BPHD 8110 Answers

Test 1

Thursday February 16^{th}

1. (20 points) Consider the following 3x3 game:

			P2	
		L	C	R
	U	$1,1$	$2,\boxed{2}$	$8,1$
P1	M	$\boxed{7},4$	$\boxed{5},\boxed{5}$	$\boxed{9},3$
	D	$3,3$	$4,\boxed{9}$	$6,7$

a (5 points) Find all pure strategy Nash equilibria to this game.

Answer:

Both players have a strictly dominant strategy: M is a strictly dominant strategy for P1 and C is a strictly dominant strategy for P2.

b (10 points) Suppose that the game is repeated infinitely, with both players having some discount factor $\delta \in [0, 1)$. Propose a set of strategies that use Nash reversion as the punishment mechanism such that the outcome (D, R) occurs each period of the game. Calculate the minimum δ for each player in order for the proposed set of strategies to be a subgame perfect Nash equilibrium.

Answer:

Proposed strategies for the game:

P1 chooses D in the initial period. P1 will always choose D as long as P2 has chosen R in every prior time period and P1 has always chosen D. If P2 ever chooses a strategy other than R, or P1 ever chooses a strategy other than D, then P1 will choose M forever. P2 chooses R in the initial period. P2 will always choose R as long as P1 has always chosen D in every prior time period and P2 has always chosen R. If P1 ever chooses a strategy other than D, or P2 ever chooses a strategy other than R, then P2 will choose C forever.

The discount rate calculation is straightforward. For P1:

$$\begin{aligned}
\frac{6}{1-\delta} &\geq 9 + \frac{5\delta}{1-\delta} \\
6 &\geq 9 - \delta + 5\delta \\
4\delta &\geq 3 \\
\delta &\geq \frac{3}{4}
\end{aligned}$$

For P2:

$$\begin{aligned}
\frac{7}{1-\delta} &\geq 9 + \frac{5\delta}{1-\delta} \\
7 &\geq 9 - \delta + 5\delta \\
4\delta &\geq 2 \\
\delta &\geq \frac{2}{4}
\end{aligned}$$

c (5 points) Suppose that P2 has a discount factor $\delta = 0.8$. If P1 observes a deviation by P2, does P1 need to punish P2 using Nash reversion for all future periods of the game, or could P1 punish P2 for a finite number of periods and then return to cooperating on (D, R)? If P1 can punish for a finite number of periods find the minimum number of periods that P1 must punish P2 using Nash reversion; if P1 must punish P2 using Nash reversion for all future periods explain why P1 cannot punish for a finite amount of periods and then return to cooperating.

Answer:

A discount factor of $\delta = 0.\dot{8}$ is higher than the minimum discount factor needed to sustain cooperation in part b, so it is possible that a less harsh punishment strategy will still sustain cooperation. We would need:

$$\sum_{i=o}^{\infty} \delta^i \Pi_{coop} \geq \Pi_{Deviate} + \sum_{i=1}^{t} \delta^i \Pi_{Nash} + \sum_{t+1}^{\infty} \delta^i \Pi_{coop}$$

in order for this less severe punishment to work. There's a formal way and an informal way to find t. Formally,

$$\begin{aligned}
\sum_{i=o}^{\infty} \delta^i \Pi_{coop} &\geq \Pi_{Deviate} + \sum_{i=1}^{t} \delta^i \Pi_{Nash} + \sum_{t+1}^{\infty} \delta^i \Pi_{coop} \\
\frac{1}{1-\delta} \Pi_{coop} &\geq \Pi_{Deviate} + \frac{\delta - \delta^{t+1}}{1-\delta} \Pi_{Nash} + \frac{\delta^{t+1}}{1-\delta} \Pi_{Coop}
\end{aligned}$$

