Problem Set 3

BPHD8110-001

Due: March 23, 2023

1. A professional card player is considering playing a game of cards with an unknown player. The unknown card player may be one of two types, a shark (good player) or a fish (bad player). The probability that the unknown player is a shark is $\frac{4}{5}$, while the probability the unknown player is a fish is $\frac{1}{5}$. Both players simultaneously decide whether or not they will play a game of cards with each other. The payoffs are as follows:

Stranger (shark)

		Play	Not play
Professional	Play	10,10	12,6
	Not Play	15,12	5,4

Stranger (fish)

		Play	Not Play
Professional	Play	20,2	10,4
	Not Play	1,12	3,10

Find all pure strategy Bayes-Nash equilibria to this game.

Answer:

The professional has 1 type while the stranger has 2 types (shark and fish), so any BNE will involve the Professional choosing one action and the Stranger choosing one action per player type as a strategy. Suppose the professional chooses Play. The shark type's best response is to choose Play, while the fish type's best response is to choose Not Play. If the shark chooses Play and the fish chooses Not Play, then if the professional chooses Play he receives:

$$\begin{aligned} E[Play] &= \frac{4}{5} * 10 + \frac{1}{5} * 10 \\ E[Play] &= 10 \end{aligned}$$

while his expected value of Not Play is:

$$\begin{aligned} E[Not\ Play] &= \frac{4}{5} * 15 + \frac{1}{5} * 3 \\ E[Not\ Play] &= \frac{63}{5} \end{aligned}$$

So this is NOT a BNE of the game.

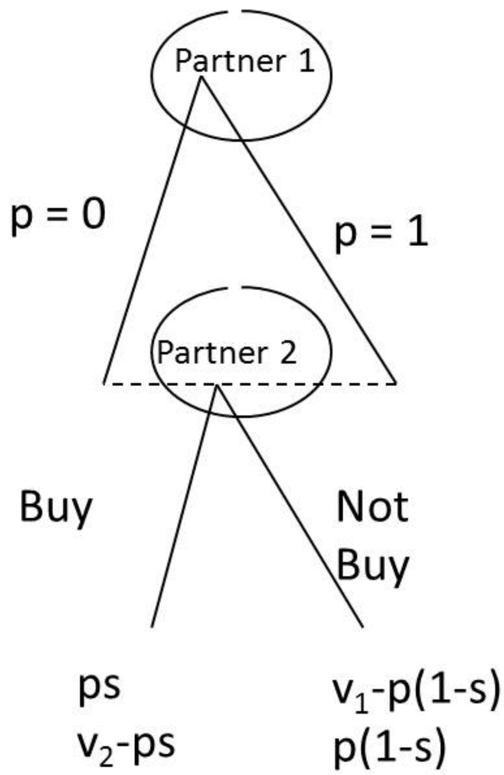
Now suppose the Professional chooses Not Play. The shark type's best response is to choose Play, and the same is true for the fish type's best response. The Professional then has an expected value from choosing Not Play of:

$$\begin{aligned} E[Not\ Play] &= 15 * \frac{4}{5} + 1 * \frac{1}{5} \\ E[Not\ Play] &= \frac{61}{5} \end{aligned}$$

If the Professional chooses Play he receives:

$$\begin{aligned} E[Play] &= 10 * \frac{4}{5} + 20 * \frac{1}{5} \\ E[Play] &= \frac{60}{5} \end{aligned}$$

Because Not Play by the Professional is a best response to the strategy: Play if shark, Play if fish, the BNE to the game is: Professional Not Play, Stranger (shark type) Play, Stranger (fish type) Play.



2. Two partners must dissolve their partnership. Partner 1 currently owns share s of the partnership, partner 2 owns share $1-s$. The partners agree to play the following game: partner 1 names a price, p, for the whole partnership, and partner 2 then chooses either to buy 1's share for ps or to sell her share to 1 for $p(1-s)$. Suppose it is common knowledge that the partners' valuations for owning the whole partnership are independently and uniformly distributed on $[0,1]$, but that each partner's valuation is private information. What is the perfect Bayesian equilibrium?

   Answer:

   This game is fairly similar to the developer-landowner game. I have drawn a tree to represent this game.

