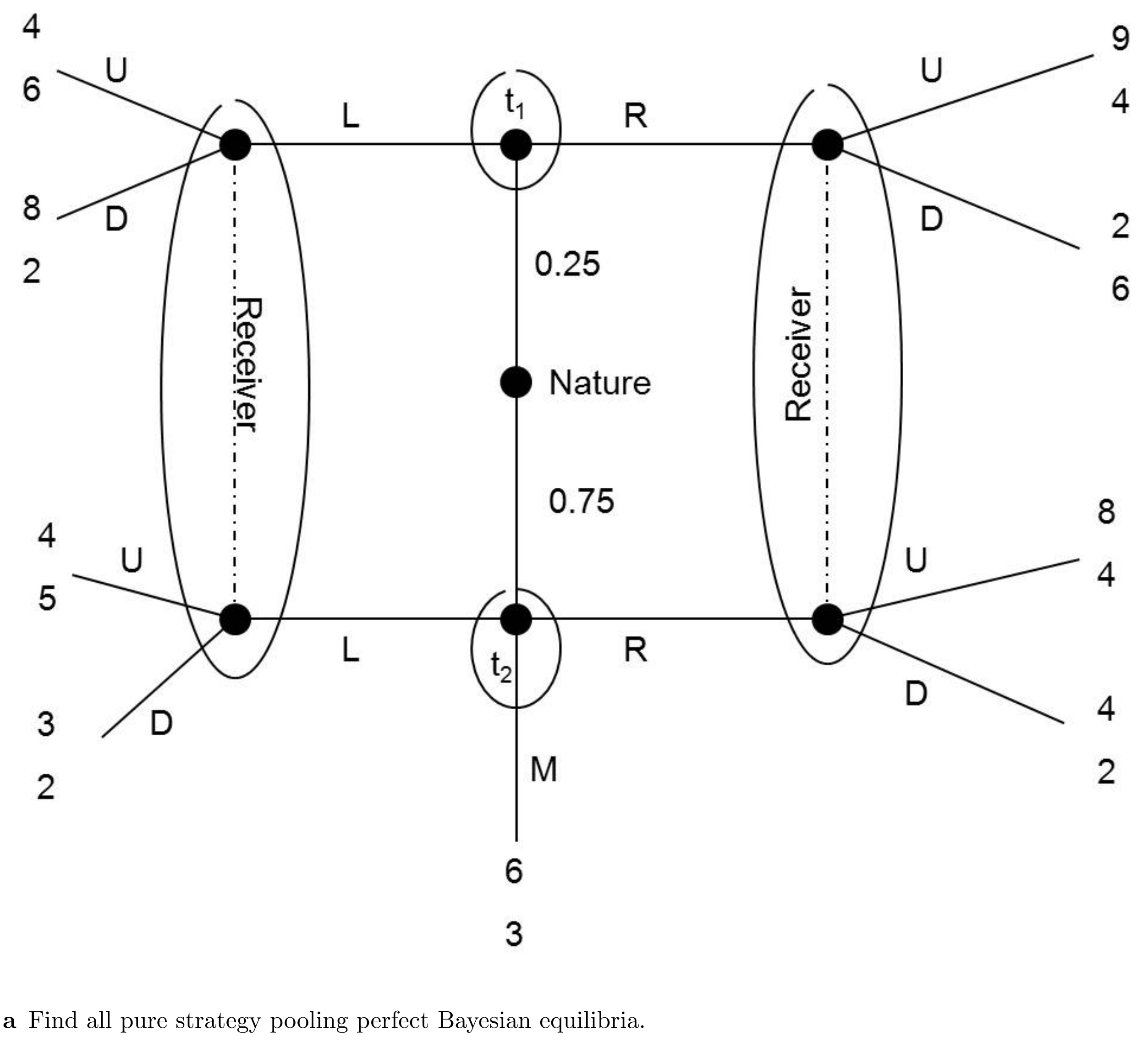
Problem Set 4 Answers

BPHD8110-001

Due: April 13

1. Consider the following Sender-Receiver game that has been slightly modified as type t_2 now has a third option, M, which ends the game without allowing the Receiver a chance to make a decision. Note that the probability of being a sender type t_1 is $\frac{1}{4}$ and the probability of being a sender type t_2 is $\frac{3}{4}$.



a Find all pure strategy pooling perfect Bayesian equilibria.

