Problem Set 2 Answers

BPHD8110-001

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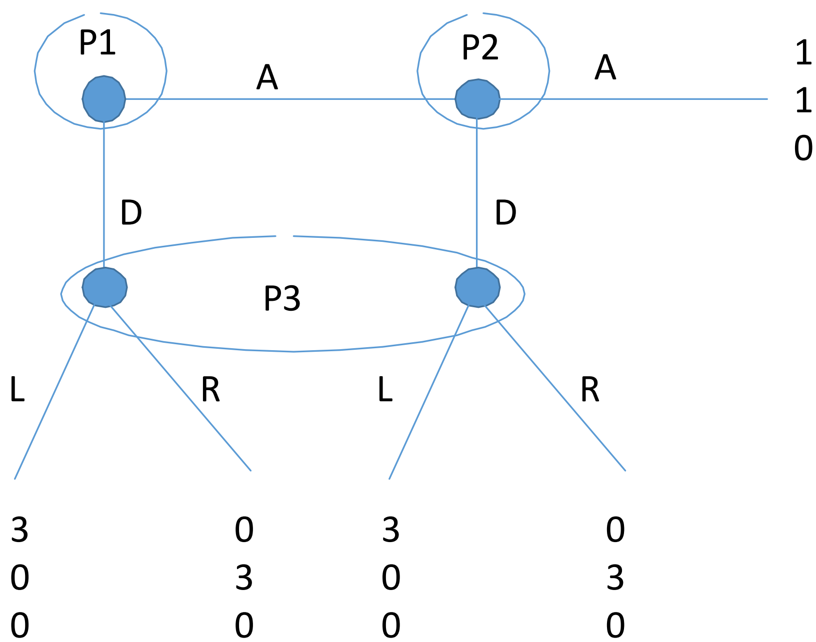
1. Consider the three-player game below, which starts with P1:



a Show that the outcome path (A, A) that results in payoffs $(1, 1, 0)$ is NOT the outcome from a Nash equilbrium.

Answer:

Assume that (A, A) is the outcome from a NE. We need to specify P3's strategy in order to completely determine the NE. P3 can choose either L or R. If P3 chooses L, then P1 would choose D. If P3 chooses R, then P2 would choose D. Regardless of what P3 chooses, either P1 or P2 would want to switch strategies, so we cannot have (A, A) as the outcome from a NE of the game.

b Find all pure strategy Nash equilibrium to this game.

Answer:

Suppose P3 choose L. Then P1 would choose D (to receive 3) while P2 could choose either A or D. Can any player switch strategies to receive a strictly higher payoff? No. P1 is receiving his highest payoff, P2 receives 0 regardless of whether A or D is chosen, and P3 receives 0 regardless of whether

L or R is chosen. Thus, D, A, L and D, D, L are PSNE. Alternatively if P3 chooses R we have that D, D, R and A, D, R are PSNE.

Alternatively, consider what a 3-player strategic form of the game would be. It would be (in this case because all players have 2 strategies) a cube. It is difficult to visualize a cube, so we break it apart into two matrices, and let one player choose which matrix to play.

P3 chooses L:

		P2: A	P2: D
P1	A	1, 1, 0	3, 0, 0
	D	3, 0, 0	3, 0, 0

P3 chooses R:

		P2: A	P2: D
P1	A	1, 1, 0	0, 3, 0
	D	0, 3, 0	0, 3, 0

2. Consider a developer who wishes to purchase k parcels of land. If the developer purchases all k parcels, the developer receives a payment of D. If the developer does not purchase all k parcels, the developer receives a payment of 0. The developer must purchase each parcel of land from the landowner who owns the land.

   Consider k landowners who each own a parcel of land. That parcel has value of v_i to the landowner, where $v_i \sim U[0, \frac{D}{k}]$. The individual landowners know their own value for the land but the developer does not. Also, the landowners do NOT know the values of other landowners.

