Problem Set 1

BPHD8110-001

Due: January 26^{th}, 2023

1. Show that any strictly dominant strategy in game $[I, \{\Delta(S_i), \{u_i(\cdot)\}\}]$ must be a pure strategy.

   Answer:

   Proof by contradiction. Assume that the strictly dominant strategy is a nondegenerate mixed strategy, σ_i, over N pure strategies. Then we must have:

   $$u_i(\sigma_i, s_{-i}) > u_i(s_i^*, s_{-i}) \ \forall \ s_i^* \in S_i \text{ and } \forall \ s_{-i} \in S_{-i}.$$

   In particular,

   $$u_i(\sigma_i, s_{-i}) > u_i\left(s_i^j, s_{-i}\right) \ \forall \ j = 1, ..., N.$$

   All this means is that the utility of playing σ_i is greater than the utility of playing any pure strategy used in σ_i against some other players' strategies s_{-i}. This implies that:

   $$u_i(\sigma_i, s_{-i}) > \sum_{j=1}^{N} \left[\sigma_i\left(s_i^j\right) * u_i\left(s_i^j, s_{-i}\right)\right] = u_i(\sigma_i, s_{-i}).$$

   This is clearly a contradiction. It's not a particularly explanatory contradiction, but it is a contradiction. Intuitively, if one is playing a mixed strategy, then this suggests that the pure strategies over which one mixes must each do better against some set of the other players' strategies.

2. Show that the two-player game below has a unique equilibrium:

		Player 2		
		L	C	R
	U	$\boxed{1}, -2$	$-2, \boxed{1}$	$0, 0$
Player 1	M	$-2, \boxed{1}$	$\boxed{1}, -2$	$0, 0$
	D	$0, 0$	$0, 0$	$\boxed{1}, \boxed{1}$

   Answer:

   The best responses are marked in the game above. Note that the only PSNE is P1 D and P2 R. Now that we know what the PSNE is, and we are told that the game has a unique equilibrium, we need to rule out MSNE.

   Let's suppose that Player 2 wanted to mix over L, C, and R with probabilities p_L, p_M, and p_R. Player 1 would then receive:

   $$\begin{aligned} E_1[U] &= p_L - 2p_M \\ E_1[M] &= -2p_L + p_M \\ E_1[D] &= p_R \end{aligned}$$

   Now we have that:

   $$\begin{aligned} E_1[U] &= E_1[M] \\ p_L - 2p_M &= -2p_L + p_M \\ 3p_L &= 3p_M \\ p_L &= p_M \end{aligned}$$

Now, before continuing, consider the payoff to each of P1's strategies under this result (whether $p_L = p_M = .5$ or $p_L = p_M = 0.3$ or any other acceptable probability). The expected value to choosing U and M will be negative, and the expected value to choosing D will be (at worst) zero. Thus, P1 would choose D regardless of the mixed strategy chosen by P2, which would then lead to P2 choosing R (because D, R is the PSNE). We can rule out P1 mixing over all 3 strategies by similar logic.

Note that the condition $p_L = p_M$ will need to hold regardless of whether or not strategy R is used in P2's mixed strategy. Thus, P2 cannot mix over just L and C. By similar logic, P1 cannot mix over just U and M.

Suppose P2 chose to mix over L and R. Now, P1 has no reason to use M as part of a mixed strategy because it is strictly dominated by both D. When M is removed, note that strategy L is strictly dominated by R, so P2 will not mix over L and R. Similar logic rules out P2 mixing over C and R, and also P1 mixing over U and D or M and D.

Now that we have ruled out all potential MSNE, we have proved that there is a unique NE to the game. We used "proof by exhaustion" to show this result,[1] so the proof is not very elegant, but it works in this case.

