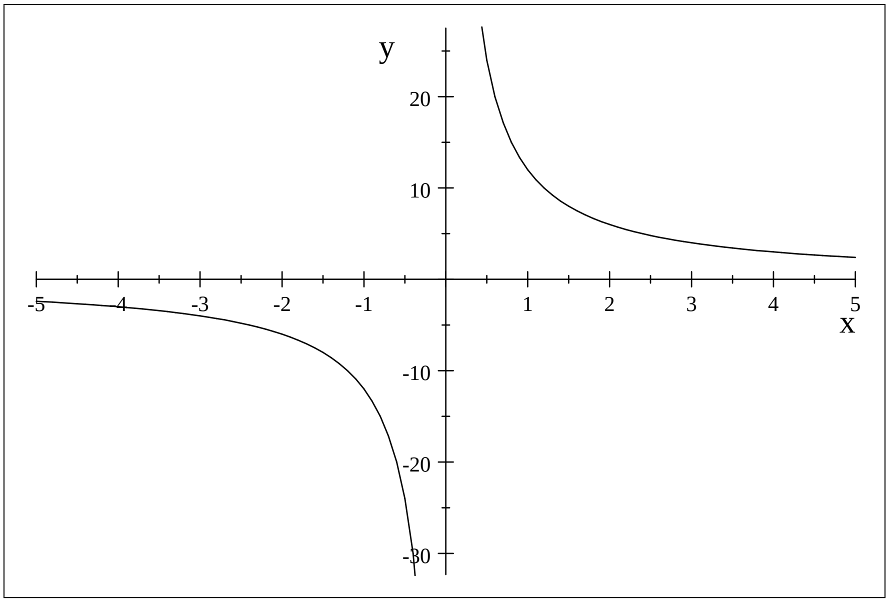


Plots of two parabolas.

It is important to note is that the sign of the slope of the parabola changes. With a positive parabola the slope, reading the graph from left to right, begins negative, then becomes zero (at the vertex or minimum), and then changes to positive. With a negative parabola we have the opposite – the slope begins positive, becomes zero at the vertex (which in this case is a maximum), and then becomes negative. In addition to the sign of the slope changing, the numerical value of the slope also changes depending upon where it is measured. This changing numerical value of the slope is different than that of a line which is constant regardless of where the slope is measured. Functions with changing slopes will be important in our study of consumer behavior. In order to determine the slope at a particular point calculus is needed and will be discussed later.

There are higher order polynomials but other than perhaps a cubic, with an x^3 term, which we might use occasionally, we do not use them that frequently for instructional purposes. We do use functions that look like slanted parabolas, which are called hyperbolas. These are not polynomial functions. A basic hyperbola can be seen from graphing an equation like $y = \frac{12}{x}$ or $y = 12x^{-1}$ or $12 = xy$ (they are all the same equation).[2] The graph below shows the result of graphing $y = \frac{12}{x}$ – note that the equation for the hyperbola is both parts of the graph, the part in Quadrant I and the part in Quadrant III. That graph is NOT two distinct equations.