At this point, all the variables are known except t:

$$\begin{aligned}
\frac{1}{1-\delta} \Pi_{coop} &\geq \Pi_{Deviate} + \frac{\delta - \delta^{t+1}}{1-\delta} \Pi_{Nash} + \frac{\delta^{t+1}}{1-\delta} \Pi_{Coop} \\
\Pi_{coop} &\geq (1-\delta) \Pi_{Deviate} + \left(\delta - \delta^{t+1}\right) \Pi_{Nash} + \delta^{t+1} \Pi_{Coop} \\
7 &\geq (0.2) * 9 + \left(0.8 - 0.8^{t+1}\right) * 5 + 0.8^{t+1} * 7 \\
7 &\geq 1.8 + 4 + (7-5) * 0.8^{t+1} \\
7 &\geq 5.8 + 2 * 0.8^{t+1} \\
1.2 &\geq 2 * 0.8^{t+1} \\
0.6 &\geq 0.8^{t+1} \\
\ln(0.6) &\geq (t+1) \ln(0.8) \\
\frac{\ln(0.6)}{\ln(0.8)} &\leq t+1 \\
2.2892 &\leq t+1 \\
1.2892 &\leq t
\end{aligned}$$

The $\geq$ switches to $\leq$ because $\ln(0.8) < 0$. As long as $t \geq 2$, which means that as long as the punishment is at least two periods, P2 should cooperate. That is a much more forgiving punishment strategy.

The informal method would be to recognize that:

$$\begin{aligned}
\sum_{i=o}^{\infty} \delta^i \Pi_{coop} &\geq \Pi_{Deviate} + \sum_{i=1}^{t} \delta^i \Pi_{Nash} + \sum_{t+1}^{\infty} \delta^i \Pi_{coop} \\
\sum_{i=o}^{t} \delta^i \Pi_{coop} &\geq \Pi_{Deviate} + \sum_{i=1}^{t} \delta^i \Pi_{Nash}
\end{aligned}$$

because those later periods of cooperation just cancel each other out. Then you could create a table and calculate payoffs for different lengths of punishment:

$t = 0 : 7 \geq 9$	Not true
$t = 1 : (7 + 0.8 * 7) = 12.6 \geq (9 + 0.8 * 5) = 13$	Not true
$t = 2 : (12.6 + 0.64 * 7) = 17.08 \geq (13 + 0.64 * 5) = 16.2$	True

That works if t is small; if t is 132 that process works less well (at least when done by hand).

2. (20 points) Consider a simultaneous Cournot game in which there are N firms and each firm chooses a quantity level q_i. Market price is given by $P(Q) = a - bQ$, with $a > 0$, $b > 0$, and $Q = \sum_{i=1}^{N} q_i$. All firms have constant marginal cost of c, with $a > c > 0$, and there are no fixed costs for any firm. The profit to firm i is:

$$\Pi_i (q_i, \Sigma q_{-i}) = (a - bq_i - b\Sigma q_{-i}) q_i - cq_i$$

a (5 points) Find the best response correspondence (or function) for a single firm.

Answer:

Firm i maximizes profit by choosing quantity:

$$\begin{aligned} \max_{q_i} \Pi_i (q_i, \Sigma q_{-i}) &= (a - bQ) q_i - cq_i \\ \frac{\partial \Pi_i}{\partial q_i} &= a - 2bq_i - b\Sigma q_{-i} - c \\ 0 &= a - 2bq_i - b\Sigma q_{-i} - c \\ 2bq_i &= a - b\Sigma q_{-i} - c \\ q_i &= \frac{a - b\Sigma q_{-i} - c}{2b} \end{aligned}$$

Technically, if the other firms produce a total quantity more than the perfectly competitive quantity then firm i should produce 0, so the best response correspondence is:

$$b_i (\Sigma q_{-i}) = Max \left[0, \frac{a - b\Sigma q_{-i} - c}{2b}\right]$$

b (10 points) Assuming a symmetric equilibrium (meaning all firms play the same strategy), what is the pure strategy Nash equilibrium to this game?