3. This question will test your understanding of Proposition 8.D.1 in MWG. Find all pure and mixed strategy Nash equilibria to the following game. If there are none of either type explain why there are none:

Player 2

		F	G	H	I	J
	A	$\boxed{18}, 11$	$\boxed{9}, \boxed{12}$	$6, 1$	$8, 0$	$7, 1$
	B	$6, 6$	$7, 8$	$5, 7$	$\boxed{9}, \boxed{11}$	$4, 5$
Player 1	C	$9, 0$	$4, 5$	$\boxed{14}, 4$	$4, 10$	$5, \boxed{16}$
	D	$3, 4$	$3, 6$	$2, 3$	$6, 7$	$1, \boxed{9}$
	E	$0, 0$	$4, 2$	$7, 1$	$7, 4$	$\boxed{8}, \boxed{6}$

Answer:

The best responses are marked in the matrix above. This leads to three PSNE:

(1) P1 choose A, P2 choose G; (2) P1 choose B, P2 choose I; (3) P1 choose E, P2 choose J.

What you should notice is that strategies F and H are both strictly dominated by G, and strategy D is strictly dominated by A, so all three of those can be removed leaving:

Player 2

		G	I	J
	A	$\boxed{9}, \boxed{12}$	$8, 0$	$7, 1$
	B	$7, 8$	$\boxed{9}, \boxed{11}$	$4, 5$
Player 1	C	$4, 5$	$4, 10$	$5, \boxed{16}$
	E	$4, 2$	$7, 4$	$\boxed{8}, \boxed{6}$

Now we can see that strategy C, once H has been removed, is strictly dominated by A so it too can be removed leaving:

Player 2

		G	I	J
	A	$\boxed{9}, \boxed{12}$	$8, 0$	$7, 1$
Player 1	B	$7, 8$	$\boxed{9}, \boxed{11}$	$4, 5$
	E	$4, 2$	$7, 4$	$\boxed{8}, \boxed{6}$

Now let's try to find an MSNE. Let g, i, and $j = 1 - g - i$ be the probabilities that Player 2 uses to choose strategies G, I, and J respectively. We need:

$$E[A] = E[B] = E[E]$$

[1] Proof by exhaustion means that we examined all of the potential equilibria and were able to show that they could not be an equilibrium. We "exhausted" all of the possibilities.

Focusing on:

$$\begin{aligned}
E[A] &= E[B] \\
9g + 8i + 7(1 - g - i) &= 7g + 9i + 4(1 - g - i) \\
9g + 8i + 7 - 7g - 7i &= 7g + 9i + 4 - 4g - 4i \\
2g + i + 7 &= 3g + 5i + 4 \\
3 - 4i &= g
\end{aligned}$$

Now focusing on:

$$\begin{aligned}
E[B] &= E[E] \\
7g + 9i + 4(1 - g - i) &= 4g + 7i + 8(1 - g - i) \\
7g + 9i + 4 - 4g - 4i &= 4g + 7i + 8 - 8g - 8i \\
3g + 5i + 4 &= -4g - i + 8 \\
7g + 6i &= 4
\end{aligned}$$

Substituting we have:

$$\begin{aligned}
7g + 6i &= 4 \\
7(3 - 4i) + 6i &= 4 \\
21 - 28i + 6i &= 4 \\
17 &= 22i \\
i &= \frac{17}{22}
\end{aligned}$$

Then we have:

$$\begin{aligned}
3 - 4i &= g \\
3 - 4 * \frac{17}{22} &= g \\
\frac{66}{22} - \frac{68}{22} &= g \\
\frac{-2}{22} &= g
\end{aligned}$$

At this point it looks like there is a problem because we are going to end up with: $g = -\frac{2}{22}$, $i = \frac{17}{22}$, and $j = \frac{7}{22}$. This is a problem because $g < 0$. The question is, do these probabilities make player 1 indifferent among the pure strategies he plays with positive probabilities?

$$\begin{aligned}
E[A] &= 9 * \left(-\frac{2}{22}\right) + 8 * \frac{17}{22} + 7 * \frac{7}{22} = \frac{167}{22} \\
E[B] &= 7 * \left(-\frac{2}{22}\right) + 9 * \frac{17}{22} + 4 * \frac{7}{22} = \frac{167}{22} \\
E[E] &= 4 * \left(-\frac{2}{22}\right) + 7 * \frac{17}{22} + 8 * \frac{7}{22} = \frac{167}{22}
\end{aligned}$$

They do – but because one of the probabilities violates the laws of probability, there cannot be an MSNE in the 3x3 game ... using all 3 strategies.[2]

But there are actually two MSNE using only 2 strategies each from the 3x3 game. Consider the following 2x2 game:

[2] For the record, if you found player 1's probabilities first you would have found that $a = \frac{18}{113}$, $b = \frac{26}{113}$, and $e = \frac{69}{113}$. These should lead to player 2 having an expected value of $\frac{562}{113}$ for each pure strategy he plays with positive probability.