   The game can be modeled as a sequential game. The developer makes an offer w_i to each landowner. Each landowner only observes his own w_i and must make a decision to accept or reject that w_i. If all k landowners accept their own offer w_i, then the landowners each receive w_i as a payment from the developer; the developer pays an amount $\sum_{i=1}^{k} w_i$, and the developer receives a payment of D. If ANY landowner chooses to reject w_i, then the developer makes no payment to any landowner and acquires no parcels of land – the developer receives 0 but pays 0. The landowners, who still own their land, receive v_i.

   For simplicity, assume the seller sets $w_i = w_j$ for all i, j. The developer maximizes expected utility, and receives $D - kw$ if aggregation is successful (which occurs only if all k landowners accept the offer) and 0 if not. Note that $\Pr(\widetilde{v} > v)$ for the uniform distribution $U\left[0, \frac{D}{k}\right]$ is $\frac{\widetilde{v}}{\frac{D}{k}}$.

   Assume the developer is risk neutral. Find a subgame perfect Nash equilibrium to this game with k landowners. Be sure to set up the developer's expected utility function correctly.

   Answer:

   Each of the landowners will accept any $w \geq v_i$ and reject any offers $w < v_i$. The developer, knowing this, then maximizes expected utility by choosing w:

$$\begin{aligned}
\max_w U_d &= (D - kw) * \left(\frac{w}{\frac{D}{k}}\right)^k \\
\frac{\partial U_d}{\partial w} &= \left(\frac{w}{\frac{D}{k}}\right)^k (-k) + (D - kw)\, k \left(\frac{w}{\frac{D}{k}}\right)^{k-1} \frac{1}{\frac{D}{k}} \\
0 &= \left(\frac{w}{\frac{D}{k}}\right)^k (-k) + (D - kw)\, k \left(\frac{w}{\frac{D}{k}}\right)^{k-1} \frac{1}{\frac{D}{k}} \\
0 &= (D - kw)\, k \frac{1}{\frac{D}{k}} - k \frac{w}{\frac{D}{k}} \\
0 &= (D - kw) - w \\
D &= kw + w \\
w &= \frac{D}{k+1}
\end{aligned}$$

The key to solving the developer's problem is to correctly specify the probability that the offer is accepted. For one individual that probability is $\frac{w}{\frac{D}{k}}$, while for all k individuals that probability is $\left(\frac{w}{\frac{D}{k}}\right)^k$. So the SPNE is for all landowners to accept any $w \geq v_i$ and to reject any $w < v_i$, while the developer offers each landowner $w = \frac{D}{k+1}$.

3. Consider 2 firms who play a simultaneous Cournot game. Market demand is given by $a - bq_1 - bq_2$, with $a > 0$ and $b > 0$. Firm 1 has constant marginal cost of c_1 and Firm 2 has constant marginal cost of c_2 with $c_1 < c_2$ and no fixed costs for either firm. The profit to firm i is:

$$\Pi_i(q_i, q_j) = (a - bq_i - bq_j)\, q_i - c_i q_i$$

These firms, however, are not solely concerned with profit, but are also concerned with inequity in production. Thus, instead of maximizing profit they maximize *utility* by choosing quantity, where utility is given by:

$$U_i(q_i, q_j) = \Pi_i(q_i, q_j) - \beta(q_i - q_j)^2$$

where $\beta > 0$ is a constant which measures how much the firm dislikes inequity in production.

a Find the best response functions for this simultaneous Cournot game.

Answer:

Player 1 wants to maximize:

$$\begin{aligned}
U_1(q_1, q_2) &= (a - bq_1 - bq_2)\, q_1 - c_1 q_1 - \beta(q_1 - q_2)^2 \\
\frac{\partial U_1}{\partial q_1} &= a - 2bq_1 - bq_2 - c_1 - 2\beta(q_1 - q_2) \\
0 &= a - 2bq_1 - bq_2 - c_1 - 2\beta(q_1 - q_2) \\
0 &= a - 2bq_1 - bq_2 - c_1 - 2\beta q_1 + 2\beta q_2 \\
2bq_1 + 2\beta q_1 &= a - bq_2 - c_1 + 2\beta q_2 \\
q_1 &= \frac{a - bq_2 - c_1 + 2\beta q_2}{2b + 2\beta}
\end{aligned}$$