Answer:

The key here is that all firms use the same strategy, so they all have the same best response function and will all produce the same quantity. So $\Sigma q_{-i} = (N - 1) q_i$. Substituting into the nonzero portion of the best response:

$$\begin{aligned} q_i &= \frac{a - b(N-1) q_i - c}{2b} \\ 2bq_i &= a - b(N-1) q_i - c \\ 2bq_i + b(N-1) q_i &= a - c \\ bq_i (2 + N - 1) &= a - c \\ bq_i (N+1) &= a - c \\ q_i &= \frac{a - c}{(N+1) b} \end{aligned}$$

Notice that if $N = 2$ then we have $q_i = \frac{a-c}{3b}$, which is the PSNE when $N = 2$.

c (5 points) Show that as $N \to \infty$, price approaches marginal cost ($P \to c$) and that $P \to c$ implies profit approaches zero ($\Pi_i \to 0$).

Answer:

The price in the market is given by:

$$P(Q) = a - bQ$$

where $Q = \sum_{i=1}^{N} q_i$. We have $q_i = \frac{a-c}{(N+1)b}$ and N firms that produce that quantity, so:

$$\begin{aligned}
P(Q) &= a - bN\left(\frac{a-c}{(N+1)b}\right) \\
P(Q) &= a - \frac{Na - Nc}{N+1} \\
P(Q) &= \frac{Na + a - Na + Nc}{N+1} \\
P(Q) &= \frac{a + Nc}{N+1} \\
P(Q) &= \frac{a}{N+1} + \frac{Nc}{N+1}
\end{aligned}$$

The limit as $N \to \infty$ for the first term, $\frac{a}{N+1}$, is zero, but the limit for the second term, $\frac{Nc}{N+1}$, is $\frac{\infty}{\infty}$. We can use l'Hopital's rule to take the ratio of the derivatives of the numerator and denominator with respect to N (which are c and 1, respectively) to find that the limit to this second term is c.

For the part about showing that profit tends to zero as $P \to c$:

$$\begin{aligned}
\Pi_i(q_i, \Sigma q_{-i}) &= (a - bq_i - b\Sigma q_{-i})\, q_i - cq_i \\
\Pi_i(q_i, \Sigma q_{-i}) &= Pq_i - cq_i
\end{aligned}$$

It should be clear that as $P \to c$, $Pq_i - cq_i \to 0$.

You can also show that firm i's profit tends to zero as $N \to \infty$ (without using the information that $P \to c$ as $N \to \infty$) though it is just more involved because there are more terms:

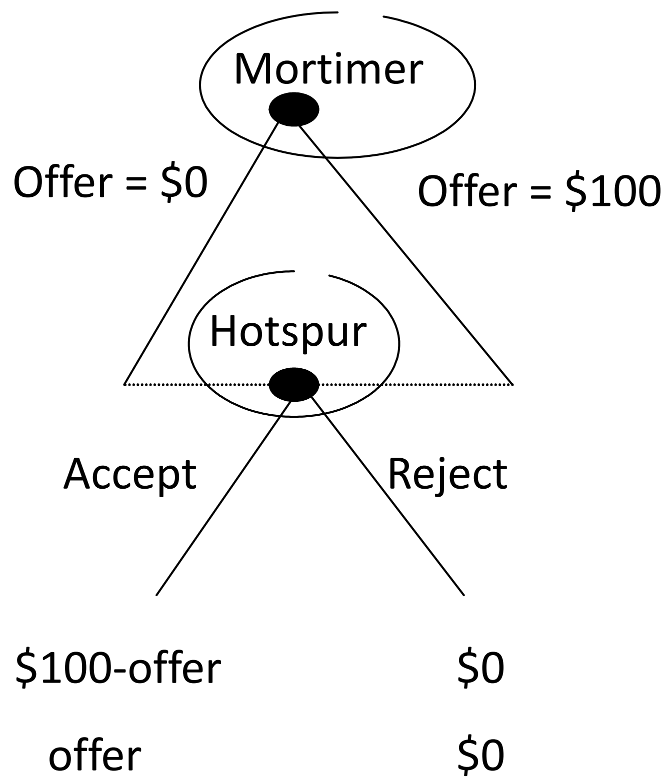
$$\begin{aligned}
\Pi_i(q_i, \Sigma q_{-i}) &= (a - bq_i - b\Sigma q_{-i})\, q_i - cq_i \\
\Pi_i(q_i, \Sigma q_{-i}) &= (a - bq_i - b(N-1)\, q_i)\, q_i - cq_i \\
\Pi_i(q_i, \Sigma q_{-i}) &= (a - bNq_i)\, q_i - cq_i \\
\Pi_i(q_i, \Sigma q_{-i}) &= (a - c)\, q_i - bNq_i^2 \\
\Pi_i(q_i, \Sigma q_{-i}) &= (a - c)\left(\frac{a-c}{(N+1)b}\right) - bN\left(\frac{a-c}{(N+1)b}\right)^2
\end{aligned}$$