   Note that Partner 1 will only make offers between 0 and 1 because he knows that the most that Partner 2's value can be is 1. Partner 2 will choose to Buy Partner 1's share if:

$$\begin{aligned} v_2 - ps &\geq p(1-s) \\ v_2 - ps &\geq p - ps \\ v_2 &\geq p \end{aligned}$$

   Now Partner 1 knows this and wants to choose p to maximize his profit. Note that this choice of p will depend on Partner 1's value, v_1, as well as Partner 1's share, s. Partner 1's expected profit is:

$$\Pi_{Partner1} = ps * \Pr(Partner\ 2\ Buy) + (v_1 - p(1-s)) * \Pr(Partner\ 2\ Not\ Buy)$$

   The probability that Partner 2 chooses to Not Buy is just p, as Partner 2's value is distributed $U[0,1]$.

If $p = \frac{1}{4}$, then $\frac{1}{4}$ of the time $v_2 < p$. The probability that Partner 2 buys is then $(1-p)$. Now:

$$\begin{aligned}
\Pi_{Partner1} &= ps * (1-p) + (v_1 - p(1-s)) * p \\
\Pi_{Partner1} &= ps - p^2 s + (v_1 - p + ps)\, p \\
\Pi_{Partner1} &= ps - p^2 s + v_1 p - p^2 + p^2 s \\
\Pi_{Partner1} &= ps + v_1 p - p^2
\end{aligned}$$

Now just differentiate with respect to p, set the first order condition equal to zero, and solve for p:

$$\begin{aligned}
\frac{\partial \Pi_{Partner1}}{\partial p} &= s + v_1 - 2p \\
0 &= s + v_1 - 2p \\
p &= \frac{s + v_1}{2}
\end{aligned}$$

So the equilibrium to this game is that Partner 1 offers Partner 2 a price of $p = \frac{s+v_1}{2}$ and Partner 2 chooses Buy if $v_2 \geq p$ and Not buy if $v_2 < p$.

3. (First-come, first-serve) Suppose that I symmetric individuals wish to acquire the single remaining ticket to a concert. The ticket office opens at 9 a.m. on Monday. Each individual must decide what time to go to get in line: the first individual to get in line will get the ticket. An individual who waits t hours incurs a (monetary equivalent) disutility of βt. Suppose also that an individual showing up after the first individual can go home immediately and so incurs no waiting cost (there are also no travel costs, so an individual who is not first in line incurs no costs at all). Individual i's value of receiving the ticket is θ_i, and each individual's θ_i is independently drawn from a uniform distribution on $[0, 1]$.

a What is the expected value of the number of hours that the first individual in line will wait?

Answer:

The structure of this problem is similar to a first-price sealed bid auction, only the bidding is done in time, not monetary units. Because all of the assumptions of the Revenue Equivalence Theorem are met, we just need to find the expected value of the second highest valued bidder, as that is how long the first individual would wait. Luckily, $\theta\tilde{\ }U[0,1]$, so we know from our discussion of order statistics that the expected value of the second highest θ is $\frac{I-1}{I+1}$, so that would be the monetary bid. Now, the actual bid is in time, so:

$$\begin{aligned}
\beta E[t] &= E[b] \\
\beta E[t] &= \frac{I-1}{I+1} \\
E[t] &= \frac{I-1}{\beta(I+1)}
\end{aligned}$$

b How does this vary when β doubles?

Answer:

When β doubles, expected time decreases by 50%.

c How does this vary when I doubles?

Answer:

If I doubles then the wait time will increase (more bidders in a first-price sealed bid auction, higher bid). However, consider the increase from 2 to 4 bidders, and then from 4 to 8 bidders.

$$\begin{aligned}
E[t|2\ bidders] &= \frac{1}{\beta 3} \\
E[t|4\ bidders] &= \frac{3}{\beta 5} \\
E[t|8\ bidders] &= \frac{7}{\beta 9}
\end{aligned}$$

To make life easy, let $\beta = 1$. Then the expected wait time increases from $\frac{1}{3}$ to $\frac{3}{5}$ to $\frac{7}{9}$. So while $E[t]$ increases when I doubles, the increase becomes smaller the larger the number of bidders is (going from 100 to 200 to 400 bidders increases wait times from $\frac{99}{101}$ to $\frac{199}{201}$ to $\frac{399}{401}$).