		Player 2	
		G	I
	A	$\boxed{9}, \boxed{12}$	$8, 0$
Player 1	B	$7, 8$	$\boxed{9}, \boxed{11}$

Now for player 1 we have:

$$
\begin{aligned}
E[A] &= E[B] \\
9g + 8(1-g) &= 7g + 9(1-g) \\
9g + 8 - 8g &= 7g + 9 - 9g \\
g + 8 &= 9 - 2g \\
3g &= 1 \\
g &= \frac{1}{3}
\end{aligned}
$$

So Player 2 would use G with probability $\frac{1}{3}$ and I with probability $\frac{2}{3}$. Now let's look at ALL of Player 1's expected values:

$$
\begin{aligned}
E[A] &= 9 * \frac{1}{3} + 8 * \frac{2}{3} = \frac{25}{3} \\
E[B] &= 7 * \frac{1}{3} + 9 * \frac{2}{3} = \frac{25}{3} \\
E[C] &= 4 * \frac{1}{3} + 4 * \frac{2}{3} = \frac{12}{3} \\
E[D] &= 3 * \frac{1}{3} + 6 * \frac{2}{3} = \frac{15}{3} \\
E[E] &= 4 * \frac{1}{3} + 7 * \frac{2}{3} = \frac{18}{3}
\end{aligned}
$$

Note that the strategies that Player 1 uses with positive probability, A and B, have an expected payoff of $\frac{25}{3}$ when Player 2 uses the MSNE $\left(0, \frac{1}{3}, 0, \frac{2}{3}, 0\right)$, while the other strategies Player 1 could use have an expected payoff less than $\frac{25}{3}$. Now we need to find Player 1's probabilities using:

$$
\begin{aligned}
E[G] &= E[I] \\
12a + 8(1-a) &= 0a + 11(1-a) \\
12a + 8 - 8a &= 11 - 11a \\
4a + 8 &= 11 - 11a \\
15a &= 3 \\
a &= \frac{3}{15} = \frac{1}{5}
\end{aligned}
$$

So if Player 1 uses A with probability $\frac{1}{5}$ and B with probability $\frac{4}{5}$ this will make Player 2 indifferent over strategies G and I. Can Player 2 deviate to another pure strategy and receive a strictly higher payoff?

$$
\begin{aligned}
E[F] &= 11 * \frac{1}{5} + 6 * \frac{4}{5} = \frac{35}{5} \\
E[G] &= 12 * \frac{1}{5} + 8 * \frac{4}{5} = \frac{44}{5} \\
E[H] &= 1 * \frac{1}{5} + 7 * \frac{4}{5} = \frac{29}{5} \\
E[I] &= 0 * \frac{1}{5} + 11 * \frac{4}{5} = \frac{44}{5} \\
E[J] &= 1 * \frac{1}{5} + 5 * \frac{4}{5} = \frac{21}{5}
\end{aligned}
$$

The answer is no, so there is an MSNE where: Player 1 chooses A with probability $\frac{1}{5}$ and B with probability $\frac{4}{5}$ while Player 2 chooses G with probability $\frac{1}{3}$ and I with probability $\frac{2}{3}$.