At this point, it should be clear that the first term in the profit function, $(a-c)\left(\frac{a-c}{(N+1)b}\right)$, tends to zero as $N \to \infty$ because the numerator is constant while the denominator goes to infinity. For the second term, $bN\left(\frac{a-c}{(N+1)b}\right)^2$, both the numerator and denominator approach infinity as $N \to \infty$, but we can use l'Hopital's rule to show that:

$$\frac{\frac{d\left(bN(a-c)^2\right)}{dN}}{\frac{d((N+1)b)^2}{dN}} = \frac{b\,(a-c)^2}{(2\,(N+1)\,b)\,b}$$

and that term tends to zero as $N \to \infty$.

3. (15 points) Suppose that we have two players, Mortimer and Hotspur, who are charged with a task of dividing \$100. Mortimer offers Hotspur an amount of the \$100, of which Hotspur would receive the offer and Mortimer would receive ($\$100 - offer$). All offers must be in integer amounts and greater than or equal to \$0 and less than or equal to the amount of the pie. This game is sequential, so Hotspur observes this offer and can choose to accept or reject the offer. If Hotspur accepts, the game ends and Hotspur receives the offer and Mortimer receives ($\$100 - offer$). If Hotspur rejects, the game ends and both players receive \$0. The extensive form of the game is:



a (5) Assume that both players have a utility function such that $u(x) = x$, which means they only care about the amount of money they receive. Find a subgame perfect Nash Equilibrium (SPNE) to this game.

Answer:

Starting with the subgame at which Hotspur makes a decision, if Hotspur rejects he receives a payoff of \$0, so he should accept any offer that is greater than zero. Assume he decides to reject an offer of zero. Knowing Hotspur's strategy, Mortimer offers the lowest possible amount that Hotspur will accept, which is \$1. So an SPNE is Mortimer offers \$1, and Hotspur accepts any offer greater than \$0 and rejects an offer of \$0. The outcome is that Mortimer receives \$99 and Hotspur receives \$1.

Alternatively, Hotspur can also accept an offer of zero because he would be indifferent between receiving zero if accept and zero if reject. In this SPNE, Mortimer offers \$0 and Hotspur accepts all offers greater than or equal to \$0. The outcome to this SPNE is that Mortimer receives \$100 while Hotspur receives \$0.

While Hotspur's strategy is rational given that he only cares about the monetary payoff, in experiments second players routinely reject offers that are far below the 50/50 split. If that happens, then it is clear that some factor other than monetary payoff (equity, spite, etc.) affects the players' utilities, which is why I stated that the utility function was such that only the monetary payoff matters.

Now, consider a modified version of the game. The beginning of the game is still the same (Mortimer makes an offer, Hotspur observes this offer and can accept or reject, if he accepts the game ends). However, now if Hotspur rejects, the game continues to a second round, where the amount of money shrinks to \$90. Hotspur now has the chance to make an offer to Mortimer. The second round is sequential so Mortimer observes this offer and can either accept or reject the offer. If Mortimer accepts, the game ends and Mortimer receives the offer and Hotspur receives ($\$90 - offer$). If Mortimer rejects, the game ends and each receives \$0.

b (10) Assume that both players have a utility function such that $u(x) = x$. Find a subgame perfect Nash Equilibrium (SPNE) to the entire game. What is the outcome to the SPNE that you found?

Hint: It may be helpful to draw the second round of the game to help you identify available actions to each player.