4. Consider a Cournot game of incomplete information. There are 2 firms in this market. Firms face the following inverse demand function, $P(Q) = 194 - Q$, where $Q = q_1 + q_2$. Firms 1 and 2 may have high or low cost and while each firm knows its own cost the other firm only knows the distribution of costs for its competitor. With probability α Firm 1 has total cost $TC_{1L} = 16q_{1L}$, where q_{1L} is the amount Firm 1 produces when it has low cost, and with probability $(1-\alpha)$ Firm 1 has total cost $TC_{1H} = 32q_{1H}$, , where q_{1H} is the amount Firm 1 produces when it has high cost. With probability θ Firm 2 has total cost $TC_{2L} = 24q_{2L}$, where q_{2L} is the amount Firm 2 produces when it has low cost and with probability $(1-\theta)$ Firm 2 has total cost $TC_{2H} = 40q_{2H}$, where q_{2H} is the amount Firm 2 produces when it has high cost. Let $\alpha = \frac{3}{4}$ and $\theta = \frac{1}{2}$. Both firms simultaneously choose a quantity of production in this market. Find a pure-strategy Bayes-Nash equilibrium to this game.

   Answer:

   A Bayes-Nash equilibrium will be a quantity of production for each firm type. We will need to set up the profit functions, find the best response functions, and then solve for the equilibrium quantities.

   The profit function for each type of firm is:

   For Firm 1 with cost TC_{1L} we have:

$$\Pi_{1L} = 178q_{1L} - (q_{1L})^2 - \frac{1}{2}q_{2L}q_{1L} - \frac{1}{2}q_{2H}q_{1L}$$

   For Firm 1 with cost TC_{1H} we have:

$$\Pi_{1H} = 162q_{1H} - (q_{1H})^2 - \frac{1}{2}q_{2L}q_{1H} - \frac{1}{2}q_{2H}q_{1H}$$

   For Firm 2 with cost TC_{2L} we have:

$$\Pi_{2L} = 170q_{2L} - (q_{2L})^2 - \frac{3}{4}q_{1L}q_{2L} - \frac{1}{4}q_{1H}q_{2L}$$

   For Firm 2 with cost TC_{2H} we have:

$$\Pi_{2H} = 154q_{2H} - (q_{2H})^2 - \frac{3}{4}q_{1L}q_{2H} - \frac{1}{4}q_{1H}q_{2H}$$

   To find the best response for each type we take the derivative of the profit function with respect to the choice variable and then set the derivative equal to zero and solve for the choice variable.

   For Firm 1 with cost TC_{1L} we have:

$$\begin{aligned}
\Pi_{1L} &= 178q_{1L} - (q_{1L})^2 - \frac{1}{2}q_{2L}q_{1L} - \frac{1}{2}q_{2H}q_{1L} \\
\frac{\partial \Pi_{1L}}{\partial q_{1L}} &= 178 - 2q_{1L} - \frac{1}{2}q_{2L} - \frac{1}{2}q_{2H} \\
0 &= 178 - 2q_{1L} - \frac{1}{2}q_{2L} - \frac{1}{2}q_{2H} \\
q_{1L} &= \frac{178 - \frac{1}{2}q_{2L} - \frac{1}{2}q_{2H}}{2}
\end{aligned}$$

For Firm 1 with cost TC_{1H} we have:

$$\begin{aligned}
\Pi_{1H} &= 162q_{1H} - (q_{1H})^2 - \frac{1}{2}q_{2L}q_{1H} - \frac{1}{2}q_{2H}q_{1H} \\
\frac{\partial \Pi_{1H}}{\partial q_{1H}} &= 162 - 2q_{1H} - \frac{1}{2}q_{2L} - \frac{1}{2}q_{2H} \\
0 &= 162 - 2q_{1H} - \frac{1}{2}q_{2L} - \frac{1}{2}q_{2H} \\
q_{1H} &= \frac{162 - \frac{1}{2}q_{2L} - \frac{1}{2}q_{2H}}{2}
\end{aligned}$$

