Mechanism Design

1 Introduction

The fundamental problem in economics is how to allocate scarce resources. Many different allocation systems or mechanisms could be used. A very simple allocation system is dictatorship – a single individual (or perhaps a committee of individuals) determines the resource allocation for all individuals. Another simple allocation system is that of "first come, first serve" which simply means that the first person to arrive gets the resource. Typically, economists like to consider how markets or prices allocate resources. The "first come, first serve" allocation system might be coupled with a payment system, like campus parking. The parking fee allows one to park on campus, but does not guarantee the consumer a particular parking space. Many posted-price markets have an element of "first come, first serve" present as well – if there is only one item remaining on the shelf (if we consider a brick and mortar location) or one item remaining in an online store (if we consider virtual locations), then the first person willing to pay the price for that item receives the item. Negotiations, or bargaining, are also systems by which resources are allocated.

While those are all interesting allocation systems, we will focus on a particular type of allocation system, which we will call a "mechanism." Consider a game of incomplete information, where one party (the principal or seller) in a transaction does not know some information (perhaps the willingness to pay) the other party (the agent or buyer) has.[1] Mechanism design can generally be thought of in three steps:

1. A seller designs a mechanism/contract/incentive scheme in which buyers send "messages" and allocations are made based upon the messages sent. An "allocation" here refers not only to the actual good(s) being transferred, but also any transfer payments that are made between buyer(s) and seller. For now our focus will be on simultaneous messages.

2. Buyers choose whether or not they want to participate in the game, or, alternatively one could think about this step as the buyers accepting or rejecting the seller's proposed mechanism. If buyers choose not to participate in the game they would typically have some reservation level of utility. This step requires consideration of the buyer's participation constraint (PC) when the buyer determines whether or not to participate.[2] There are some instances in which the "seller" can force the "buyer" to participate, thus removing the need for a participation constraint. Consider the government imposing a tax scheme on individuals or corporations. These economic agents must participate, even if "participating" means "not filing taxes properly and suffering the consequences." That scenario is different than eBay, which has no authority to force individuals to bid in its auctions or punish them for failing to bid in its auctions.

3. The final step is that buyers who choose to participate in the game submit their messages, and then an allocation is made based upon the submitted messages and the mechanism design.

Another key constraint that the seller must consider when designing the mechanism is the buyer's incentive compatibility (IC) constraint. Sellers want buyers to behave in a certain manner, and a properly designed mechanism can elicit certain types of responses from buyers. Alternatively, in a principal-agent framework, the principal would like to structure a contract to align the incentives of the agent with those of the principal.

[1] It may be helpful to have a particular example in mind. If so, consider the auctions we have discussed, as we will discuss them in more detail throughout this section of notes.

[2] Some texts refer to this constraint as the individual rationality (IR) constraint.

As a specific example of a mechanism consider the first-price sealed bid auction we discussed. The allocation rule is that the bidder who submits the highest bid receives the item and pays an amount equal to his or her submitted bid, while all other bidders make no payment. The "messages" in this mechanism are the bids submitted by the bidders. All messages are used in determining the final allocation and payment because they must all be compared to each other to determine which is the highest bid.

2 Two bidder, two type auction example[3]

Consider a seller with a single good for sale and two bidders who are *ex ante* identical. Bidder values can be one of two possibilities: $\underline{\theta}$ with probability $\underline{p}$ and $\overline{\theta}$ with probability $\overline{p}$. Let s_1 and s_2 be the realizations of the bidder strategies., $X_i(s_1, s_2)$ be the probability the good is transferred to bidder i, and $T_i(s_1, s_2)$ be the transfer payment of bidder i to the seller. Comparing the 1^{st}-price and 2^{nd}-price sealed bid auctions:

	1^{st}-price	2^{nd}-price
	$X_i(s_1, s_2) = 1$	$X_i(s_1, s_2) = 1$
if $s_i > s_j$	$T_j(s_1, s_2) = 0$	$T_j(s_1, s_2) = 0$
	$T_i(s_1, s_2) = s_i$	$T_i(s_1, s_2) = s_j$