Answer:

There are two potential pooling equilibria, t_1 and t_2 choose L or t_1 and t_2 choose R. Hopefully the first thing you noticed is that t_2 will NEVER choose L because t_2 is always guaranteed to be better off by choosing M. Thus, the only pooling equilibrium that needs to be considered is where both t_1 and t_2 choose R.

If t_1 and t_2 choose R then the Receiver believes:

$$\begin{aligned}\Pr(t_1|R) &= 0.25\\ \Pr(t_2|R) &= 0.75\\ \Pr(t_1|L) &= q\\ \Pr(t_2|L) &= 1-q\end{aligned}$$

If R is observed the Receiver's expected value from U is:

$$E[U|R] = \frac{1}{4} * 4 + \frac{3}{4} * 4 = \frac{16}{4}$$

The Receiver's expected value from D is:

$$E[D|R] = \frac{1}{4} * 6 + \frac{3}{4} * 2 = \frac{12}{4}$$

So the Receiver will choose U if R is observed because $\frac{16}{4} > \frac{12}{4}$.

Now, when will the Receiver choose U if L is observed (there are two ways to think about this – we will use the q and $1-q$ first).

$$\begin{aligned}E[U|L] &= 6q + 5(1-q) = 6q + 5 - 5q = q + 5\\ E[D|L] &= 2q + 2(1-q) = 2\end{aligned}$$

As long as $q + 5 > 2$ or $q > -3$ the Receiver will choose U if L. This should make sense because if the Receiver chooses D if L he gets 2 regardless of which type chooses L, and if he chooses U he gets at least 5 regardless of which type chooses L. So the Receiver will always choose U if L.

Now, will either t_1 or t_2 switch? Type t_1 receives 9 from choosing R and would receive 4 from switching to L so t_1 will not switch. Type t_2 receives 8 from choosing R, would receive 6 from switching to M, and would receive 4 from switching to L, so t_2 will not switch. Thus we have a pure strategy pooling equilibrium.

$$\left.\begin{matrix} t_1 \text{ choose } R \\ t_2 \text{ choose } R \end{matrix}\right\} \text{Sender's strategy}$$

$$\left.\begin{matrix} \Pr(t_1|R) = 0.25 \\ \Pr(t_2|R) = 0.75 \\ \Pr(t_1|L) = q \in [0,1] \\ \Pr(t_2|L) = 1-q \end{matrix}\right\} \text{Receiver's beliefs}$$

$$\left.\begin{matrix} U \text{ if } R \\ U \text{ if } L \end{matrix}\right\} \text{Receiver's strategy}$$

Note that this is a WPBE. If we wanted a strong pooling PBE then we would have $\Pr(t_1|L) = 1$ and $\Pr(t_2|L) = 0$ because the Receiver should know that type t_2 will never choose L.

b Find all pure strategy separating perfect Bayesian equilibria.

Answer:

There are four potential separating equilibria here:

(1) t_1 choose L, t_2 choose R

(2) t_1 choose L, t_2 choose M

(3) t_1 choose R, t_2 choose L

(4) t_1 choose R, t_2 choose M

We can rule out (3) because t_2 will never choose L.

Also, recall from part a that the Receiver will always choose U if L regardless of who chooses L.

Starting with (1), if t_1 chooses L the Receiver chooses U and if t_2 chooses R the Receiver will also choose U. Now, t_1 will switch to R because $9 > 4$ so this is not an equilibrium.

Jumping to (4), if t_1 chooses R the Receiver chooses D and even though L is not chosen by any Sender we know the Receiver will chooses U if L is observed. However, this means that the t_1 type will switch to L because t_1 will receive 4 from choosing L and only receives 2 from choosing R. So (4) is not a separating equilibrium.

That leaves (2).