Now consider the 2x2 game:

		Player 2	
		G	J
Player 1	A	$\boxed{9}, \boxed{12}$	$7, 1$
	E	$4, 2$	$\boxed{8}, \boxed{6}$

Setting:

$$\begin{aligned}
E[A] &= E[E] \\
9g + 7(1-g) &= 4g + 8(1-g) \\
9g + 7 - 7g &= 4g + 8 - 8g \\
2g + 7 &= 8 - 4g \\
6g &= 1 \\
g &= \frac{1}{6}
\end{aligned}$$

So if Player 2 uses G with probability $\frac{1}{6}$ and J with probability $\frac{5}{6}$ then Player 1's expected values (for all 5 pure strategies) are:

$$\begin{aligned}
E[A] &= 9 * \frac{1}{6} + 7 * \frac{5}{6} = \frac{44}{6} \\
E[B] &= 7 * \frac{1}{6} + 4 * \frac{5}{6} = \frac{27}{6} \\
E[C] &= 4 * \frac{1}{6} + 5 * \frac{5}{6} = \frac{24}{6} \\
E[D] &= 3 * \frac{1}{6} + 1 * \frac{5}{6} = \frac{8}{6} \\
E[E] &= 4 * \frac{1}{6} + 8 * \frac{5}{6} = \frac{44}{6}
\end{aligned}$$

so Player 1 would not want to switch to either B, C, or E. Now to find Player 1's probabilities:

$$\begin{aligned}
E[G] &= E[J] \\
12a + 2(1-a) &= 1a + 6(1-a) \\
12a + 2 - 2a &= 1a + 6 - 6a \\
10a + 2 &= 6 - 5a \\
15a &= 4 \\
a &= \frac{4}{15}
\end{aligned}$$

So if Player 1 uses A with probability $\frac{4}{15}$ and E with probability $\frac{11}{15}$ then Player 2's expected values are:

$$\begin{aligned}
E[F] &= 11 * \frac{4}{15} + 0 * \frac{11}{15} = \frac{44}{15} \\
E[G] &= 12 * \frac{4}{15} + 2 * \frac{11}{15} = \frac{70}{15} \\
E[H] &= 1 * \frac{4}{15} + 1 * \frac{11}{15} = \frac{15}{15} \\
E[I] &= 0 * \frac{4}{15} + 4 * \frac{11}{15} = \frac{44}{15} \\
E[J] &= 1 * \frac{4}{15} + 6 * \frac{11}{15} = \frac{70}{15}
\end{aligned}$$

so that Player 2 would not want to switch to either F, H, or I. Thus, we have a second MSNE: Player 1 chooses A with probability $\frac{4}{15}$ and E with probability $\frac{11}{15}$ while Player 2 chooses G with probability $\frac{1}{6}$ and J with probability $\frac{5}{6}$.

Now, let's consider the last remaining 2x2:

		Player 2 I	J
Player 1	B	$\boxed{9},\boxed{11}$	$4,5$
	E	$7,4$	$\boxed{8},\boxed{6}$

Setting:

$$\begin{aligned} E[B] &= E[E] \\ 9i + 4(1-i) &= 7i + 8(1-i) \\ 9i + 4 - 4i &= 7i + 8 - 8i \\ 5i + 4 &= 8 - i \\ 6i &= 4 \\ i &= \frac{2}{3} \end{aligned}$$

Looking at Player 1's expected values:

$$\begin{aligned} E[A] &= 8 * \frac{2}{3} + 7 * \frac{1}{3} = \frac{23}{3} \\ E[B] &= 9 * \frac{2}{3} + 4 * \frac{1}{3} = \frac{22}{3} \\ E[C] &= 4 * \frac{2}{3} + 5 * \frac{1}{3} = \frac{18}{3} \\ E[D] &= 6 * \frac{2}{3} + 1 * \frac{1}{3} = \frac{13}{3} \\ E[E] &= 7 * \frac{2}{3} + 8 * \frac{1}{3} = \frac{22}{3} \end{aligned}$$

Now, what do we notice here? We notice that $E[A] > E[B] = E[E]$. Thus, while Player 2's choice of I with probability $\frac{2}{3}$ and J with probability $\frac{1}{3}$ makes Player 1 indifferent between B and E, it doesn't really matter because Player 1 would choose A. This is the reason we do not have an MSNE for the 3x3 game. Thus, there is no MSNE for this 2x2 so there are 5 total Nash equilibria.

3 PSNE:

(1) P1 choose A, P2 choose G

(2) P1 choose B, P2 choose I

(3) P1 choose E, P2 choose J.