We will restrict our discussion to truth telling mechanisms, the reason for which will be discussed shortly. A truth telling mechanism means that the bidders will reveal their true type (as in the equilibrium of a 2^{nd}-price sealed bid auction). Let $\overline{X}$ be the probability of receiving the good and $\overline{T}$ be the expected payment when the value is $\overline{\theta}$. Let $\underline{X}$ be the probability of receiving the good and $\underline{T}$ be the expected payment when the value is $\underline{\theta}$. The seller will structure the mechanism to determine those four parameters based on the submitted signals. We have the following constraints:

$$\begin{aligned} PC_1 &: \underline{\theta X} - \underline{T} \geq 0 \\ PC_2 &: \overline{\theta X} - \overline{T} \geq 0 \end{aligned}$$

$$\begin{aligned} IC_1 &: \underline{\theta X} - \underline{T} \geq \underline{\theta}\overline{X} - \overline{T} \\ IC_2 &: \overline{\theta X} - \overline{T} \geq \overline{\theta}\underline{X} - \underline{T} \end{aligned}$$

Consider what each of these constraints implies. The participation constraints mean that the expected payoff for each type must be at least zero, otherwise they would not participate. The incentive compatibility constraints mean that each type (represented by $\underline{\theta}$ and $\overline{\theta}$) must be better off submitting a message that reveals his or her type. If PC_1 are IC_2 are satisfied, then:

$$\begin{aligned} \underline{\theta X} &\geq \underline{T} \\ \overline{\theta X} - \overline{T} &\geq \overline{\theta}\underline{X} - \underline{T} \end{aligned}$$

$$\begin{aligned} \overline{\theta X} - \overline{T} &\geq \overline{\theta X} - \overline{T} - (\underline{\theta X} - \underline{T}) \geq \overline{\theta}\underline{X} - \underline{\theta X} \\ \left(\overline{\theta} - \underline{\theta}\right)\underline{X} &\geq 0 \\ \overline{\theta X} - \overline{T} &\geq 0 \end{aligned}$$

The next to last step follows because $\underline{X} \geq 0$ and $\overline{\theta} > \underline{\theta}$. Thus, PC_2 is also satisfied and will not be binding unless the seller does not sell to the low type. Now we will show that PC_1 and IC_2 are *binding* constraints. Recall that if the constraints are binding that they can be set to equalities (like the budget constraint in a standard consumer optimization problem). If PC_1 is not binding, then the seller could increase $\overline{T}$ and $\underline{T}$ by the same amount, make more money, and not violate any constraints. So $\underline{\theta X} = \underline{T}$ and a low value bidder does not expect to earn any surplus in equilibrium. Now consider IC_2. If IC_2 were not binding then the seller could increase $\overline{T}$, earn more money, and not violate any constraints. We can now determine

[3] Straightforward adaption from Fudenberg and Tirole, pgs. 246-253. Also see Wolfstetter (1999), pgs. 218-221.

the expected payments from PC_1 and IC_2:

$$\underline{\theta X} = \underline{T}$$

$$\begin{aligned}
\overline{\theta X} - \overline{T} &= \overline{\theta}\underline{X} - \underline{T} \\
\overline{\theta X} - \overline{T} &= \overline{\theta}\underline{X} - \underline{\theta X} \\
\overline{\theta X} - \overline{\theta}\underline{X} + \underline{\theta X} &= \overline{T} \\
\overline{\theta}\left(\overline{X} - \underline{X}\right) + \underline{\theta X} &= \overline{T}
\end{aligned}$$

So we have found the payments made, at least in terms of the probabilities. Now we need to find the probabilities.

Now consider the seller's problem. Letting $E\mu_0$ be the seller's expected utility from the mechanism, then:

$$E\mu_0 = \underline{p}\underline{T} + \overline{p}\overline{T}.$$

Note that $E\mu_0$ is just the payment the seller expects to receive based upon the exogenous probabilities of $\underline{\theta}$ and $\overline{\theta}$ and the expected payments. Substituting for the payments:

$$\begin{aligned}
E\mu_0 &= \underline{p}\underline{\theta X} + \overline{p}\overline{\theta}\left(\overline{X} - \underline{X}\right) + \underline{\theta X} \\
E\mu_0 &= \left(1 - \overline{p}\right)\underline{\theta X} + \overline{p}\left(\overline{\theta}\left(\overline{X} - \underline{X}\right) + \underline{\theta X}\right) \\
E\mu_0 &= \underline{\theta X} + \overline{p}\overline{\theta}\left(\overline{X} - \underline{X}\right) \\
E\mu_0 &= \left(\underline{\theta} - \overline{p}\overline{\theta}\right)\underline{X} + \overline{p}\overline{\theta X}
\end{aligned}$$