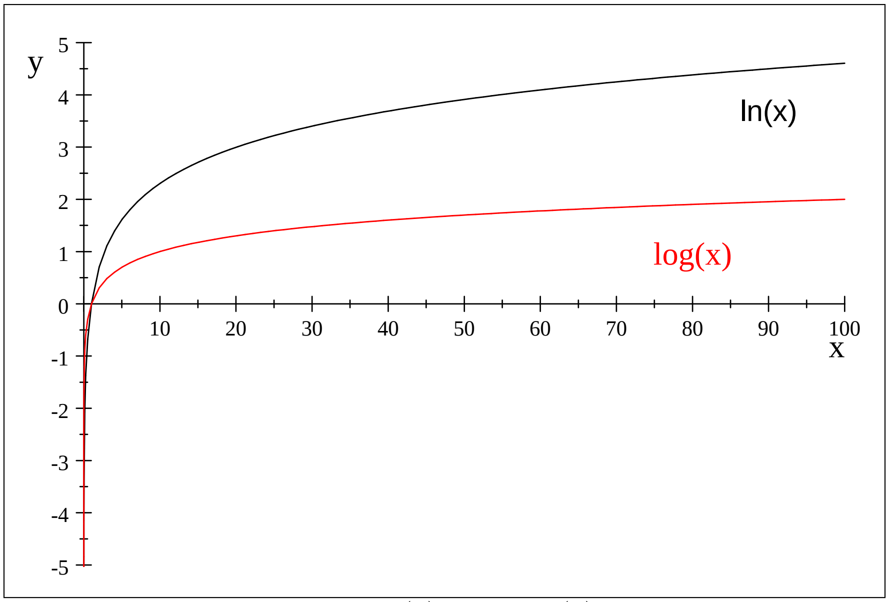
[2] The more general functional form for a hyperbola is more complicated and those details are not needed for our discussion.



Plot of a hyperbola.

Typically we will only be concerned with the portion of the graph in Quadrant I. Comparing to the line and the parabola, like the line the slope of this hyperbola always has the same sign (it is negative regardless of the portion of the hyperbola on which it is measured, unless the measurement is taken at zero at which the slope does not exist because neither x nor y can equal zero) but like the parabola the numerical value of the slope changes when measured at different points.

The final nonlinear function for discussion is the logarithmic function, and particularly the natural log function, $y = \ln(x)$. The natural log function is useful in many economic applications because it can be used to measure growth rates. Any logarithmic function needs a base. We tend to use base 10 regularly in our everyday lives (counting). Computers tend to use binary (base 2). We could write $y = \log_{10}(x)$, but what does that mean? What we are trying to solve is $10^y = x$ and the logarithmic transformation makes it easy to write that equation in terms of y. If $y = 1$, then $x = 10$; if $y = 2$, then $x = 100$. Importantly, if $y = 0$, then $x = 1$ because any positive number raised to the power of 0 is one. With the natural logarithm, the base is understood to be the exponential, or e. So $y = \ln(x)$ is the same as $y = \log_e(x)$. For the natural log, we are solving $e^y = x$. While e looks like variable, it is actually a specific mathematical number in the same way that π is a very specific mathematical number. Both numbers are irrational, which means they cannot be written as the ratio of two integers, but while people tend to be familiar with $\pi \approx 3.14$ fewer are familiar with $e \approx 2.718$. The plot below shows plots of $\ln(x)$ and $\log(x)$.



Plot of $\ln(x)$ and $\log_{10}(x)$.

It is important to note that x is always positive; it can never be zero or negative. However, y will be zero when $x = 1$ and will be negative when $x < 1$. Returning to the regular logarithmic function, if $y = -1$ we have $10^{-1} = x$ and in that case $x = \frac{1}{10^1} = \frac{1}{10}$. That result is an application of the rules of exponents. Like the hyperbola, the slope of the natural log function has the same sign throughout but the numerical value changes depending upon where it is measured.

Rules of exponents An exponent (or power) is a shorthand method of writing the product of the same number or variable with itself. Rather than writing $2 * 2 * 2 * 2 * 2$ we simplify that expression and write 2^5, which can be read as "two raised to the power of five" or "two raised to the fifth power." The exponent provides information on how many times that number is multiplied by itself. If we have x^4 that is just $x * x * x * x$.

There are some basic rules regarding exponents. First, any number or variable, other than zero, raised to the power of zero is 1. So $100^0 = \frac{1}{10}^0 = 3^0 = x^0 = -1^0 = -10,000^0 = 1$.