We know that if t_1 choose L the Receiver will choose U. All that we need to do now is find the probabilities that lead to the Receiver choosing U or D when R is chosen because the Receiver never observes R in equilibrium (2). But if we think carefully, if the Receiver chooses U if R then both t_1 and t_2 will switch to R because both Sender types' highest payoffs are when they choose R and the Receiver chooses U. This just leaves us to find the beliefs that the Receiver uses to choose D if R.

$$
\begin{aligned}
E[U|R] &= 4q + 4(1-q) = 4 \\
E[D|R] &= 6q + 2(1-q) = 6q + 2 - 2q = 4q + 2 \\
& \quad \text{so} \\
E[D|R] &\geq E[U|R] \\
4q + 2 &\geq 4 \\
q &\geq \frac{1}{2}
\end{aligned}
$$

So the only separating equilibrium occurs when:

$$
\left.\begin{array}{c} t_1 \text{ choose } L \\ t_2 \text{ choose } M \end{array}\right\} \text{Sender's strategy}
$$

$$
\left.\begin{array}{c} \Pr(t_1|R) = q \geq \frac{1}{2} \\ \Pr(t_2|R) = 1-q \\ \Pr(t_1|L) = 1 \\ \Pr(t_2|L) = 0 \end{array}\right\} \text{Receiver's beliefs}
$$

$$
\left.\begin{array}{c} D \text{ if } R \\ U \text{ if } L \end{array}\right\} \text{Receiver's strategy}
$$

2. An individual named B.B. invented a device for monitoring the effort level of employees. This device takes precise measurements of the effort of employees and the Dept. of Justice has certified that its measurements are admissible and valid in court proceedings. B.B. now has a problem: how does he price his wonderful new invention?

   Assume that he is trying to sell lit to one particular person named Xavier (X). X has risk neutral preferences and wishes to contract with an Agent, A, to have A sell some books for him. A is risk averse, has utility function of mu$(w, e) = \sqrt{w} - e$ and can choose either $e_h > e_l$, or not work at all and receive reservation utility of $\overline{u}$. There are 3 possible outcomes for A's efforts. She can either sell many books, a few books, or no books (x_m, x_f, x_0). If e_h is chosen then these states occur with probabilities $(0.75, 0.20, 0.05)$. If e_l is chosen then these states occur with probabilities $(0.20, 0.30, 0.50)$.

   Let $x_m = 600$, $x_f = 200$, $x_0 = 0$, $e_h = 5$, $e_l = 0$, and $\overline{u} = 15$.

   a What contract would X offer to A if X could monitor A perfectly? In other words, what wage would X offer when observing e_h and when observing e_l? Also note the profit X receives in this case.

Answer:

First, what is e_l worth to X? If the agent provides low effort then X gets:

$$0.2 * 600 + 0.3 * 200 + 0.5 * 0 = 180$$

So the expected value of low effort to X is \$180. Now, if X offers to pay A the entire surplus of 180, then A's utility is:

$$\sqrt{180} - 0 \approx 13.42.$$

Note that this is below A's reservation utility of 15, so there is no contract offered for e_l because it is not profitable. Alternatively, we could look at the wage needed to induce the agent to exert low effort:

$$\begin{aligned} \sqrt{w} - e_l &\geq \overline{u} \\ \sqrt{w} - 0 &\geq 15 \\ w &\geq 225 \end{aligned}$$

The wage the agent would need to be offered is \$225, which results in a loss of \$45 to X. It is just a different way to show that X will not pay for low effort.

For e_h, it is worth:

$$0.75 * 600 + 0.2 * 200 + 0.05 * 0 = 490$$

to X. The minimum offer X must make to A is 400, because X needs to get A to participate:

$$\begin{aligned} \sqrt{w} - e &\geq \overline{u} \\ \sqrt{w} - 5 &\geq 15 \\ \sqrt{w} &\geq 20 \\ w &\geq 400 \end{aligned}$$

The profit to A in this case is 90. Thus, there is no contract offer for e_l, and a wage of 400 for e_h, with a profit of 90 to X when A uses e_h.

There is no need for an incentive compatibility constraint because effort is observable.

b In the imperfect information case, X must offer a wage based on the outcomes observed (x_m, x_f, x_0). Thus X's problem, in order to achieve high effort from the employee, is:

$$\min \left(0.75w_m + 0.2w_f + 0.05w_m\right)$$

What are the incentive compatibility and participation constraints needed for this problem?