The last part of the problem involves putting constraints on the bidders' probabilities $\underline{X}$ and $\overline{X}$. Note that if one bidder receives the good then the other bidder does not, so ex ante:

$$\underline{p}\underline{X} + \overline{p}\overline{X} \leq \frac{1}{2}$$

This equation should probably be explained a little more. Recall that $\overline{p}$ and $\underline{p}$ are exogenous, and that $\overline{p} + \underline{p} = 1$. Consider that $\overline{p} = 1$, so that both bidders have value $\overline{\theta}$. If that is the case, then, if the item is awarded when $\overline{\theta}$ is revealed, we have $\overline{X} \leq \frac{1}{2}$. Now, $\overline{\theta}$ will always be revealed, and there are two bidders, so $\overline{X} = \frac{1}{2}$. A similar argument could be made for $\underline{p}$. Now assume $\overline{p} = \underline{p} = \frac{1}{2}$. We then have $\frac{1}{2}\underline{X} + \frac{1}{2}\overline{X} \leq \frac{1}{2}$, or $\underline{X} + \overline{X} \leq 1$. Again, $\underline{X}$ and $\overline{X}$ are probabilities, but note that the seller does not have to award the item to a bidder, so the sum could be less than 1 (but never, of course, more than 1).

However there is more to specify. Considering

$$E\mu_0 = \left(\underline{\theta} - \overline{p}\overline{\theta}\right)\underline{X} + \overline{p}\overline{\theta X}$$

there are two potential cases based upon the exogenous parameters $\underline{\theta}$, $\overline{p}$, and $\overline{\theta}$. It is possible that $\underline{\theta} \leq \overline{p}\overline{\theta}$ or $\underline{\theta} > \overline{p}\overline{\theta}$. This relationship will determine the optimal probabilities $\underline{X}$ and $\overline{X}$. If $\underline{\theta} \leq \overline{p}\overline{\theta}$ then $E\mu_0$ is decreasing in $\underline{X}$ and the seller wants to set $\underline{X} = 0$. In that case:

$$E\mu_0 = \overline{p}\overline{\theta X}$$

The constraint in this scenario is that if both bidders have type $\overline{\theta}$ then they each must win with probability $\frac{1}{2}$. Recall that $\overline{X}$ is the probability of receiving the item if the bidder reveals $\overline{\theta}$. Thus, the bidder will always win if the other bidder reveals the low type or will win one-half of the time when the other bidder reveals the high type. So $\overline{X} = \underline{p} + \overline{p}/2$ because the seller wants to maximize the payment $\overline{T}$. The optimal mechanism in this scenario awards the item to no one if both announce $\underline{\theta}$, to the bidder who reveals $\overline{\theta}$ if one reveals the low type and the other the high type, and to each bidder with probability $\frac{1}{2}$ if they both reveal $\overline{\theta}$.

When $\underline{\theta} > \overline{p}\overline{\theta}$ then $E\mu_0$ is strictly increasing in $\underline{X}$ and $\overline{X}$. Then $\underline{p}\underline{X} + \overline{p}\overline{X} = \frac{1}{2}$ and

$$E\mu_0 = \frac{1}{2\underline{p}}\left(\underline{\theta} - \overline{p}\overline{\theta}\right) + \frac{\overline{p}}{\underline{p}}\left(\overline{\theta} - \underline{\theta}\right)\overline{X}.$$

While more complicated than $E\mu_0 = \overline{p}\overline{\theta}\overline{X}$, the form is the same, so $\overline{X} = \underline{p} + \overline{p}/2$. Using $\underline{p}\underline{X} + \overline{p}\overline{X} = \frac{1}{2}$ we can see that $\underline{X} = \underline{p}/2$. In this scenario, if both announce the low type then they each receive the item with probability $\frac{1}{2}$, if one bidder announces the high type and the other bidder announces the low type then the bidder who announces the high type receives the item, and if both announce the high type then they each receive the item with probability $\frac{1}{2}$.

3 General Results

Among the many results for Bayesian games there are two that are used quite a bit. The first result is the Revelation Principle and the second result is the Revenue Equivalence Theorem.

3.1 Revelation Principle

Suppose that a seller wishes to sell an object using some mechanism – the precise mechanism is left unspecified as long as the following conditions are met:

1. The buyers simultaneously make claims about their types. Buyer i can claim to be any type from his feasible set of types.
2. Given the buyers' claims, buyer i pays an amount that is a function of all the reported types and receives the good with some probability based upon the reported types (in an auction the bidder with the highest reported type receives the good with probability 1).