Player 2 maximizes a similar utility function:

$$\begin{aligned}
U_2(q_1, q_2) &= (a - bq_2 - bq_1)\, q_2 - c_2 q_2 - \beta(q_2 - q_1)^2 \\
\frac{\partial U_2}{\partial q_2} &= a - 2bq_2 - bq_1 - c_2 - 2\beta(q_2 - q_1) \\
0 &= a - 2bq_2 - bq_1 - c_2 - 2\beta(q_2 - q_1) \\
0 &= a - 2bq_2 - bq_1 - c_2 - 2\beta q_2 + 2\beta q_1 \\
2bq_2 + 2\beta q_2 &= a - bq_1 - c_2 + 2\beta q_1 \\
q_2 &= \frac{a - bq_1 - c_2 + 2\beta q_1}{2b + 2\beta}
\end{aligned}$$

Technically, the best response functions are:

$$\begin{aligned}
q_1 &= Max\left[0, \frac{a - bq_2 - c_1 + 2\beta q_2}{2b + 2\beta}\right] \\
q_2 &= Max\left[0, \frac{a - bq_1 - c_2 + 2\beta q_1}{2b + 2\beta}\right]
\end{aligned}$$

b Find the pure strategy Nash equilibrium for these firms which dislike inequity.

Answer:

To find this simply substitute in for either q_1 or q_2:

$$\begin{aligned}
(2b+2\beta)\,q_1 &= a-c_1+2\beta q_2-bq_2\\
(2b+2\beta)\,q_1 &= a-c_1+(2\beta-b)\,q_2\\
(2b+2\beta)\,q_1 &= a-c_1+(2\beta-b)\left(\frac{a-bq_1-c_2+2\beta q_1}{2b+2\beta}\right)\\
(2b+2\beta)^2\,q_1 &= 2ba+2\beta a-2bc_1-2\beta c_1+(2\beta-b)\,(a-bq_1-c_2+2\beta q_1)\\
(2b+2\beta)^2\,q_1 &= \begin{array}{c}2ba+2\beta a-2bc_1-2\beta c_1\\ +2\beta a-2\beta bq_1-2\beta c_2+4\beta^2q_1-ba+b^2q_1+bc_2-2\beta bq_1\end{array}\\
(2b+2\beta)^2\,q_1+4\beta bq_1-4\beta^2q_1-b^2q_1 &= 2ba+2\beta a-2bc_1-2\beta c_1+2\beta a-2\beta c_2-ba+bc_2\\
(2b+2\beta)^2\,q_1+4\beta bq_1-4\beta^2q_1-b^2q_1 &= ba+4\beta a-2bc_1-2\beta c_1-2\beta c_2+bc_2\\
4b^2q_1+8b\beta q_1+4\beta^2q_1+4\beta bq_1-4\beta^2q_1-b^2q_1 &= ba+4\beta a-2bc_1-2\beta c_1-2\beta c_2+bc_2\\
3b^2q_1+12b\beta q_1 &= ba+4\beta a-2bc_1-2\beta c_1-2\beta c_2+bc_2\\
q_1 &= \frac{ba+4\beta a-2bc_1-2\beta c_1-2\beta c_2+bc_2}{3b^2+12b\beta}\\
q_1 &= \frac{ba-2bc_1+bc_2}{3b^2+12b\beta}+\frac{4\beta a-2\beta c_1-2\beta c_2}{3b^2+12b\beta}\\
q_1 &= \frac{a-2c_1+c_2}{3b+12\beta}+\frac{4\beta a-2\beta c_1-2\beta c_2}{3b^2+4b\beta}
\end{aligned}$$

Now you do not need the result in this form, it just makes it easy to see that if $\beta=0$ the NE quantity matches that in part **c** below. For Firm 2 we will have a similar quantity, only replacing the c_1's and c_2's in Firm 1's NE quantity with c_2's and c_1's. So:

$$q_2=\frac{a-2c_2+c_1}{3b+12\beta}+\frac{4\beta a-2\beta c_2-2\beta c_1}{3b^2+12b\beta}$$

c In the standard asymmetric cost Cournot model without inequity the PSNE is $q_i=\frac{a-2c_i+c_j}{3b}$ for $i=1,2$. Let q_1^* be Firm 1's equilibrium production in the standard model and $\widetilde{q}_1^*$ be Firm 1's equilibrium production in the model with inequity. Show that if $q_1^*>\widetilde{q}_1^*$ then c_1 must be less than c_2.