2 MSNE:

(4) Player 1 chooses A with probability $\frac{1}{5}$ and B with probability $\frac{4}{5}$ while Player 2 chooses G with probability $\frac{1}{3}$ and I with probability $\frac{2}{3}$.

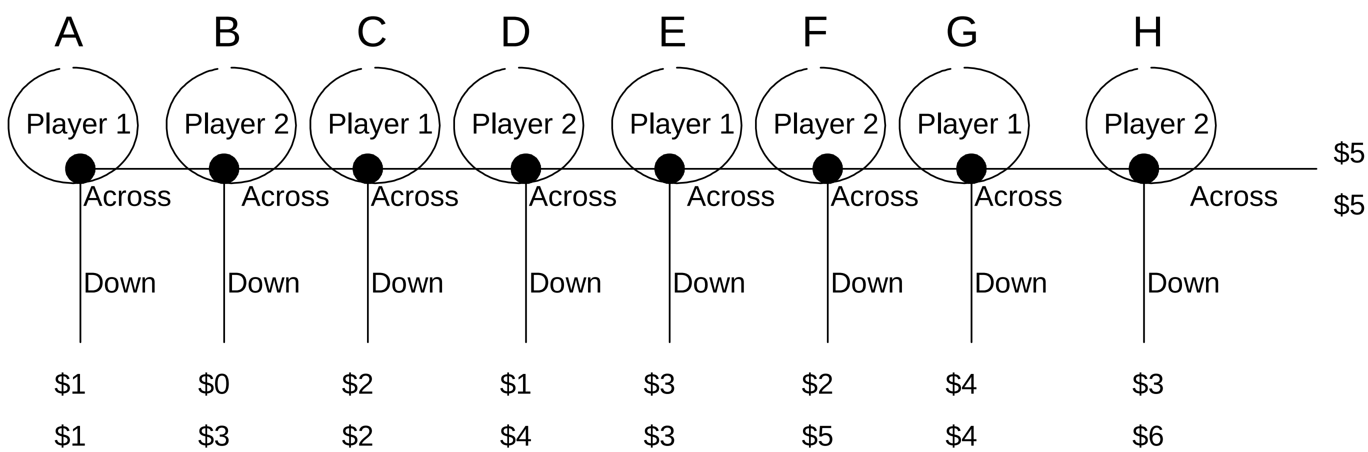
(5) Player 1 chooses A with probability $\frac{4}{15}$ and E with probability $\frac{11}{15}$ while Player 2 chooses G with probability $\frac{1}{6}$ and J with probability $\frac{5}{6}$.

4. Consider the following two-player sequential game. In period 1 each player is given \$1. Player 1 has the opportunity to end the game right away by choosing Down and both players will get \$1. However, Player 1 also have the option of sending the game to a second period by choosing Across, where \$1 will be taken from Player 1 and \$2 will be given to the Player 2. Player 2 then has the opportunity to end the game (with Player 1 getting nothing and Player 2 getting \$3) by choosing Down or to send the game to a third period by choosing Across where \$1 will be taken from him and \$2 dollars will

be given to Player 1. Player 1 can then end the game by choosing Down or continue the process by choosing Across. If the game gets to period 8 and Player 2 decides to send the game onto the next period by choosing Across, the game ends with both players getting \$5.

a Draw the extensive form version (game tree) of this game. Be sure to include all the components of the game in your diagram.

Answer:



I have added one extra feature in this game. Above each of the decision nodes (alternatively, because there is one decision node for each information set, above each information set) I placed a letter which identifies the decision node. This will facilitate describing the SPNE in part **b**.

b Find the subgame perfect Nash equilibrium to this game.

Answer:

To solve this start from the smallest subgame (there are 8 subgames including the entire game) which is the one at decision node H. Player 2 should choose Down because \$6 > \$5. At decision node G Player 1 knows this and will then have to choose between Down (\$4) and Across (\$3) and will choose Down. This game then unravels back to the beginning with both players choosing Down whenever they get the chance. Thus, the SPNE is for Player 1 to choose Down at decision nodes A, C, E, and G and for Player 2 to choose Down at decision nodes B, D, F, and H.

c What is the outcome if the subgame perfect Nash equilibrium is played?