For Firm 2 with cost TC_{2L} we have:

$$\begin{aligned}
\Pi_{2L} &= 170q_{2L} - (q_{2L})^2 - \frac{3}{4}q_{1L}q_{2L} - \frac{1}{4}q_{1H}q_{2L} \\
\frac{\partial \Pi_{2L}}{\partial q_{2L}} &= 170 - 2q_{2L} - \frac{3}{4}q_{1L} - \frac{1}{4}q_{1H} \\
0 &= 170 - 2q_{2L} - \frac{3}{4}q_{1L} - \frac{1}{4}q_{1H} \\
q_{2L} &= \frac{170 - \frac{3}{4}q_{1L} - \frac{1}{4}q_{1H}}{2}
\end{aligned}$$

For Firm 2 with cost TC_{2H} we have:

$$\begin{aligned}
\Pi_{2H} &= 154q_{2H} - (q_{2H})^2 - \frac{3}{4}q_{1L}q_{2H} - \frac{1}{4}q_{1H}q_{2H} \\
\frac{\partial \Pi_{2H}}{\partial q_{2H}} &= 154 - 2q_{2H} - \frac{3}{4}q_{1L} - \frac{1}{4}q_{1H} \\
0 &= 154 - 2q_{2H} - \frac{3}{4}q_{1L} - \frac{1}{4}q_{1H} \\
q_{2H} &= \frac{154 - \frac{3}{4}q_{1L} - \frac{1}{4}q_{1H}}{2}
\end{aligned}$$

Technically, all of these should be either the function we found or 0 if the quantity is too low.

To find the Nash equilibrium to this game I will begin by substituting the best response functions for q_{2H} and q_{2L} into the best response functions for q_{1L} and q_{1H}, so that q_{1L} is solely a function of q_{1H} and q_{1H} is solely a function of q_{1L}.

$$\begin{aligned}
q_{1L} &= \frac{178 - \frac{1}{2}\left(\frac{170 - \frac{3}{4}q_{1L} - \frac{1}{4}q_{1H}}{2}\right) - \frac{1}{2}\left(\frac{154 - \frac{3}{4}q_{1L} - \frac{1}{4}q_{1H}}{2}\right)}{2} \\
2q_{1L} &= 178 - \frac{1}{2}\left(\frac{170 - \frac{3}{4}q_{1L} - \frac{1}{4}q_{1H}}{2}\right) - \frac{1}{2}\left(\frac{154 - \frac{3}{4}q_{1L} - \frac{1}{4}q_{1H}}{2}\right) \\
8q_{1L} &= 712 - 170 + \frac{3}{4}q_{1L} + \frac{1}{4}q_{1H} - 154 + \frac{3}{4}q_{1L} + \frac{1}{4}q_{1H} \\
8q_{1L} &= 388 + \frac{3}{2}q_{1L} + \frac{1}{2}q_{1H} \\
16q_{1L} &= 776 + 3q_{1L} + q_{1H} \\
13q_{1L} &= 776 + q_{1H}
\end{aligned}$$

For q_{1H} we have:

$$\begin{aligned}
q_{1H} &= \frac{162 - \frac{1}{2}\left(\frac{170-\frac{3}{4}q_{1L}-\frac{1}{4}q_{1H}}{2}\right) - \frac{1}{2}\left(\frac{154-\frac{3}{4}q_{1L}-\frac{1}{4}q_{1H}}{2}\right)}{2} \\
2q_{1H} &= 162 - \frac{1}{2}\left(\frac{170-\frac{3}{4}q_{1L}-\frac{1}{4}q_{1H}}{2}\right) - \frac{1}{2}\left(\frac{154-\frac{3}{4}q_{1L}-\frac{1}{4}q_{1H}}{2}\right) \\
8q_{1H} &= 648 - 170 + \frac{3}{4}q_{1L} + \frac{1}{4}q_{1H} - 154 + \frac{3}{4}q_{1L} + \frac{1}{4}q_{1H} \\
8q_{1H} &= 324 + \frac{3}{2}q_{1L} + \frac{1}{2}q_{1H} \\
16q_{1H} &= 648 + 3q_{1L} + q_{1H} \\
15q_{1H} &= 648 + 3q_{1L} \\
q_{1H} &= \frac{648 + 3q_{1L}}{15}
\end{aligned}$$

Now, substituting in for q_{1H} we have:

$$\begin{aligned}
13q_{1L} &= 776 + \frac{648 + 3q_{1L}}{15} \\
195q_{1L} &= 11640 + 648 + 3q_{1L} \\
192q_{1L} &= 12288 \\
q_{1L} &= 64
\end{aligned}$$