Answer:

We know, from part 1, that A will want X to participate and exert high effort. We need the expected value of the wage contract, which specifies a wage based on each outcome of books that could occur, high enough so that the agent participates and exerts low effort. So the relevant participation constraint is:

$$0.75 * \sqrt{w_m} + 0.2 * \sqrt{w_f} + 0.05 * \sqrt{w_0} - 5 \geq 15.$$

We know that in order for X to want to exert high effort over low effort the following incentive compatibility constraint will need to be satisfied:

$$0.75 * \sqrt{w_m} + 0.2 * \sqrt{w_f} + 0.05 * \sqrt{w_0} - 5 \geq 0.2 * \sqrt{w_m} + 0.3 * \sqrt{w_f} + 0.5 * \sqrt{w_0}.$$

The left-hand side of this equation is the same as the left-hand side of the participation constraint because X wants the agent to participate and use high effort, but the right-hand side is X's expected outcome if the agent chooses low effort based on the wage schedule.

c Because B.B. is very good at solving optimization problems, once you have set it up he tells you that the solution to this one is $w_m = 433.22$, $w_f = 378.11$, $w_0 = 100.14$. Verify that he is right.

Answer:

For this we just need to make sure that (1) X makes a profit, (2) A's participation constraint is satisfied, and (3) A's incentive compatibility constraint is satisfied.

For X's profit, he gets 490 if the agent exerts high effort (this is from part 1), and his cost is:

$$0.75 * 433.22 + 0.2 * 378.11 + 0.05 * 100.14 = 405.54$$

because it is just the product of the probability X pays each wage with the specific wage for that outcome. So X's profit is $490 - 405.54 = 84.46 > 0$.

For A we need to have the participation constraint satisfied:

$$\begin{aligned} 0.75 * \sqrt{w_m} + 0.2 * \sqrt{w_f} + 0.05 * \sqrt{w_0} - 5 &\geq 15 \\ 0.75 * \sqrt{w_m} + 0.2 * \sqrt{w_f} + 0.05 * \sqrt{w_0} &\geq 20 \\ 0.75 * \sqrt{433.22} + 0.2 * \sqrt{378.11} + 0.05 * \sqrt{100.14} &= 20 \\ \text{so } 20 &\geq 20 \end{aligned}$$

and that constraint is satisfied. And now the incentive compatibility constraint satisfied:

$$\begin{aligned} \text{with } e_h &: \quad 0.75 * \sqrt{433.22} + 0.2 * \sqrt{378.11} + 0.05 * \sqrt{100.14} - 5 = 15 \\ \text{with } e_l &: \quad 0.2 * \sqrt{433.22} + 0.3 * \sqrt{378.11} + 0.5 * \sqrt{100.14} = 15 \\ \text{so } 15 &\geq \quad 15 \end{aligned}$$

and the incentive compatibility constraint is also satisfied.

d Now B.B. knows everything he needs to know in order to set his price for his invention. What is the maximum amount B.B can charge X for the use of his wonderful new device? Explain.

Answer:

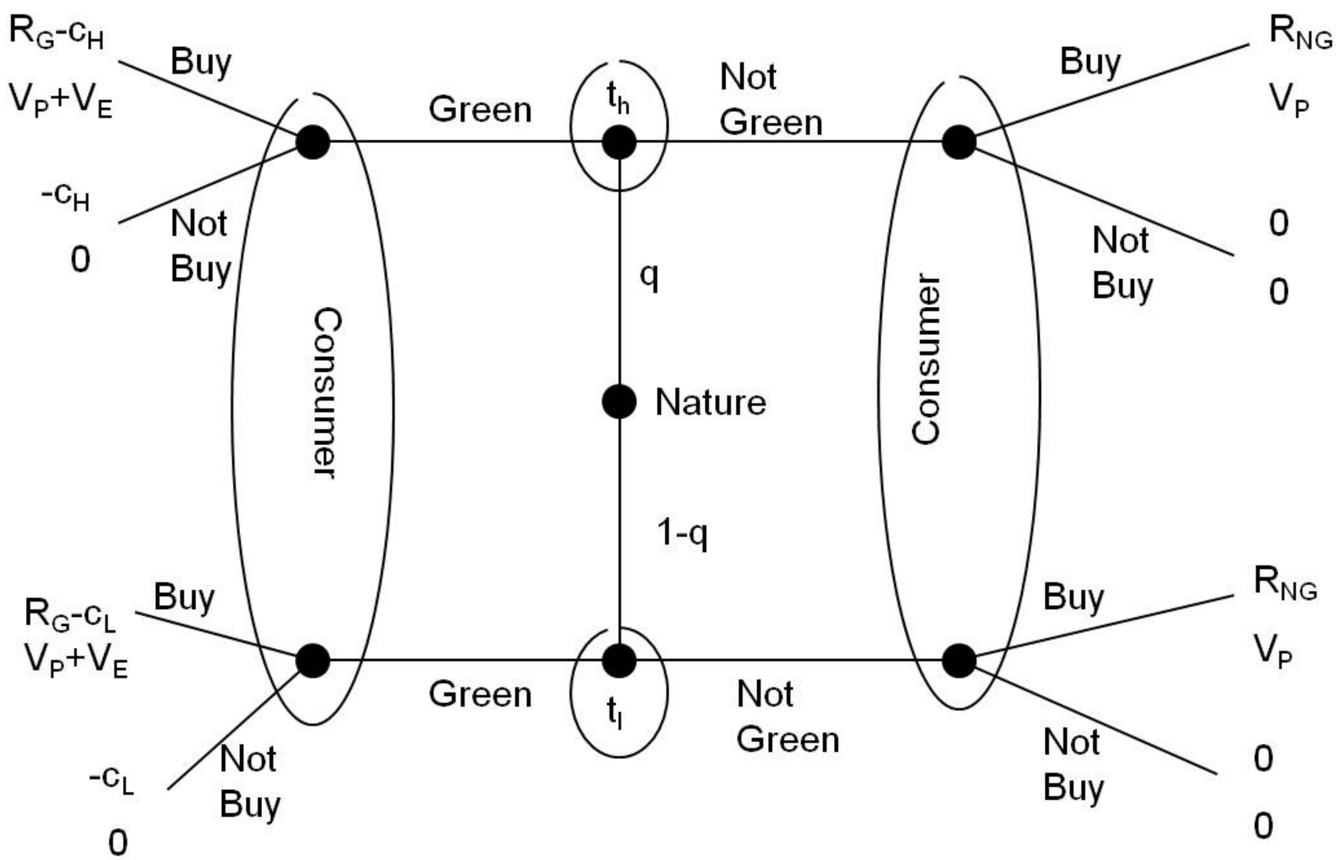
When effort is unobservable X faces a situation of imperfect information (parts **b** and **c**). B.B's device turns imperfect information into perfect information (part **a**). We know that the principal will always be at least as well off under perfect information as under imperfect information otherwise the principal could offer the imperfect information contract, so we should expect profit in part **a** to be higher than that in parts **b**/**c**. In part **a**, under perfect information X makes a profit of 90 because high effort from the agent is worth 490 to X but X pays the agent 400. Under imperfect information X makes an expected profit of 84.46 based on the wages X pays conditional on the outcome. If X earns a profit of 90 in the perfect information case and 84.46 in the imperfect information case, the most B.B. can charge is 5.54, otherwise X wouldn't pay for the device.

3. Consider a game between a firm and a consumer. The firm may be one of two types, high cost (t_h) or low cost (t_l), where the cost of the firm switching to "green" (or environmentally friendly) production. The firm knows its type and can choose to either switch to using green production or not. The consumer does not observe the firm's type, only the production decision made by the firm. The consumer initially believes that with probability q the firm is a high cost type and with probability $1 - q$ the firm is a low cost type, where $0 \leq q \leq 1$. The consumer's decision is whether or not to buy the product. The consumer has a value of V_P for the product, regardless of whether or not the firm uses green technology. If the firm uses green technology, then the consumer receives additional utility V_E due to the fact that the consumer gets some utility from having purchased from a green producer (so the consumer's total utility is $V_P + V_E$ in this instance). If the consumer chooses not to buy then the consumer receives a value of 0.

   If the firm does not switch to green production then the firm receives a profit of R_{NG} if the consumer purchases (regardless of the firm's type) and 0 if the consumer does not. If the firm makes a claim about using green production then the firm receives a profit of $R_G - c_L$ if the firm is a low cost type and the consumer buys, $R_G - c_H$ if the firm is a high cost type and the consumer buys, $-c_L$ if the firm is a low cost type and does not buy, and $-c_H$ if the firm is a high cost type and the consumer does not buy.

a Draw a game tree of this dynamic game of incomplete information.