If two terms have the same base number and are multiplied together then the exponents can be added together:

$$\begin{aligned} 2^5 * 2^3 &= 2^8 \\ (2*2*2*2*2)*(2*2*2) &= 2^8 \\ & \quad or \\ x^3 * x^2 &= x^5 \\ (x*x*x)*(x*x) &= x^5 \end{aligned}$$

Similarly, if two terms have the same base and one is divided into the other then the exponents can be subtracted:

$$\begin{aligned} \frac{2^5}{2^3} &= 2^2 \\ \frac{2*2*2*2*2}{2*2*2} &= 2*2 \\ 2*2 &= 2^2 \end{aligned}$$

If the exponent is negative, such as in 3^{-3}, then it can be written as:

$$3^{-3} = \frac{1}{3^3}$$

That rule is just a straightforward application of the rule about division of exponents and the rule that any number raised to the power of zero is 1.

$$\frac{1}{3^3} = \frac{3^0}{3^3}$$
Choosing wisely, $3^0 = 1$
$$\frac{3^0}{3^3} = 3^{-3}$$
Using the rule of subtraction of exponents

If there are two different base numbers or variables but they are raised to the same power and those terms are multiplied together than the two base numbers can be multiplied together and raised to that power:

$$\begin{aligned}
2^3 * 5^3 &= 10^3 \\
(2*2*2)*(5*5*5) &= (2*5)*(2*5)*(2*5) \\
(2*5)*(2*5)*(2*5) &= 10*10*10 \\
10*10*10 &= 10^3 \\
& \quad \textit{or} \\
x^2 * y^2 &= (xy)^2 \\
(x*x)*(y*y) &= (x*y)*(x*y) \\
(x*y)*(x*y) &= xy*xy \\
xy*xy &= (xy)^2
\end{aligned}$$

If there is a term raised to a power that is raised to another power, then the exponents can be multiplied together:

$$\begin{aligned}
\left(3^2\right)^3 &= 3^6 \\
\left(3^2\right)^3 &= (3*3)^3 \\
(3*3)^3 &= (3*3)*(3*3)*(3*3) \\
(3*3)*(3*3)*(3*3) &= 3^6
\end{aligned}$$

Rules of logarithms The focus is on the natural logarithm, $\ln(x)$. These rules will be stated without much explanation.

If there is a term where the x variable is raised to a power, such as $\ln\left(x^7\right)$, then the power can be placed in front of the natural log:

$$\ln\left(x^7\right) = 7\ln(x)$$

If the natural log is taken over the product of two (or more) numbers or variables, then the result is the same as the natural log of the sum of each individual number or variable:

$$\ln(7xy) = \ln 7 + \ln(x) + \ln(y)$$

Similarly, if the natural log is taken over the ratio of two (or more) numbers or variables, then the result is the same as the natural log of the difference between those variables:

$$\begin{aligned}
\ln\left(\frac{7x}{z}\right) &= \ln(7x) - \ln(z) \\
& \quad \textit{or} \\
\ln(7x) - \ln(z) &= \ln 7 + \ln(x) - \ln(z)
\end{aligned}$$

2 Solving systems of equations

At times we have multiple unknown variables and multiple equations and we would like to find the solution of those equations. Sometimes those solutions will be unique (there is only one correct numerical answer) and at other times there will be multiple solutions. It is important to note that the same number of equations and variables are needed in order to find a unique solution; if there are more equations than variables or more variables than equations then it will either not be possible to find a solution at all or not be possible to find a unique solution.

There are a few different methods for solving systems of equations. The most commonly used method is to solve one equation for one variable and then substitute that solution into another equation in the system. Suppose we have $y = 2x + 10$ and $y = -6x + 14$. We know that $y = 2x + 10$ from the first equation so we can substitute into the second equation:

$$\begin{aligned}
&\text{The goal is to find } x\\
&y = -6x + 14\\
&\text{Substitute } 2x + 10 \text{ for } y\\
&2x + 10 = -6x + 14\\
&\text{Add } 6x \text{ to both sides}\\
&8x + 10 = 14\\
&\text{Subtract 10 from both sides}\\
&8x = 4\\
&\text{Divide both sides by 8}\\
&x = \frac{4}{8} = \frac{1}{2}
\end{aligned}$$