Answer:

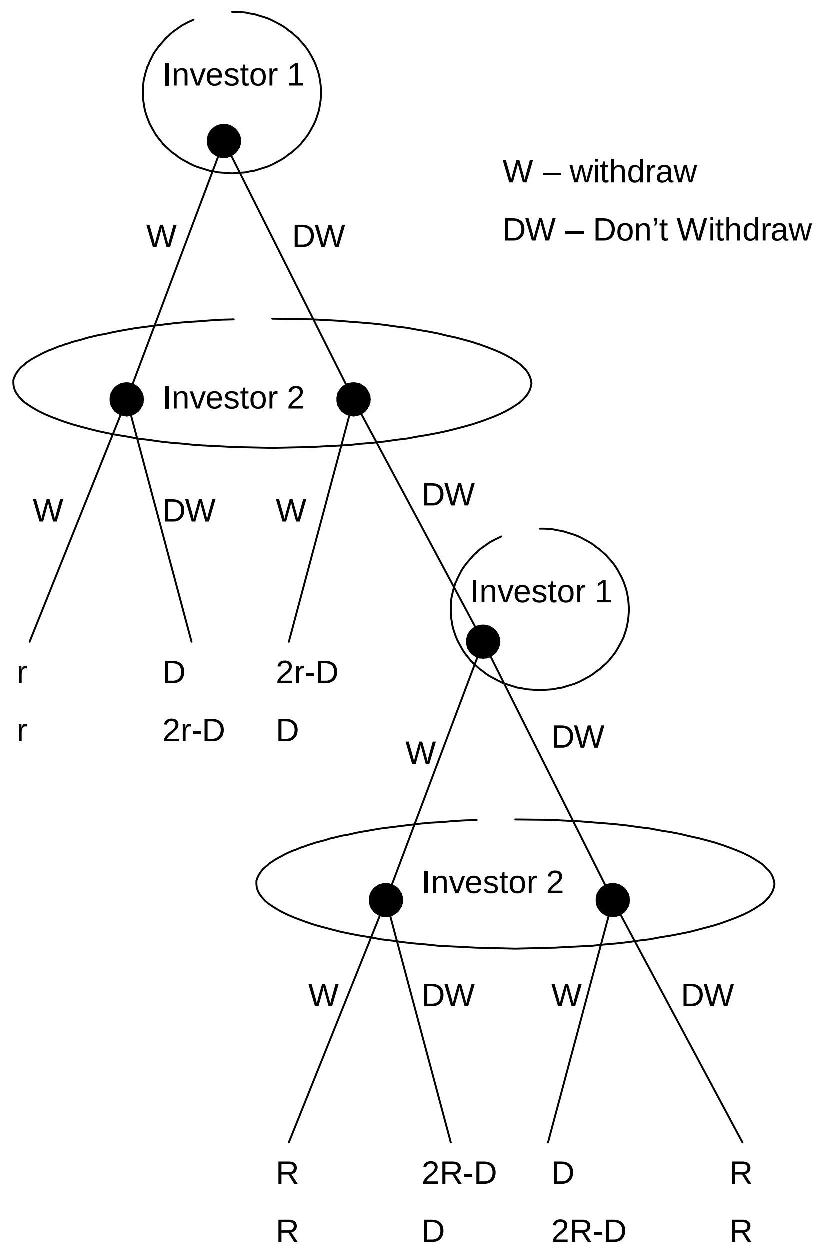
The outcome if the SPNE is played is both players receive \$1 because the game ends instantly as Player 1 chooses Down at decision node A.

5. Two investors have each deposited D with a bank. The bank has invested these deposits in a long-term project. If the bank is forced to liquidate its investment before the project matures, a total of $2r$ can be recovered, where $D > r > \frac{D}{2}$. If the bank allows the investment to reach maturity, however, the project will pay out a total of $2R$, with $R > D$.