Answer:



b Find the restrictions on the parameters such that there is a separating perfect Bayesian equilibrium where the low cost types choose green production and the high cost types do not, while the consumers choose to buy regardless of type.

Answer:

First, specify what the separating equilibrium would look like. The equilibrium would be:

$$
\begin{aligned}
& & \text{Type } t_h \text{ choose Not Green} \\
& & \text{Type } t_l \text{ choose Green} \\
\Pr(t_h|Green) &=& 0 \\
\Pr(t_l|Green) &=& 1 \\
\Pr(t_h|Not\ Green) &=& 1 \\
\Pr(t_l|Not\ Green) &=& 0 \\
& & \text{Consumer buys if Green} \\
& & \text{Consumer buys if Not Green}
\end{aligned}
$$

In order for the consumer to buy regardless we need $V_P + V_E \geq 0$ as well as $V_P \geq 0$. The beliefs follow from the strategies of the firm. In order for type t_l to choose Green instead of Not Green, we need $R_G - c_L \geq R_{NG}$. In order for type t_h to choose Not Green instead of Green, we need $R_{NG} \geq R_G - c_H$.

Thus, as long as the cost of producing using green technology is low enough the low cost firm will use green technology, and if it is too high then the high cost firm will not.

c Find the restrictions on the parameters such that there is a pooling perfect Bayesian equilibrium where all types choose green production and where the consumers choose to buy regardless of type.

Answer:

First, specify what the pooling equilibrium would look like. The equilibrium would be:

$$\begin{aligned} & & \text{Type } t_h \text{ choose Green} \\ & & \text{Type } t_l \text{ choose Green} \\ \Pr(t_h|Green) &=& q \\ \Pr(t_l|Green) &=& 1-q \\ \Pr(t_h|Not\ Green) &=& p \in [0,1] \\ \Pr(t_l|Not\ Green) &=& 1-p \\ & & \text{Consumer buys if Green} \\ & & \text{Consumer buys if Not Green} \end{aligned}$$

First, the reason I have specified that $\Pr(t_h|Not\ Green)$ can be any number between 0 and 1 is because the consumer will always buy, regardless of which type of firm produces, as long as $V_P \geq 0$. Thus it doesn't matter what beliefs the consumer has about which firm has chosen Not Green if the consumer happens to see a Not Green decision. Also, as long as $V_P + V_E \geq 0$ the consumer will buy if a Green decision is made. Since the consumer is always buying, in order for type t_h to choose Green we need $R_G - c_H \geq R_{NG}$. In order for type t_l to choose Green we need $R_G - c_L \geq R_{NG}$. As you can see, the separating and pooling equilibria depend upon the cost of switching to Green production.

4. Suppose that members of Congress believe that the US legal system needs to be reformed to reduce expenditures on legal costs. Currently both parties pay their own legal costs (call this the Current System). Congress proposes that the loser of the lawsuit pays the winner an amount equal to the loser's costs, so that the loser would have to pay double his costs (call this the Proposed System). The thought is that if the cost to the loser increases, the loser will think twice about going to court because it will cost him more money.

   Let us set up the problem as follows. Assume that each party in the lawsuit has a privately known value of winning the lawsuit relative to losing, and that this value is independently drawn from a common probability distribution over the range $[\underline{v}, \overline{v}]$. Also assume that parties simultaneously and independently decide how much to spend on legal expenses and that whoever spends the most will win the lawsuit, and that the parties are risk-neutral.

   Would we expect the Proposed System to reduce legal expenses relative to the Current System? Clearly explain why, citing specific reasons. **Note**: You should not need a lot of formal mathematics here, nor should you need to use much opinion.

   Answer:

   First of all, just note that while the loser might spend less, the winner will likely spend more, because he gets part of his costs returned to him. So one cannot say that because the loser's costs will increase that parties to the lawsuit will be less likely to go to court (or spend money) in the Proposed System (because they could be the winner).