Games that satisfy these criteria are known as direct mechanisms, because the only action is to submit a claim about a type. A 1^{st}-price sealed bid auction and a 2^{nd}-price sealed bid auction are direct mechanisms; the all-pay auction example/experiment we did in class is a direct mechanism; the ascending and descending clock auctions, as well as the oral outcry example we did, are not direct mechanisms. I am going to state a variety of forms of the revelation principle from multiple sources to give you all an idea of how it has been used:

Proposition 1 *(MWG pg. 493) Denote the set of possible states by Θ. In searching for an optimal contract, the owner can without loss restrict himself to contracts of the following form:*

1. *After the state θ is realized, the manager is required to announce which state has occurred.*
2. *The contract specifies an outcome $\left[w\left(\widehat{\theta}\right), e\left(\widehat{\theta}\right)\right]$ for each possible announcement $\widehat{\theta} \in \Theta$. (Note that w and e are just wages and efforts.)*
3. *in every state $\theta \in \Theta$, the manager finds it optimal to report the state truthfully.*

Proposition 2 *(MWG pg. 884) Suppose that there exists a mechanism $\Gamma = (S_1, ..., S_I, g(\cdot))$ that implements social choice function $f(\cdot)$ in Bayesian Nash equilibrium. Then $f(\cdot)$ is truthfully implementable in Bayesian Nash equilibrium.*

Proposition 3 *(Fudenberg and Tirole, pg. 255) The principal can content herself with "direct" mechanisms, in which the message spaces are the type spaces, all agents accept the mechanism in step 2 regardless of their types, and the agents simultaneously and truthfully announce their types in step 3.*

Proposition 4 *(Gibbons, 1992, pg. 165) Any Bayesian Nash equilibrium of any Bayesian game can be represented by an incentive-compatible direct mechanism.*

Proposition 5 *(Wolfstetter, 1999, pg. 214) For any equilibrium of any auction game, there exists an equivalent incentive-compatible direct auction that leads to the same probabilities of winning and expected payments.*

Alternatively, one could consult Myerson (1979). Incentive Compatibility and the Bargaining Problem. *Econometrica* 47, 61-73. You may want to read it just to see the simplest theorem ever. Theorem 2: $F^{**} = F^{*}$.

Those are just five statements of the revelation principles, some of which are a little general. I highly suggest reading the text surrounding the statement of the theorems to understand what assumptions are being made (for example, in Wolfstetter, what constitutes an "auction game"). Why is the revelation principle useful and important? It is useful because it might be difficult to determine the equilibrium in one game, but we know that if an equilibrium exists in that game then we can find a direct, truth-telling mechanism that has the same general properties. In the next section results we examine the 1^{st}-price sealed bid auction (direct mechanism, not truth-telling) and the 2^{nd}-price sealed bid auction (direct mechanism, truth-telling) in more detail.

3.2 Revenue Equivalence

In a previous set of notes I provided a comparison between the expected revenue of a 1^{st}-price sealed bid auction and a 2^{nd}-price sealed bid auction. That comparison was made under a specific set of assumptions. A more general statement about the expected revenue from auctions follows:

Proposition 6 *23.D.3 (Revenue Equivalence Theorem) Consider an auction setting with I risk-neutral buyers, in which buyer i's valuation is drawn from an interval* $\left[\underline{\theta}_i, \overline{\theta}_i\right]$ *with* $\underline{\theta}_i \neq \overline{\theta}_i$ *and a strictly positive density* $\phi_i(\cdot) > 0$*, and in which buyer's types are statistically independent. Suppose that a given pair of Bayesian Nash equilibria of two different auction procedures are such that for every buyer* i *: (i) For each possible realization of* $(\theta_i, ..., \theta_I)$*, Buyer* i *has an identical probability of getting the good in the two auctions; and (ii) Buyer* i *has the same expected utility level in the two auctions when his valuation for the object is at its lowest possible level. Then these equilibria of the two auctions generate the same expected revenue for the seller.*

This result relies upon the previous result (revelation principle). The general idea is that there are direct mechanisms that are not truth-telling mechanisms, but they have the same Bayes-Nash equilibrium as a truth-telling mechanism by the revelation principle. We can then compare the expected revenue from the truth-telling mechanisms because the expected revenue relies on the underlying probability distribution of values.

It is important to consider the assumptions that are embedded in that proposition beyond the two that are explicitly enumerated (a buyer who has a specific type in each mechanism has the same probability of winning in each auction and buyers with the lowest possible value have the same expected utility in each auctions). Buyers are risk-neutral. Values are drawn from the same probability distribution. Value draws are statistically independent. Those assumptions are just the SIPV-RN environment mentioned earlier.