Once we have found the solution for one variable it can be substituted into the original equations to determine the value of the other variable.

$$\begin{aligned}
y &= 2x + 10\\
y &= 2\left(\frac{1}{2}\right) + 10\\
y &= 1 + 10\\
y &= 11\\
&\text{and}\\
y &= -6x + 14\\
y &= -6\left(\frac{1}{2}\right) + 14\\
y &= -3 + 14\\
y &= 11
\end{aligned}$$

I solved both equations because the answers should be the same; if they are not then there is an error somewhere. So the solution to $y = 2x + 10$ and $y = -6x + 14$ is $(x, y) = \left(\frac{1}{2}, 11\right)$.

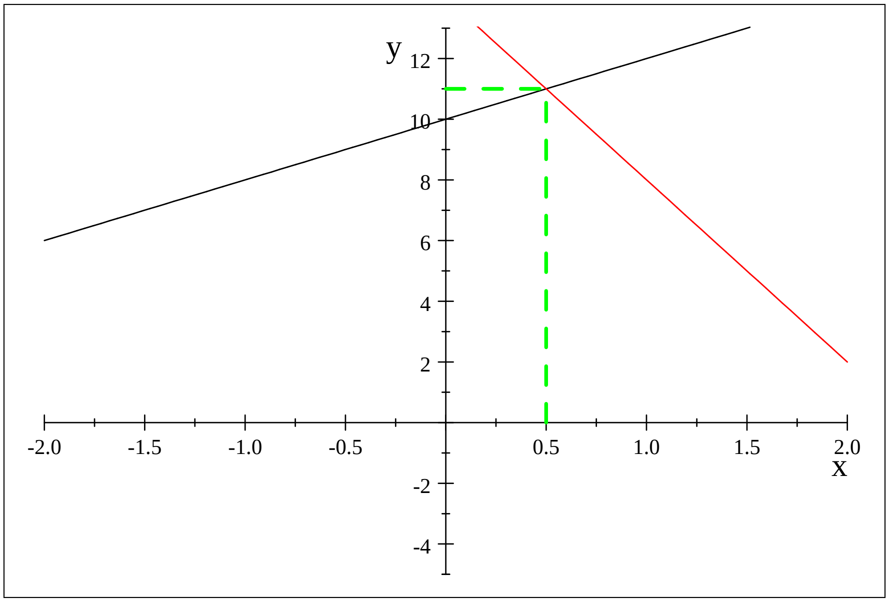
Another solution method is to use addition (or subtraction). Essentially the two equations are stacked like two numbers and either the addition or subtraction operation is performed.

$$\begin{array}{rr}
 & y = 2x + 10\\
- & y = -6x + 14\\
\hline
 & 0 = 8x - 4\\
 & 4 = 8x\\
 & \frac{4}{8} = x\\
 & \frac{1}{2} = x
\end{array}$$

We know that $x = \frac{1}{2}$ is correct from our previous solution and that $y = 11$.

Finally, we can use matrix algebra to solve linear equations. Matrices are particularly helpful when solving large systems of equations, and it is even more helpful to have a computer to solve those systems. When a linear regression model is estimated, the computer is solving a matrix algebra problem. We will not get into those details.

Perhaps more importantly than being able to solve the system of equations is what the results mean. The solution to the system of $y = 2x + 10$ and $y = -6x + 14$ is $(x, y) = \left(\frac{1}{2}, 11\right)$. That result tells us where the two lines intersect. Graphing those two lines we have:



The solution simply tells us where the two lines intersect. In economics, intersection points are one of two types of important points when graphing equations. Plotting the equations is another way to solve the system of equations but it is not very precise when done by hand and even when using software it can be difficult to determine the intersection point accurately.