Answer:

Now we need to show that:

$$\begin{aligned}
\frac{a-2c_1+c_2}{3b} &> \frac{ba+4\beta a-2bc_1-2\beta c_1-2\beta c_2+bc_2}{3b^2+12b\beta}\\
(a-2c_1+c_2)\left(3b^2+12b\beta\right) &> (ba+4\beta a-2bc_1-2\beta c_1-2\beta c_2+bc_2)\,3b\\
3b^2a-6b^2c_1+3b^2c_2+12b\beta a-24b\beta c_1+12b\beta c_2 &> 3b^2a+12b\beta a-6b^2c_1-6\beta bc_1-6\beta bc_2+3b^2c_2\\
-24b\beta c_1+12b\beta c_2 &> -6\beta bc_1-6\beta bc_2\\
18\beta bc_2 &> 18\beta bc_1\\
c_2 &> c_1
\end{aligned}$$

4. Consider a capacity-constrained duopoly pricing game. Firm j's capacity is q_j for $j=1,2$, and each firm has the same constant cost per unit of output of $c\geq 0$ up to this capacity limit. Assume that the market demand function $x(p)$ is continuous and strictly decreasing at all p such that $x(p)>0$ and that there exists a price $\widetilde{p}$ such that $x(\widetilde{p})=q_1+q_2$. Suppose also that $x(p)$ is concave. Let $p(\cdot)=x^{-1}(\cdot)$ denote the inverse demand function.

   Given a pair of prices charged, sales are determined as follows: consumers try to buy at the low-priced firm first. If demand exceeds this firm's capacity, consumers are served in order of their valuations, starting with high-valuation consumers. If prices are the same, demand is split evenly unless one firm's demand exceeds its capacity, in which case the extra demand spills over to the other firm. Formally, the firms' sales are given by the functions $x_1(p_1, p_2)$ and $x_2(p_1, p_2)$ satisfying:

$$\text{If } p_j > p_i: \qquad \begin{cases} x_i(p_1, p_2) = Min\{q_i, x(p_i)\} \\ x_j(p_1, p_2) = Min\{q_j, Max\{x(p_j) - q_i, 0\}\} \end{cases}$$

$$\text{If } p_2 = p_1 = p: \qquad \left\{ x_i(p_1, p_2) = Min\left\{q_i, Max\left\{\frac{x(p)}{2}, x(p) - q_j\right\}\right\}\right\}$$

a Suppose that $q_1 < b_c(q_2)$ and $q_2 < b_c(q_1)$, where $b_c(\cdot)$ is the best-response function for a firm with constant marginal costs of c. Show that $p_1^* = p_2^* = p(q_1 + q_2)$ is a Nash Equilibrium of this game.

Answer:

Consider that Firm 1 lowers its price, so that it chooses $p_1 < p_1^* = p(q_1 + q_2)$. If Firm 1 does this, then Firm 1 still sells q_1 units (since it was selling its capacity at p_1^*), but now sells all the units it can for a lower price. This is not an optimal change by Firm 1.

Now consider that Firm 1 raises its price, so that it chooses $p_1 > p_1^* = p(q_1 + q_2)$. We know that Firm 2 will produce q_2 units because it produces q_2 units when both Firm 1 and Firm 2 choose $p(q_1 + q_2)$, so it will produce q_2 units when it chooses $p_2 = p(q_1 + q_2)$ and Firm 1 chooses $p_1 > p(q_1 + q_2)$. Firm 1's best response to Firm 2 producing q_2 is given by $b_c(q_2)$, but by assumption this is more than Firm 1 can produce. The question then becomes what amount should Firm 1 produce if it cannot produce $b_c(q_2)$, and this amount is q_1. How does Firm 1 produce this amount? By choosing $p_1 = p(q_1 + q_2)$. This leads us right back to where we started. A similar pair of arguments can be made for Firm 2 to show that $p_1 = p_2 = p(q_1 + q_2)$ is a NE.

b Argue that if either $q_1 > b_c(q_2)$ or $q_2 > b_c(q_1)$, then no PSNE exists.