While these assumptions are restrictive, they establish a benchmark for comparing more realistic settings. What happens if buyers are not risk-neutral? What happens if the distribution is not symmetric? What happens if value draws are not independent? What happens if value draws are not private?

3.3 Common Value Auctions

Suppose that I am auctioning off a jar of coins. The jar is see-through, so that you all can see there are coins in the jar. I tell you they are all U.S. coins from 1965-present (prior to 1965 some U.S. coins, notably dimes and quarters, are made of silver and are worth more than their monetary denomination) and you can see various coins (pennies, nickels, dimes, and quarters) in the jar. However, no one is allowed to look insider the jar or take the coins out of the jar. I conduct a 1^{st}-price sealed bid auction for the jar of coins, where the winner gets the coins. Clearly, the monetary amount that each individual would receive is the same because the coins do not depend on the winning bidder. Bidders may have different utility for the coins because perhaps they do not want to carry around coins to spend, but let us assume that they are all students who will happily take money in coin form. Alternatively, we can consider that the bidders have no disutility from the monetary unit and only care about the value of the money. How are individuals' values formed for this jar of coins?

This auction is different than the ones we have discussed previously. In the prior auctions bidders had different values for the same item. It is fairly simple to motivate that example - there are plenty of goods for which you and your friends would pay different amounts. However, in the jar of coins example, all bidders have the same value for the same item, but they likely have different estimates of the item's value. They will not know if their estimate is correct unless they win the item and take possession of it. The jar of coins example seems a bit contrived – after all, who would auction off a jar of coins? But there are plenty of examples that fit this particular type of value determination. Consider a seller who has discovered that there is oil on her property. The seller wants to sell because she does not know much about extracting oil and refining it, but the bidders will not know exactly how much oil is in the deposit until they own the property and can begin to extract it. Perhaps more relevant to finance students, consider a target firm that is for sale. Other firms would like to buy this target firm, and they have an estimate of the target firm's value based on observable information, but they will not truly know the target firm's value until it is acquired.

Auctions of this type are known as "common value auctions." They are different than "private value auctions" because bidders now have a signal about the items value, but do not know the true value until it is purchased. Making the concept slightly more formal:[4]

1. There is a common value V for the item, which is drawn from some underlying probability distribution with support $(\underline{v}, \underline{v})$.

2. Bidders receive a signal S_i of V prior to bidding. The signal S_i is drawn from some probability distribution $(V - \varepsilon, V + \varepsilon)$. Thus, each bidder's signal is dependent on the common value V, but they all have (potentially) different signals.

Consider an example assuming a 1^{st}-price sealed bid auction. If we assume a symmetric equilibrium, then the bidder with the highest signal will win. However, bidders who win know that they have the highest signal and, the more bidders in the auction or the larger ε is, the more likely that signal is an overestimate of V. Thus, they need to shade their bid not only to receive a surplus (as they would in a private value 1^{st}-price sealed bid auction) but also because they realize their signal, if they win, is likely an overestimate of V. In equilibrium, bidders in a common value auction make positive profit. However, bidders who are not familiar with the common value auction may (likely) end up bidding too much for the item, and ultimately lose money because the pay some price $P > V$. Overbidding in a common value auction and losing money is know as the winner's curse. The phrase "winner's curse" appears in finance journals and you all should be familiar with it. The term really applies to common value auctions, though some individuals use it (likely incorrectly) with private value auctions. In a private value auction individuals know their values, and can avoid the winner's curse by making sure they do not submit bids that would lead to payments greater than their value, whereas in common value auctions bidders will likely be submitting a bid below their signal, but that bid might still be above the common value V.[5]

3.3.1 Wallet Game

Consider that it is the 1990s, when students in class had physical money (cash) in their wallets or purses. The game is as follows: two students are asked to bid in an ascending clock auction for the combined cash contents of the two wallets/purses. The common value V is the sum of the two amounts of cash v_1 and v_2, so $V = v_1 + v_2$. Each student has their own private signal, S_i, and part of that signal is that they know how much money they have themselves. The winner pays the amount on the clock, p, and receives V from the auctioneer and the loser receives a payoff of zero. We want to find a symmetric equilibrium so that both players are using the same strategy. Specifically, we want to find some $b_i(v_i)$ for all i that is an equilibrium.