   There are 2 dates at which the investors can make withdrawals from the bank: date 1 is before the bank's investment matures; date 2 is after. For simplicity, assume that there is no discounting. If both investors make withdrawals at date 1 then each receives r and the game ends. If only one investor makes a withdrawal at date 1 then that investor receives D, the other receives $2r - D$, and the game ends. Finally, if neither investor makes a withdrawal at date 1 then the project matures and both investors make withdrawal decisions at date 2. If both investors make withdrawals at date 2 then each receives R and the game ends. If only one investor makes a withdrawal at date 2 then that investor receives $2R - D$, the other receives D, and the game ends. If neither investor makes a withdrawal at date 2 then the bank returns R to each investor and the game ends. Note that neither player observes the withdrawal decision of the other player at either date (in other words, at date 1 the players simultaneously choose to withdraw or not, and the same at date 2 – obviously once date 2 is reached both players know what the other player chose at date 1).

a Draw the extensive form version of this game.

Answer:



b There are 2 subgames in this game, one of which is the entire game and the other of which is the game that begins at date 2. Write down the normal form version of the subgame that begins at date 2.

Answer:

The normal form version of the subgame starting at date 2 is:

Date 2 subgame		Investor 2	
		Withdraw	Don't Withdraw
Investor 1	Withdraw	R, R	$2R - D, D$
	Don't Withdraw	$D, 2R - D$	R, R

c Find the Nash equilibrium to the date 2 subgame in part **b**.

Answer:

The Nash equilibrium to the date 2 subgame in part **b** is that both investors choose Withdraw. Note that Withdraw is a strictly dominant strategy for both investors because $R > D$.

d Find the subgame perfect Nash equilibria to this game.

Answer:

We know that in the date 2 subgame both investors will choose Withdraw because it is a strictly dominant strategy. Using this information we can create a 2x2 normal form game for the date 1 subgame, because the payoff when both players choose Don't Withdraw is R, R (this follows from part **c**). The payoffs from any other choice of actions at date 1 is known because the game ends unless both investors choose Don't Withdraw. The following normal form game results because the investors both choose Withdraw at date 2.

Date 1 subgame		Investor 2	
		Withdraw	Don't Withdraw
Investor 1	Withdraw	r, r	$D, 2r - D$
	Don't Withdraw	$2r - D, D$	R, R

There are two (pure strategy) Nash equilibria to this game, one where both investors choose Withdraw and one where both investors choose Don't Withdraw. In a sense it is like the Boxing-Opera game. Of course, the SPNE to the entire game is Investor 1 chooses Withdraw at date 1 and Withdraw at date 2 and Investor 2 chooses Withdraw at date 1 and Withdraw at date 2. Alternatively, another SPNE is that Investor 1 chooses Don't Withdraw at date 1 and Withdraw at date 2 and Investor 2 chooses Don't Withdraw at date 1 and Withdraw at date 2. Thus, there are 2 pure strategy SPNE to this game – one where both players run to the bank, and one where both players wait for the project to mature.

e Write out the strategic (normal) form of the entire game and find all pure strategy NE.

Answer:

The strategic form will be a 4x4 game. Note that each player has two information sets and two actions at each information set, so four total strategies. I will list strategies as (first period action, second period action) for simplicity.

		Investor 2			
		W, W	W, DW	DW, W	DW, DW
	W, W	$\boxed{r}, \boxed{r}$	$\boxed{r}, \boxed{r}$	$D, 2r - D$	$D, 2r - D$
Investor 1	W, DW	$\boxed{r}, \boxed{r}$	$\boxed{r}, \boxed{r}$	$D, 2r - D$	$D, 2r - D$
	DW, W	$2r - D, D$	$2r - D, D$	$\boxed{R}, \boxed{R}$	$\boxed{2R - D}, D$
	DW, DW	$2r - D, D$	$2r - D, D$	$D, \boxed{2R - D}$	R, R

The best responses are marked in the matrix. Recall that there were two SPNE: (1) $W, W; W, W$ and $DW, W; DW, W$. Note that there are three other PSNE – basically, both players would play Withdraw in the first stage and it does not matter what they do in the second stage because they never reach the second stage. Note that they cannot exploit the other player's second stage strategy when both Withdraw in the first stage because they would need to switch to DW and would also need the other player to switch to DW.