Answer:

This is only true if $p(q_1 + q_2) > c$. If $p(q_1 + q_2) > c$ then either firm can guarantee itself a positive profit if it charges a low price (if $p(q_1 + q_2) = 60$ and $c = 9$, either firm (or both) can charge a price of 10 and guarantee a positive profit for itself). Thus, any NE would have both firms making some positive sales since they can guarantee a payoff better than 0 by charging a price close to c.

Suppose $p_i < p_j$. If Firm j is making positive sales, Firm i must be selling at capacity. Firm i can then raise its price slightly and still sell at capacity. This will increase Firm i's profit, meaning that the two firms charging different prices cannot be a NE.

This leaves $p_i = p_j$ for some level of p as the only potential NE of the game. There are 3 cases that are possible:

a If $p_1 = p_2 > p(q_1 + q_2)$, then at least one firm sells below capacity and this firm would be better off by slightly lowering its price and either selling at full capacity or stealing all the customers from the other firm.

b If $p_1 = p_2 < p(q_1 + q_2)$, then both firms sell at full capacity. Each firm could gain by increasing its price to $p(q_1 + q_2)$ which would still enable it to sell at full capacity.

c If $p_1 = p_2 = p(q_1 + q_2)$. This leads us back to the NE of the first part of this question. However, now we have either $q_1 > b_c(q_2)$ or $q_2 > b_c(q_1)$ by assumption Thus, the best response for Firm 1 to Firm 2 producing q_2 is to produce less than full capacity, which means that it needs to raise its price. We have already seen that both firms charging different prices is not a NE, and that both firms charging a price above $p(q_1 + q_2)$ is not a NE.

5. Consider the following simultaneous game:

		Player 2			
		w	x	y	z
	w	$\boxed{4},\boxed{4}$	$5,3$	$6,2$	$1,1$
Player 1	x	$3,5$	$6,6$	$7,\boxed{10}$	$2,7$
	y	$2,6$	$\boxed{10},7$	$8,8$	$2,\boxed{11}$
	z	$1,1$	$7,2$	$\boxed{11},2$	$\boxed{3},\boxed{3}$

a Find all pure strategy Nash equilibria to the stage game.

Answer:

There are two PSNE to this game: (1) both choose w and (2) both choose z.

For parts **b-d** assume the game is repeated infinitely.

b Find a strategy profile that results in an outcome path in which both players choose x in every period and the strategy profile you have found is a subgame perfect Nash equilibrium (SPNE).

Answer:

Player 1 chooses x in period 0 (or 1) and continues to choose x unless a deviation he observes a deviation by player 2. Define a deviation as any choice by player 2 other than x. If a deviation occurs player 1 then chooses z for the remainder of the game (note that there are two PSNE so two potential punishment strategies, choosing w or choosing z). Player 2 uses the same strategy. If this is the set of strategies, they will be a SPNE if:

$$\begin{aligned}
\sum_{i=0}^{\infty} 6\delta^i &\geq 10 + \sum_{i=1}^{\infty} 3\delta^i \\
\frac{6}{1-\delta} &\geq 10 + \frac{3\delta}{1-\delta} \\
6 &\geq 10 - 10\delta + 3\delta \\
7\delta &\geq 4 \\
\delta &\geq \frac{4}{7}
\end{aligned}$$

Note that there are other strategy profiles that would also work – I'm sure you all have thought of some that I did not.

c Find a strategy profile that results in an outcome path in which both players choose x in every odd period and y in every even period and the strategy profile you have found is a subgame perfect Nash equilibrium.