In the private value auction we know that each bidder i should stop bidding when $p = v_i$. In this common value auction, should both bidders drop out if $p < v_i$, where v_i is the amount they have themselves? To take a more specific example, should they drop out when $p = \frac{v_i}{2}$ or $b_i(v_i) = \frac{v_i}{2}$? I would argue that both players using $b_i = \frac{v_i}{2}$ (and really any symmetric strategy where both players bid below their own v_i) is not an equilibrium because each player knows that the minimum amount of V is their own v_i. If I have \$10 in my wallet, and I choose to drop out at \$5, and the other player is using the same strategy then I know they

[4] See Wolfstetter (1999), pgs. 226-229 for more information.
[5] Wilson (1977) provides a discussion of equilibrium derivation in common value auctions.

have at least \$10 if $p = \$5$ so I know V is at least \$20. Using the strategy $b_i(v_i) = \frac{v_i}{2}$ means that I would only be willing to pay up to \$5 to win \$20, so I should probably adjust my strategy because I am leaving some surplus on the table. If the other player uses $b_j(v_j) = \frac{v_j}{2}$, I can use $b_i(v_i) = v_i$ and be better off. I am not claiming that $b_i(v_i) = v_i$ is a best response to $b_j(v_j) = \frac{v_j}{2}$, just that $b_i(v_i) = v_i$ is a better response to $b_j(v_j) = \frac{v_j}{2}$ than $b_i(v_i) = \frac{v_i}{2}$. As long as we can find some strategy that is better for one player then the proposed set of strategies cannot be an equilibrium.

Should both players use $b_i = v_i$ as they do in the private value ascending auction? Again, if my value is \$10 and I choose to drop out at \$10 and the other player is using the same strategy then I know they have at least \$10 so if $p = \$10$, then I know V is at least \$20. Again, I am only paying \$10 for a prize that I know is at least \$20, so I am leaving some surplus on the table with this strategy.

In both of these two scenarios, if I have \$10 and both players use the same $b_i \leq v_i$ strategy I know that V is at least \$20 if the price reaches my drop out price and my strategy states that I am willing to pay less than \$20 for \$20 when I should be willing to pay \$20 (or at least \$19.99). Well, $\$20 = 2 * \$10$, which is $2v_i$, so perhaps $b_i = 2v_i$ is an equilibrium. Now if my value is \$10 and I choose to drop out at \$20 and the other player is using the same strategy, $b_j(v_j) = 2v_j$, then I know that there is at least \$20 because the other player must have at least \$10 to be willing to bid \$20. I certainly do not want to bid less than $2v_i$ because of the reasons already discussed. Do I want to bid more than $2v_i$? Suppose I use $b_i(v_i) = 3v_i$ when the other player uses $b_j(v_j) = 2v_j$. Suppose the other player has a value of \$11; that player will drop out at \$22 but I will remain in at \$22 and win the item. I now pay \$22 but receive only $\$10 + \$11 = \$21$, meaning I lose one dollar. That is not an improvement when the other player has a value between \$10.01 and \$15 and makes no difference when $v_j \leq \$10$ or $v_j > \$15$. For $b_i(v_i) = \alpha v_i$, the same result occurs for any $\alpha > 2$. The unique symmetric equilibrium is $b_i^*(v_i) = 2v_i$ for $i = 1, 2$. Essentially, all bidders bid as if the other bidder has an amount of money equal to their own.

The wallet game is from Klemperer (1998) and discussed in Wolfstetter (1999). The strategy $b_i^*(v_i) = 2v_i$ is proposed as an equilibrium and then shown to be an equilibrium as follows. If both players use $b_i^*(v_i) = p = 2v_i$, then the actual value is:

$$
\begin{aligned}
V &= v_i + v_j \\
V &= v_i + \frac{1}{2}p
\end{aligned}
$$

It follows that $v_j = \frac{1}{2}p$ because $p = 2v_j$. Now we want to find the p such that bidder i is not losing money:

$$
\begin{aligned}
V - p &\geq 0 \\
V &\geq p \\
v_i + \frac{1}{2}p &\geq p \\
v_i &\geq \frac{1}{2}p \\
2v_i &\geq p
\end{aligned}
$$

As long as $p \leq 2v_i$, bidder i is earning a positive profit; if $p > 2v_i$, then bidder i earns a loss, as shown above in the example. Note that the bidders also avoid the winner's curse.

The analysis to show $b_i^*(v_i) = 2v_i$ is a symmetric equilibrium is fairly straightforward, but neither Klemperer nor Wolfstetter discuss the thought process to arrive at that equilibrium. There is no optimization problem provided to solve through, the equilibrium is just proposed. I went through the thought process (which may or may not be what Klemperer used) to give you an idea of how working from an example might lead to some intuition about what the equilibrium might be.[6]

Klemperer and Wolfstetter also both show that there are asymmetric equilibria (which is why I specifically focused on the symmetric equilibrium) to the two player game where $b_i^*(v_i) = \alpha v_i$ and $b_j^*(v_j) = \frac{\alpha}{\alpha - 1} v_j$ for $\alpha \geq 2$. When $\alpha = 2$ we have the symmetric equilibrium.