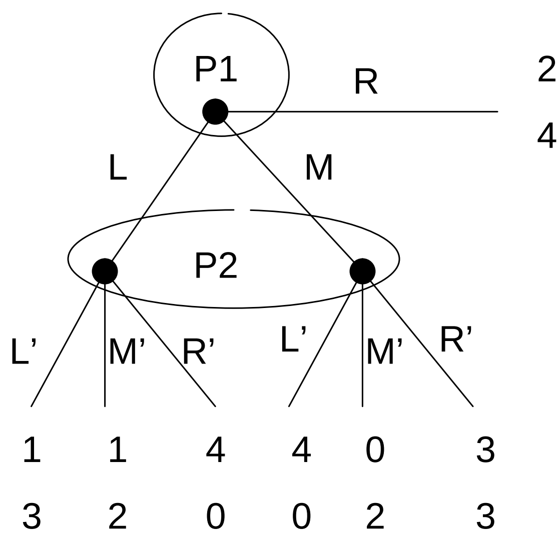
   We have two "mechanisms" if you will, the Proposed System and the Current System. In these mechanisms we have an environment where values are drawn from the same distribution (symmetric), they are independently drawn (independent), and they are private (each player only observes his own value draw). Also, the players are risk neutral. So we are in the SIPV-RN environment, which is a first step.

We are told that whoever spends the most will win the lawsuit. That is a second step.

The last thing to think about is how much a player who draws $\underline{v}$ will spend. In the Current System, he pays an amount equal to his expenditure. In the Proposed System, he pays twice his expenditure. In either system, the bidder who draws $\underline{v}$ will lose (in equilibrium), and will end up with a negative payoff if he spends anything at all. He can guarantee himself a zero surplus if he spends zero.

Thus, we have two mechanisms. In both mechanisms we have the SIPV-RN environment. Both mechanisms are efficient (by assumption). In both mechanisms the loser expects a surplus of zero. Thus, the revenue equivalence theorem holds. Note that revenue in this case is expenditure by both parties of the lawsuit.

5. Consider the following game:



a Write down the normal form (or matrix) for this game.

Answer:

The normal form of this game is:

		Player 2 L'	M'	R'
Player 1	L	1, 3	1, 2	4, 0
	M	4, 0	0, 2	3, 3
	R	2, 4	2, 4	2, 4

b Find all pure strategy Nash equilibria (PSNE), subgame perfect Nash equilibria (SPNE), and perfect Bayesian equilibria (PBE) in this game.

Answer:

First, note that there is only one subgame to this game (the entire game), so that all the PSNE and SPNE will be the same. Using the matrix we see:

		Player 2 L'	M'	R'
Player 1	L	$1,\boxed{3}$	$1,2$	$\boxed{4},0$
	M	$\boxed{4},0$	$0,2$	$3,\boxed{3}$
	R	$2,\boxed{4}$	$\boxed{2},\boxed{4}$	$2,\boxed{4}$

there is only one PSNE (and so only one SPNE as well), which is for Player 1 to choose R and Player 2 to choose M'. Also, Player 1 choose R and Player 2 choose M' is the ONLY PSNE to this game.

So it will be the only PBE as well, only we need to find the beliefs (probabilities) such that Player 2 would choose M'.

To find the beliefs, let q be Player 2's belief that Player 1 chooses L and $(1-q)$ be Player 2's belief that Player 1 chooses M. Player 2's expected value of each strategy is:

$$\begin{aligned} E[L'] &= 3*q+0*(1-q)=3q \\ E[M'] &= 2*q+2*(1-q)=2 \\ E[R'] &= 0*q+3*(1-q)=3-3q \end{aligned}$$

We can tell that Player 2 will choose L' if:

$$\begin{aligned} 3q &> 2 \\ 3q &> 3-3q \end{aligned}$$

or

$$\begin{aligned} q &> \frac{2}{3} \\ q &> \frac{1}{2} \end{aligned}$$

We need both to be true, so if $q > \frac{2}{3}$ Player 2 will choose L'. Player 2 will choose M' if:

$$\begin{aligned} 2 &> 3q \\ 2 &> 3-3q \end{aligned}$$

or

$$\begin{aligned} q &< \frac{2}{3} \\ q &> \frac{1}{3} \end{aligned}$$

So if $q \in \left(\frac{1}{3}, \frac{2}{3}\right)$ Player 2 will choose M'. Finally, Player 2 will choose R' if:

$$\begin{aligned} 3-3q &> 3q \\ 3-3q &> 2 \end{aligned}$$

or

$$\begin{aligned} q &< \frac{1}{2} \\ q &< \frac{1}{3} \end{aligned}$$

Again, we need both to be true, so if $q < \frac{1}{3}$ Player 2 will choose R'.