Now that we have $q_{1L} = 64$, we know $q_{1H} = 56$ because $q_{1H} = \frac{648+3q_{1L}}{15}$. Now that we have q_{1H} and q_{1L}, we know that:

$$\begin{aligned}
q_{2L} &= \frac{170 - \frac{3}{4}q_{1L} - \frac{1}{4}q_{1H}}{2} \\
q_{2L} &= \frac{170 - \frac{3}{4} * 64 - \frac{1}{4} * 56}{2} \\
q_{2L} &= \frac{170 - 48 - 14}{2} \\
q_{2L} &= 54
\end{aligned}$$

and:

$$\begin{aligned}
q_{2H} &= \frac{154 - \frac{3}{4}q_{1L} - \frac{1}{4}q_{1H}}{2} \\
q_{2H} &= \frac{154 - \frac{3}{4} * 64 - \frac{1}{4} * 56}{2} \\
q_{2H} &= \frac{154 - 48 - 14}{2} \\
q_{2H} &= 46
\end{aligned}$$

So the Nash equilibrium to this game is: $q_{1L} = 64$, $q_{1H} = 56$, $q_{2L} = 54$, and $q_{2H} = 46$.

You can use matrices to solve for these quantities – either way gets the same answer.

5. Consider a simultaneous Bertrand pricing game with two firms, I and J. Each firm's demand, conditional on their price relative to the other firm, is given as follows:

if	q_i	q_j
$p_i, p_j > r$	0	0
$p_i > r \geq p_j$	0	M
$r \geq p_i > p_j$	0	M
$r \geq p_i = p_j$	$\frac{M}{2}$	$\frac{M}{2}$

This firm demand function tells us that if one firm prices below the other firm and below some level $r > 0$, then the firm with the lower price captures the entire market and sells the quantity $M > 0$. If the firms both choose the same price and it is less than or equal to r then the firms each sell one half of the market ($\frac{M}{2}$). Note that r and M are fixed amounts that are related in the following way – if the market price is less than or equal to r, then consumers will purchase M units of the good. If the market price is greater than r, then the consumers will purchase 0 units. Note that the price space is continuous, so $p_i, p_j \in [0, \infty)$. Figure 1 shows the demand function and the relationship between r, c_H, and c_L, with the thick solid line representing market demand.

a Suppose that both firms have constant average and marginal cost equal to $c > 0$. Also assume all firms that there is complete information about this cost. What is the pure strategy Nash equilibrium of this game?

Answer:

This game is simply a standard Bertrand pricing game so the PSNE of this game is for both firms to choose $p_i = p_j = c$.

b Now consider the case when Firm I has cost c_L and Firm J has cost c_H, with $c_L < c_H < r$. Again assume complete information about these costs. What is the pure strategy Nash equilibrium of this game?

Answer:

The PSNE for this game is also fairly straightforward. The lowest price that Firm J can charge and make nonnegative profit is $p_j = c_H$. At this price Firm J earns zero profit. However, Firm I can price slightly below Firm J and capture the entire market. So the equilibrium to this game is that Firm J chooses $p_j = c_H$ and Firm I chooses $p_i = c_H - \varepsilon$, where ε is some small positive number.

c Now consider the case where the cost for each firm may take one of two values $\{c_L, c_H\}$ where $r > c_H > c_L$ (so there is incomplete information about cost for both firms). Cost c_L occurs with probability ρ and cost c_H occurs with probability $1 - \rho$. The equilibrium to this game involves one type playing a pure strategy and the other type playing a mixed strategy.

- Which type plays a pure strategy and what is that pure strategy?

Answer:

The type that plays a pure strategy is any firm that receives a cost draw of c_H and any firm with cost c_H chooses $p_k = c_H$. The way to think about this is to use parts **a** and **b**. Assume that Firm J draws cost c_H. If Firm I draws cost c_L then Firm J will never produce in this market because it will be in Firm I's best interest to never price above c_H. So Firm J is unconcerned about what Firm I does when I has a lower cost draw. What if Firm I also draws c_H? Then the two firms have the same cost and end up playing the equilibrium to the standard Bertrand pricing game which leads to $p_i = p_j = p_H$ when both firms have cost c_H.