[6] I have not checked, but based on the thought process I would guess that if there were k bidders in the wallet game that the symmetric equilibrium is $b_i^*(v_i) = kv_i$ for $k = 1, ..., k$.

3.3.2 Takeover bids

The wallet game may seem to be a bit of a contrived example to provide an introduction to common value auctions, but it does hint at some useful results. Of more relevance, particularly for finance students, would be takeover auctions or mergers. It is possible that takeover auctions have some common value components (which apply to any firm that attempts to acquire the target firm) as well as private value components (which may be specific to which firm acquires the target firm) and we will discuss that possibility shortly. But first a brief, and certainly not comprehensive, discussion of some early work on takeover auctions.

Grossman and Hart (1980a, 1980b, 1981) provide some seminal discussion and formulation of the takeover auction using game theoretic and mechanism design approaches. Grossman and Hart (1980b) analyze how exclusionary devices can be built into the corporate charter to overcome the free-rider problem that occurs among shareholders due to the shareholders having limited power (due to holding a small number of shares) and thus little to no individual incentive to attempt to push managers to act in the shareholders' interest. The corporate charter or firm constitution is where the choice of mechanism can be put in place to avoid the free-rider problem. Grossman and Hart (1980a) examine the role of disclosure laws in takeover bids, and reference their model in 1980b. Grossman and Hart (1981) focus on the case of asymmetric information, where only one bidder has specific information about the value of the target firm. These papers tend to be a response to popular arguments at the time to limit takeover auctions. While there have been many critiques and extensions of these articles (they have been cited about 6,000 times combined), I point out two papers here to compare approaches in criticizing mathematical models. Bebchuk (1989) shows that a key proposition in Grossman and Hart (1980b), "that successful bids must be made at or above the expected value of minority shareholders", does not always hold once the assumption that "the only successful bids are those whose success could have been predicted with certainty" is dropped. Deman (1994) also critiques Grossman and Hart (1980b). A key difference between Bebchuk and Deman is that Bebchuk is much clearer about which key assumption is being changed and how that affects the Grossman and Hart result.

Giammarino and Heinkel (1986) develop a model of takeovers with three players: a target firm, an informed bidder, and an uninformed bidder. A target firm may provide synergies with the acquiring firm; if so, there are two possible synergy levels, G_1 and G_3, with $G_3 > G_1 > 0$. Only the informed bidder can determine whether there are synergies, but once the informed bidder determines that there are synergies and makes a bid both the target firm and uninformed bidder know that there are synergies. The synergy is the same for either firm, but the informed bidder only receives a signal about the synergy. The signal may or may not perfectly reveal the synergy. The key mechanism design aspect is that the target firm should find restricted bidding preferred to unrestricted bidding. In particular, restrictive bidding allows the informed bidder to make its initial (and only) bid, the target firm accepts or rejects the bid, and then the uniformed bidder can choose to make a bid but any bid that firm makes would have to be strictly greater than the bid by the informed bidder. The rationale for this mechanism is that if unrestricted bidding is allowed then the informed bidder will always have an advantage over the uninformed bidder, the uninformed bidder will never bid because they will only win if they overbid (bid as if there is a high synergy when there is actually a low synergy) and that the informed bidder, knowing the uninformed bidder will never bid, will always submit the lowest possible bid. Even though restrictive bidding at times results in the takeover bid failing, the target firm still finds restrictive bidding more profitable. Stein (1988) has a similar model though the focus of that paper is on managerial myopia, which is that managers tend to favor short-term profits over long-term goals.

Eckbo, Malenko, and Thorburn (2020) provide a review of more recent developments about strategic decisions in takeover auctions. They provide a discussion of the basic auction framework (which we will discuss), auctions vs. negotiations, strategic concerns in deal initiation, toehold decisions, jump (preemptive) bidding and markup pricing, and choice of payment method. They also provide some empirical evidence that, depending upon the theoretical model, may be consistent or inconsistent with theoretical predictions. Hirshleifer and Titman (1990) have a section in their paper discussing empirical implications of their model. While they do not test those implications in that paper, that section does provide a framework for future empirical researchers. Gilberto and Varaiya (1989) provide an empirical test of the winner's curse from failed bank auctions. They find that bids decrease when there is more uncertainty and increase when there are more bidders, and those findings are consistent with other studies (primarily laboratory experiments at the time). However, they also find that bidders do not sufficiently adjust their bids to account for potential overvaluing when they receive a high signal, which seems unlikely to occur with experienced bidders, so they advise caution in interpreting their results. Boone and Mulherin (2008) also provide an empirical test of the winner's curse in the corporate takeover market. They find that it is competitive pressure, and not the winner's curse, that leads to breakeven returns. Eckbo (2009) provides a review of empirical findings up to that time.