Answer:

Now Mortimer is the one who would accept any offer greater than \$0 in the second stage; Hotspur knowing this should offer Mortimer \$1 and take \$89 for himself. Once we have solved that second round we can then reanalyze the first round. Hotspur knows that if he rejects an offer he can guarantee that he receives \$89 in the second round and Mortimer also knows this. So Hotspur should accept any offer that is greater than or equal to \$89 and reject any offers less than \$89. Mortimer knows that if the game goes to a second round he will only receive \$1 and he can do better than that by offering Hotspur more money in the first round because the pie does not shrink. So Mortimer offers Hotspur the minimum amount that Hotspur will accept, which is \$89 and keeps \$11 for himself. The SPNE:

Mortimer (first round): Offer \$89

Hotspur (first round): Accept any offers greater than or equal to \$89; reject any offer less than \$89.

Hotspur (second round): Offer \$1

Mortimer (second round): Accept any offers greater than or equal to \$1: reject any offer less than \$1

The outcome ends up being that Mortimer receives \$11 and Hotspur \$89 because the game ends after the first round; there's another SPNE where the outcome is Mortimer receives \$10 and Hotspur \$90 that relies on Mortimer accepting any offer in the second round.

4. (15 points) There are two players, Milo and Otis, who bargain over how to split \$100. Both players simultaneously name shares that they would like to have, s_1 and s_2, where $0 \leq s_1 \leq \$100$ and $0 \leq s_2 \leq \$100$. If $s_1 + s_2 \leq \$100$, then the players receive the shares they submitted; if $s_1 + s_2 > \$100$, then both players receive \$0. Both players have a utility function $u(x) = x$ meaning they are only concerned with the monetary payoff. Assume that the shares s_i can be any real number between $[0, 100]$ (they are not just restricted to be integers).

a (5 points) There are two existence theorems listed on the last page of the exam. Can either be applied to guarantee that this game has a mixed strategy Nash equilibrium or a pure strategy Nash equilibrium? Explain.

Answer:

The first proposition is about an MSNE existing if there are a finite number of players and each player has a strategy set with a finite number of elements. It is clear that there are a finite number of players because there are two. It is less clear that there are a "finite" number of strategies – after all, we have an interval $[0, 100]$ in which any real number can be chosen, which contains an infinite number of elements. If we had an interval $[0, 100]$ in which only integers could be chosen then we would certainly have a guarantee of existence of an MSNE.

The second proposition requires that the strategy space be compact, convex, and nonempty. A closed interval in which any real number can be chosen meets that definition. However the second part of the proposition requires that the utility function be continuous in the strategy space. Like with the Bertrand game, the payoff function in this game does not meet that criterion. To see that the payoff function is discontinuous in the strategy space, consider Otis choosing a strategy of \$50. If Milo chooses \$49 he receives \$49; if he chooses \$49.01 he receives \$49.01; etc.; if he chooses \$50 he receives \$50. But if he chooses $\$50 + \varepsilon$, where ε is a very small amount, his payoff drops from \$50 to zero.

b (5 points) Is there a pure strategy Nash equilibrium (or equilibria) to this game? If so what is the equilibrium (or the set of equilibria) and if not explain why not.

Answer:

There is a set of PSNE where $s_1 + s_2 = \$100$; three examples of the PSNE are $s_1 = \$1$ and $s_2 = \$99$ or $s_1 = s_2 = \$50$ or $s_1 = \$58$ and $s_2 = \$42$. If either player increases their share request then both players will receive zero; if either player decreases their share request then that player is just sacrificing some payoff that they could be receiving.

We cannot have $s_1+s_2 < \$100$ because both players would have the incentive to increase their requests because they could earn a higher payoff by increasing their request given what the other has chosen.

We cannot (generally) have $s_1 + s_2 > \$100$ because both players would have the incentive to decrease their requests so that they receive a positive nonzero payoff. There is one exception – if both players submit $s_i = \$100$, then they both receive zero and neither player has the incentive to deviate because if either player changes their strategy then they still receive \$0 because $s_1 + s_2$ is still greater than \$100. However, we cannot have one player submit $s_1 = \$100$ and the other player submit $s_2 = \$60$. While there is no change that player 2 could make to earn a nonzero payoff given the share submitted by player 1, given the share that player 2 has submitted player 1 could reduce their share to \$40 and both would receive a nonzero payoff.

c (5 points) Question 3 (with Mortimer and Hotspur) and this question (with Milo and Otis) both involve two players splitting an amount of money, though one game is sequential and the other simultaneous. Explain how the difference in game structure (sequential or simultaneous) affects the number of equilibria and the expected outcome of the game.