To see that a pure strategy price choice $p_j \in (c_H, r]$ will not work, consider that if Firm J chooses r when it has p_H then Firm I will choose $r - \varepsilon$ when it has p_H. This leads to both firms undercutting the other until they reach $p_i = p_j = c_H$.

What about a mixed strategy randomizing between $[c_H, r]$? That might work if there were only high cost firms, but not with the possibility of low cost firms. To see this we need to know what the low cost firm's equilibrium strategy is.

- The type that plays a mixed strategy chooses its price randomly from a uniform distribution over a particular interval. What is that interval?

Answer:

The low cost firm chooses to randomize by choosing $p_i \in [\bar{c}, c_H]$, where $\bar{c}$ is the expected cost of the opposing firm, or $\bar{c} = \rho c_L + (1 - \rho) c_H$. Why randomize over this interval rather than $[c_L, c_H]$? For

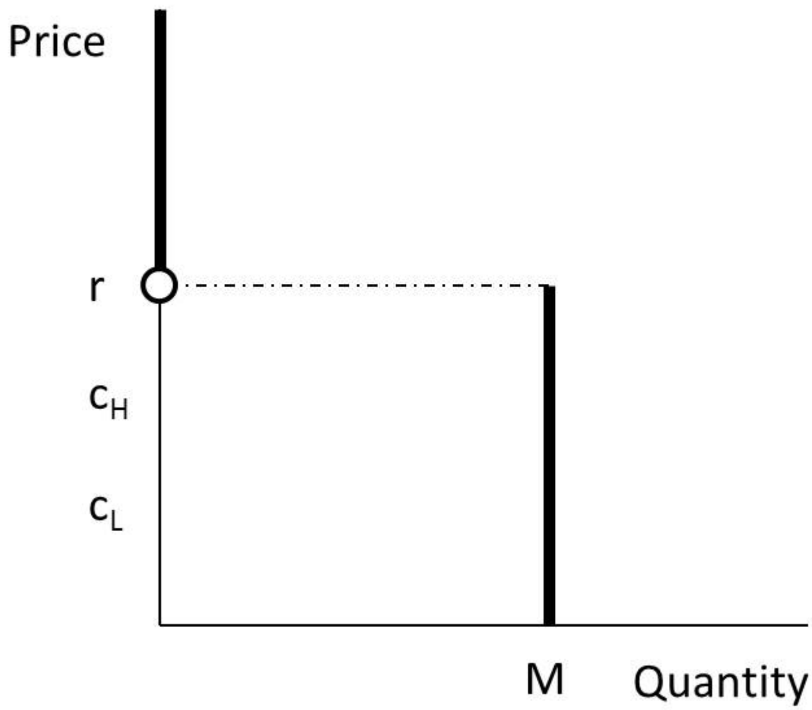


Figure 1: Demand function for Bertrand game

opponent prices below $\bar{c}$ the best response is NOT to undercut price but to increase price up to c_H (or $c_H - \varepsilon$). The idea being that capturing the entire market by undercutting the other firm when it is the low cost type is not the best response if the low cost type is pricing too low, because now you are also pricing low even if it is the high cost type. An example might make this clearer.

Let $\rho = \frac{1}{2}$, $c_H = 20$, $c_L = 10$, and $M = 100$. If all high cost types are pricing at $c_H = 20$, and the other player uses $p_L = 12$ when it is the low cost, your best response is NOT to choose $p_L = 12 - \varepsilon$, but to choose $p_L = 20 - \varepsilon$. Why? If you choose $p_L = 12 - \varepsilon$, you get (as ε goes to zero) $100 * (12 - 10) = 200$ because you always have the lowest price. But if you were to choose $p_L = 20 - \varepsilon$, you get (again as ε goes to zero) $100 * (20 - 10) * \frac{1}{2} = 500$. You only end up having the lowest price against the high cost firm, but you have a really high price against that firm, so while you only produce half of the time you earn more money than you do by lowering your price to 11.99 and capturing the entire market.

Now, if you choose a price of $20 - \varepsilon$ when the other firm chooses a price of 12 this is also not an equilibrium because the other firm will the raise price to $20 - \varepsilon - \varepsilon$. And then the firms start undercutting again until they get to $\bar{c}$. And then they cycle back up to $20 - \varepsilon$, so there is no pure strategy equilibrium for the firms with low cost.