Eckbo, Malenko, and Thorburn (2020) provide a general framework that allows for private value components and common value components. Assume two bidders, indexed by i, and one target firm. Each bidder knows there own private signal s_i, which is relevant for their private value and possibly for the common value. This structure is similar to the wallet game discussed earlier. Each s_i is an independent random draw from the uniform distribution over $[0, 1]$. In this model, the value of the target firm to bidder i is given by:

$$v\left(s_i, s_{-i}\right) = \alpha s_i + (1 - \alpha) s_{-i}$$

for $\alpha \in \left[\frac{1}{2}, 1\right]$. When $\alpha = 1$ then this model is the independent private value model; when $\alpha = \frac{1}{2}$, the bidder's have the same value for the item so the auction is a common value auction. Note that this structure is slightly different than the general structure we discussed earlier. The general structure had some unknown value V and signals of that value were randomly distributed around that value. In this structure, the signals themselves determine the value, which is similar to the wallet game. The difference is subtle because in the case where the value is not the average of the signals both players may receive a signal that is above or below the true value; when the value is the average of the signals ($\alpha = \frac{1}{2}$), if the signals are not the same, then one bidder must receive a signal that is higher than the true value and the other a signal that is lower than the true value.

For an English (ascending) auction setting, we have seen that if we are in the private value setting ($\alpha = 1$) that $b_i\left(s_i\right) = s_i$ because bidders can do no better than dropping out when the price reaches their value. In this model structure, where the value is a weighted average of the signals, there is still a symmetric equilibrium where both players remain in the auction until the price reaches their signal. The logic is similar to that of the wallet auction – if both bidders are using the strategy $b\left(s_i\right) = s_i$, if the price reaches bidder i's value, then bidder i knows that bidder j has a value of at least s_i.

For a first-price sealed bid auction, let $b(s)$ denote the equilibrium bid of a bidder. Let $b(s')$ be the bidder's strategy for some signal s'. Assume a symmetric monotone increasing equilibrium (so that bidders use the same strategy and bidders with higher signals place higher bids). The bidder's expected payoff is:

$$\begin{gathered}\Pr(win) * Payoff \\ s'\left[\alpha s + (1-\alpha)\frac{s'}{2} - b\left(s'\right)\right]\end{gathered}$$

The probability of winning is s' because the signals are distributed uniformly on the unit interval. The value of the item, conditional on winning, is $\alpha s + (1-\alpha)\frac{s'}{2}$. The term αs is from the bidder's own signal s, and the expectation of the portion of the value determined by the other bidder's signal is $(1-\alpha)\frac{s'}{2}$ because the other bidder's signal will be distributed uniformly between $[0, s']$. The payment made is $b(s')$. Differentiating with respect to s' yields:

$$\begin{aligned}\alpha s + (1-\alpha)\frac{s'}{2} - b\left(s'\right) + s'\left(\frac{1-\alpha}{2} - b'\left(s'\right)\right) &= 0 \\ \alpha s + (1-\alpha)\frac{s'}{2} - b\left(s'\right) + s'\frac{1-\alpha}{2} - s'b'\left(s'\right) &= 0 \\ \alpha s + (1-\alpha)s' - b\left(s'\right) - s'b'\left(s'\right) &= 0\end{aligned}$$

Imposing that $s' = s$ because the maximum must be reached there, we have:

$$\begin{aligned}\alpha s + (1-\alpha)s - b(s) - sb'(s) &= 0 \\ s - b(s) - sb'(s) &= 0 \\ \frac{s - b(s)}{s} &= b'(s)\end{aligned}$$

The initial condition is $b(0) = 0$ because otherwise a bidder with the lowest possible signal would receive a negative payoff, leading to the following solution:

$$b(s) = \frac{s}{2}.$$

Note that bidding one-half of the signal is the same general form of the equilibrium that we had for a first-price private value auction (bidders bid one-half of their value in that case in a two bidder auction). As with the English auction, this result assumes that the true value of the item is the weighted average of the signals.

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