Answer:

The Milo and Otis game has a continuum of equilibria which creates coordination problems among the players. While we assume that they will play an equilibrium, given the multiplicity of equilibria it is very possible that they miscoordinate and either (1) end up receiving no money because the sum of their shares are greater than \$100 or (2) end up not receiving the entire \$100 because the sum of their shares are less than \$100. The sequential nature of the Mortimer and Hotspur game removes that uncertainty because it allows players to credibly commit to an offer that is then either accepted or rejected. We can see that there is less uncertainty because there are many fewer SPNE in the M/H game than there are PSNE in the M/O game. While we do not know which equilibrium will be played in the M/O game, one might guess that both players may submit a share close to the midpoint of \$50; there are plenty of other equilibria, but I would guess that to be the most likely outcome. Alternatively, one could say on average both players in the M/O game should get \$50 if the average of all the PSNE were calculated. The M/H game has a much more unequal split of the money as an outcome (at least theoretically) – essentially one player receives most of the money and the other player receives very little. However, as I mentioned in the response to part a of the third question (and this information is not part of the answer to this question but additional information), in practice many of the very uneven splits in the sequential game are rejected. There are ways to structure the game such that those unequal offers are accepted, but in just a standard game that is presented a 90/10 offer would likely be rejected. Why that happens is an open question – it could be because people have some type of preference for equity or people will give up a small amount of money to punish someone who they believe acts unfairly or a host of other reasons.

5. (30 points) Consider the following very general coordination games, where $a > 0$, $b > 0$, and $c > 0$.

Game 1

Player 1 \ Player 2	A	B
A	$\boxed{a},\boxed{a}$	$0,0$
B	$0,0$	$\boxed{b},\boxed{b}$

Game 2

Player 1 \ Player 2	A	B	C
A	$\boxed{a},\boxed{a}$	$0,0$	$0,0$
B	$0,0$	$\boxed{b},\boxed{b}$	$0,0$
C	$0,0$	$0,0$	$\boxed{c},\boxed{c}$

a (5 points) Are there any strictly dominant or strictly dominated strategies in Game 1 or Game 2? Explain.

Answer:

There are no strictly dominant or strictly dominated strategies. I have marked the payoffs to the best responses in the matrices and every strategy for both players is a best response to some strategy choice of the other player.

b (5 points) Find all pure strategy Nash equilibrium (PSNE) to Game 1 and Game 2. If there are no PSNE, explain why there are none.

Answer:

In Game 1, the PSNE are both players choose A and both players choose B.

In Game 2, the PSNE are both players choose A, both players choose B, and both players choose C.

c (5 points) Using the general payoffs, find the mixed strategy Nash equilibrium for Game 1. **Hint**: The game is symmetric.

Answer:

The hint about symmetry is useful because we only have to find the probabilities for one player (the other player will have the exact same probabilities because that second player will solve the exact same system of equations as the first). Letting σ_{2A} and σ_{2B} be the probabilities for Player 2 using strategies A and B, respectively, we know that Player 1's expected value of choosing A must equal the expected value of choosing B:

$$\begin{aligned} E_1[A] &= E_1[B] \\ a\sigma_{2A} &= b\sigma_{2B} \end{aligned}$$

Using $\sigma_{2B} = 1 - \sigma_{2A}$, we have:

$$\begin{aligned} a\sigma_{2A} &= b(1 - \sigma_{2A}) \\ a\sigma_{2A} &= b - b\sigma_{2A} \\ a\sigma_{2A} + b\sigma_{2A} &= b \\ \sigma_{2A} &= \frac{b}{a+b} \end{aligned}$$

Using $\sigma_{2B} = 1 - \sigma_{2A}$, we know that $\sigma_{2B} = 1 - \frac{b}{a+b} = \frac{a}{a+b}$. By symmetry we know the MSNE is Players 1 and 2 chooses A with probability $\frac{b}{a+b}$ and B with probability $\frac{a}{a+b}$.

d (10 points) Using the general payoffs, find the mixed strategy Nash equilibrium for Game 2. Assume all strategies that are not strictly dominated are used in the MSNE. **Hint(s)**: The game is symmetric. Also, I know I said not to simplify results, but it may be helpful to simplify in this problem.