Answer:

Players 1 and 2 choose x in every odd period and y in every even period unless a deviation occurs. A deviation occurs when anything but x is chosen in an odd period and anything but y is chosen in an even period. If a deviation occurs then the punishment will be to play z every period for the remainder of the game. This is an SPNE if:

$$\begin{aligned}
\sum_{i=0}^{\infty} 8\left(\delta^2\right)^i + \sum_{i=0}^{\infty} 6\delta\left(\delta^2\right)^i &\geq 11 + \sum_{i=1}^{\infty} 3\delta^i \\
\frac{8}{1-\delta^2} + \frac{6\delta}{1-\delta^2} &\geq 11 + \frac{3\delta}{1-\delta} \\
\frac{8+6\delta}{(1-\delta)(1+\delta)} &\geq 11 + \frac{3\delta}{1-\delta} \\
\frac{8+6\delta}{1+\delta} &\geq 11 - 11\delta + 3\delta \\
\frac{8+6\delta}{1+\delta} &\geq 11 - 8\delta \\
8 + 6\delta &\geq 11 + 11\delta - 8\delta - 8\delta^2 \\
8\delta^2 + 3\delta - 3 &\geq 0
\end{aligned}$$

You can solve this to show that if $\delta \geq\approx 0.453$ then the players will prefer not to deviate. Also, we can check to make sure that no deviation occurs in the periods when x is chosen. Then we would essentially have:

$$\begin{aligned}
\sum_{i=0}^{\infty} 6\left(\delta^2\right)^i + \sum_{i=0}^{\infty} 8\delta\left(\delta^2\right)^i &\geq 10 + \sum_{i=1}^{\infty} 3\delta^i \\
\frac{6}{1-\delta^2} + \frac{8\delta}{1-\delta^2} &\geq 10 + \frac{3\delta}{1-\delta} \\
\frac{6+8\delta}{(1-\delta)(1+\delta)} &\geq 10 + \frac{3\delta}{1-\delta} \\
\frac{6+8\delta}{1+\delta} &\geq 10 - 10\delta + 3\delta \\
\frac{6+8\delta}{1+\delta} &\geq 10 - 7\delta \\
6 + 8\delta &\geq 10 + 10\delta - 7\delta - 7\delta^2 \\
7\delta^2 + 5\delta - 4 &\geq 0
\end{aligned}$$

You can solve this to show that if $\delta \geq\approx 0.479$ then the players will prefer not to deviate. Thus, the players would need a $\delta \geq 0.479$ in order for this set of strategies to be an SPNE. For $0.453 \leq \delta \leq 0.479$ the players would cooperate for the first period (assuming that the first period is 0 and is even) but then would deviate from the suggested strategies in the second period.

Again, there are other SPNE that can be used to support this as an outcome.

d Assume that $\delta = 0.4$, where δ is the discount factor. Find a strategy profile that results in an outcome path in which both players choose y in every period and the strategy profile you have found is a subgame perfect Nash equilibrium.

Answer:

Similar to part **b**, but now both players choose y instead of x unless they observe a deviation. A deviation is any choice of strategy other than y by the other player. If a deviation occurs, then the other player will punish by choosing z forever. This is an SPNE if:

$$\begin{aligned}
\sum_{i=0}^{\infty} 8\delta^i &\geq 11 + \sum_{i=1}^{\infty} 3\delta^i \\
\frac{8}{1-\delta} &\geq 11 + \frac{3\delta}{1-\delta} \\
8 &\geq 11 - 11\delta + 3\delta \\
8\delta &\geq 3 \\
\delta &\geq \frac{3}{8}
\end{aligned}$$

As long as $\delta \geq 0.375$ this will be an SPNE to the game. Since $\delta = 0.4$ it is an SPNE. What if the players had decided to use w (the other PSNE to the game) as their punishment strategy? Then we would have:

$$\begin{aligned}
\sum_{i=0}^{\infty} 8\delta^i &\geq 11 + \sum_{i=1}^{\infty} 4\delta^i \\
\frac{8}{1-\delta} &\geq 11 + \frac{4\delta}{1-\delta} \\
8 &\geq 11 - 11\delta + 4\delta \\
7\delta &\geq 4 \\
\delta &\geq \frac{4}{7}
\end{aligned}$$

This would NOT be an SPNE to the game because $\delta = 0.4$ and we would need $\delta \geq 0.\overline{428571}$ in order for it to be an SPNE. So the choice of punishment strategy makes a difference in this particular setting with $\delta = 0.4$.