Answer:

The same basic process as in part **b** can be used, only now we let σ_{2A}, σ_{2B}, and σ_{2C} be the probabilities for Player 2 using strategies A, B and C, respectively. We will need two equations where expected values are set equal and the fact that probabilities sum to 1 to solve for the equilibrium.

$$\begin{aligned} E_1[A] &= E_1[B] \\ a\sigma_{2A} &= b\sigma_{2B} \\ a\sigma_{2A} &= b(1 - \sigma_{2A} - \sigma_{2C}) \\ a\sigma_{2A} &= b - b\sigma_{2A} - b\sigma_{2C} \\ a\sigma_{2A} + b\sigma_{2A} &= b - b\sigma_{2C} \\ b\sigma_{2C} &= b - a\sigma_{2A} - b\sigma_{2A} \\ \sigma_{2C} &= \frac{b - a\sigma_{2A} - b\sigma_{2A}}{b} \end{aligned}$$

and

$$\begin{aligned}
E_1[B] &= E_1[C] \\
b\sigma_{2B} &= c\sigma_{2C} \\
b(1-\sigma_{2A}-\sigma_{2C}) &= c\sigma_{2C} \\
b-b\sigma_{2A}-b\sigma_{2C} &= c\sigma_{2C} \\
b-b\sigma_{2A} &= c\sigma_{2C}+b\sigma_{2C} \\
\frac{b-b\sigma_{2A}}{c+b} &= \sigma_{2C}
\end{aligned}$$

Now we have:

$$\begin{aligned}
\frac{b-a\sigma_{2A}-b\sigma_{2A}}{b} &= \frac{b-b\sigma_{2A}}{c+b} \\
bc-ac\sigma_{2A}-bc\sigma_{2A}+b^2-ab\sigma_{2A}-b^2\sigma_{2A} &= b^2-b^2\sigma_{2A} \\
bc-ac\sigma_{2A}-bc\sigma_{2A}-ab\sigma_{2A} &= 0 \\
bc &= ac\sigma_{2A}+bc\sigma_{2A}+ab\sigma_{2A} \\
\frac{bc}{ac+bc+ab} &= \sigma_{2A}
\end{aligned}$$

Using our prior result:

$$\begin{aligned}
\frac{b-b\sigma_{2A}}{c+b} &= \sigma_{2C} \\
b-b\sigma_{2A} &= (c+b)\,\sigma_{2C} \\
b-b\left(\frac{bc}{ac+bc+ab}\right) &= (c+b)\,\sigma_{2C} \\
\frac{bac+b^2c+b^2a-b^2c}{ac+bc+ab} &= (c+b)\,\sigma_{2C} \\
\frac{bac+b^2a}{ac+bc+ab} &= (c+b)\,\sigma_{2C} \\
\frac{ab(c+b)}{(ac+bc+ab)}\frac{1}{(c+b)} &= \sigma_{2C} \\
\frac{ab}{ac+bc+ab} &= \sigma_{2C}
\end{aligned}$$

Then using $\sigma_{2B}=1-\sigma_{2A}-\sigma_{2C}$ we have:

$$\sigma_{2B}=\frac{ac}{ac+bc+ab}$$

So the MSNE is that Players 1 and 2 play A with probability $\frac{bc}{ac+bc+ab}$, B with probability $\frac{ac}{ac+bc+ab}$, and C with probability $\frac{ab}{ac+bc+ab}$. In expectation, each player will receive $\frac{abc}{ac+bc+bc}$ in expected value from playing any pure strategy.

e (5 points) Now consider the following 5x5 matrix, Game 3:

Game 3 — Player 2

		A	B	C	D	E
Player 1	A	$1,1$	$0,0$	$0,0$	$0,0$	$0,0$
	B	$0,0$	$2,2$	$0,0$	$0,0$	$0,0$
	C	$0,0$	$0,0$	$3,3$	$0,0$	$0,0$
	D	$0,0$	$0,0$	$0,0$	$4,4$	$0,0$
	E	$0,0$	$0,0$	$0,0$	$0,0$	$5,5$

Find the MSNE to this game. **Hint**: You can work through all the math to find this MSNE, but it may be helpful to look at the pattern from parts **c** and **d** and use that to aid in your answer.

Answer:

The point of doing parts **b** and **c** was to set up this question. We know (or could probably guess) that the general probability that Players 1 and 2 would use strategy A would be:

$$\sigma_{2A} = \frac{bcde}{abcd + abce + abde + acde + bcde}$$

And then the following for strategies B, C, D, and E:

$$\begin{aligned}
\sigma_{2B} &= \frac{acde}{abcd + abce + abde + acde + bcde} \\
\sigma_{2C} &= \frac{abde}{abcd + abce + abde + acde + bcde} \\
\sigma_{2D} &= \frac{abce}{abcd + abce + abde + acde + bcde} \\
\sigma_{2E} &= \frac{abcd}{abcd + abce + abde + acde + bcde}
\end{aligned}$$

Using the numbers we have:

$$\begin{aligned}
bcde &= 120 \\
acde &= 60 \\
abde &= 40 \\
abce &= 30 \\
abcd &= 24
\end{aligned}$$

The denominator is 274. The probabilities are:

$$\begin{aligned}
\sigma_{2A} &= \frac{120}{274} \\
\sigma_{2B} &= \frac{60}{274} \\
\sigma_{2C} &= \frac{40}{274} \\
\sigma_{2D} &= \frac{30}{274} \\
\sigma_{2E} &= \frac{24}{274}
\end{aligned}$$

You can check that the expected value of either player choosing to play any PSNE is $\frac{120}{274}$, which satisfies the theorem we had in class. This problem is more time consuming than difficult, but I wanted to use it to get you all to think about how starting with easy games (a 2x2 game and a 3x3 game) to find patterns can lead to a more general form for games with larger strategy spaces.

(Extra information – you didn't have to do this): Note that you should be able to find a mixed strategy in the 5x5 game for any combination of two, three, and four pure strategies. Consider the mixed strategy in part **c** where the players mix over A and B. We had $\sigma_{iA} = \frac{b}{a+b}$ and $\sigma_{iB} = \frac{a}{a+b}$ for $i = 1, 2$. Using the payoffs from part **e** we would have $\sigma_{iA} = \frac{2}{3}$ and $\sigma_{iB} = \frac{1}{3}$. If one player uses the pure strategy of A against this mix then they receive an expected payoff of $\frac{2}{3}$; if they use a pure strategy of B against this mix then they receive an expected payoff of $\frac{2}{3}$; however, if they use a pure strategy of either C, D, or E against this mix then they receive an expected payoff of 0 because they are always getting zero, so neither player would ever want to use C, D, or E (despite the higher payoffs) if the other player is mixing over A and B. Having the payoff be zero when there is a mismatch of strategies should hopefully make that relatively easy to see when there is not the time pressure of an exam.

Proposition 1 *Every game* $\Gamma_N = [I, \{\Delta(S_i)\}, \{u_i(\cdot)\}]$ *in which the sets* $S_1, ..., S_I$ *have a finite number of elements has a mixed strategy Nash Equilibrium (note that it could be degenerate, like the Prisoner's Dilemma).*

Proposition 2 *A Nash Equilibrium exists in game* $\Gamma_N = [I, \{S_i\}, \{u_i(\cdot)\}]$ *if for all* $i = 1, ..., I$

1. S_i *is a nonempty, convex, and compact subset of some Euclidean space* $\mathbb{R}^M$
2. $u_i(s_1, ..., s_I)$ *is continuous in* $(s_1, ..., s_I)$ *and quasiconcave in* s_i