AMBIGUITY AVERSION AND THE DECLINING PRICE ANOMALY: THEORY AND ESTIMATION

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ABSTRACT. Using data from online auctions of train tickets in Sweden, we study the "declining price anomaly" in sequential auctions. First, we study a model of sequential second-price auctions with independent private values that closely matched the auctions for train tickets and assume bidders have *maxmin* expected utilities over multiple priors. In the unique symmetric equilibrium, bidders use their worst-case conditional beliefs to evaluate their payoffs in each round of the auction. This makes bidders underestimate their future payoffs and thus bid relatively more aggressively in earlier rounds and causing prices to decline on average. Also, equilibrium, in general, is history-dependent even in the independent private values paradigm, which is symptomatic of dynamic inconsistency, a common feature of dynamic problems with ambiguity. We show that this latter implication can distinguish the ambiguity aversion explanation from other theoretical explanations of the anomaly depending on the direction of the dependence. A reduce form analysis of the bidding behavior in the train ticket auctions shows that bidding in the auctions is positively history-dependent, which provides evidence in favor of our approach over other models.

Finally, we show that using dynamic bidding data we can identify and disentangle bidders worst-case beliefs from the true distribution of valuations, even in the presence of dynamic inconsistency, using a novel technique that exploits bidders' inter-temporal first order conditions. Employing our identification we estimate the true distribution of valuations and bidders' worst-case beliefs. Our estimation uncovers a first-order stochastic dominance relationship between beliefs and the true distribution as well as changing of beliefs over time, both consistent with ambiguity aversion. Our counterfactuals show that, while ambiguity increases the seller's revenue by at least 18% compared to the common prior case, switching to sequential first-price auctions would further increase revenue by at least 11%.

Keywords: declining price anomaly, ambiguity, estimation, train ticket auctions

JEL classification: C11, C44, C57, D44

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1. Introduction

Sequential auctions are one of the oldest and most commonly used auction formats for selling multiple units of a good. In a canonical version of this format, goods are sold sequentially in auctions to the highest bidder in each round. In much the same way, in 2010-11 the Swedish rail road company (SJ) sold train tickets in over 6,500 online sequences of auctions where tickets sold in a sequence were identical. Prices in these auctions on average declined 8.5% between consecutive rounds which is a curious, but common, pattern for sequential auctions with identical objects. In fact, following the seminal work on wine auctions by Ashenfelter (1989), many studies have documented instances of such "declining price anomalies," which contradict the theoretical predictions in standard sequential auctions (Milgrom and Weber (2000)).[1] Informally, prices for identical goods in a sequential auction are predicted to be a martingale, in the standard model, to avoid arbitrage opportunities.[2]

In this paper we study sequential auctions theoretically and empirically using Swedish train ticket auctions. These auctions used an eBay style online auction format that closely resembles sequential second price auctions with price announcements. Thus, we study a theoretical model of sequential second-price auctions (sSPAs) with independent private values but augment the standard model by allowing for ambiguity regarding bidders' valuation distribution. That is bidders know their own values but may not precisely know the distribution of valuations. Bidders are modeled as being averse to such ambiguity and thus maximize their *worst-case* payoff. That is they have *maxmin* preferences a lá Gilboa and Schmeidler (1989).

Using a solution concept called Consistent Planning Equilibrium (Strotz (1955), Siniscalchi (2011)) that generalizes weak Perfect Bayesian Equilibrium to an environment with ambiguity we prove the existence and uniqueness of a symmetric equilibrium in which bidders in each round use their worst-case conditional beliefs to calculate their probability of winning, where beliefs are conditioned on previous round prices. We show that under simple conditions prices decline in equilibrium due to bidders' inter-temporal pessimism resulting from ambiguity aversion: since bidders use their worst-case beliefs, they underestimate their 'option value' of participating in future rounds. Hence they bid more aggressively on average in the current round causing prices to decline on average in future rounds as lower valuation bidders, who are relatively less aggressive than the current round winner, will win future rounds in a monotone equilibrium.

[1] See Ashenfelter and Genovese (1992) (condominiums), McAfee and Vincent (1993) (wines), Beggs and Graddy (1997) (art), Van den Berg et al. (2001) (roses), Lambson and Thurston (2006) (fur), and Snir (2006) (computers).

[2] More specifically, there are two forces that affect subsequent prices in sequential auctions: a valuation affect pushing prices down, since lower valued bidders win later rounds in any equilibrium that is monotone in values and a competition affect pushing prices affect due to diminishing relative supply. The standard auction model with independent private values shows that these two effects cancel each other.

Furthermore, since bidders use worst-case conditional beliefs to evaluate their payoffs, bidding can be history-dependent, as worst-case beliefs can change round to round depending on the right truncation of the valuation support. That is in each round bidders can back out information about the highest possible value in the current round from the previous round price due to monotonicity of the bid functions. The upper end of the support will affect bidder's worst-case beliefs. This history-dependence is a novel feature of sequential auctions with ambiguity and private values. With private values, previous round prices do not inform remaining bidders anything about their valuations, as is the case in models with affiliation, but rather about the worst-case competition they face. Such history-dependent bidding is symptomatic of dynamic inconsistency, a regular feature in dynamic problems with ambiguity.[3]

Besides ambiguity aversion, there have been many theoretical explanations for the declining price anomaly over the years and we discuss them in detail in the literature section. However, to the best of our knowledge, not many papers have empirically evaluated the different explanations or carried out an estimation exercise in the presence of declining prices. Thus, in the empirical part of the paper we first evaluate the reduced form implications of the existing formalized theoretical explanations for this puzzle using our data set, and then structurally estimate our sequential auction model with ambiguity averse bidders, which best fits the reduced form evidence.

There are three "preference-based" explanations of the declining price anomaly in the literature: *risk aversion* or *aversion to price risk* (McAfee and Vincent (1993) and Mezzetti (2011)), *loss aversion* (Rosato (2023)) and *ambiguity aversion* (Ghosh and Liu (2021) and this paper). While all three models predict declining prices, they have different implications regarding the relationship between bids and previous round prices in the independent private value paradigm: risk aversion predicts history-independence, loss aversion predicts negative history-dependence and ambiguity aversion allows for history dependence that can be positive or negative (depending on the set of bidders' priors). We show that bidding in the train ticket auctions has a slight positive history dependence: controlling for all other factors, a one percent increase in the price in round $k-1$ corresponds to $0.04-0.08$ percent increase in the bids in round k. Only a model with ambiguity averse bidders is consistent with this pattern since ambiguity aversion allows for positive history-dependence. In a model with ambiguity, history-dependence comes from changing of beliefs. The model does not restrict how these beliefs change and how they affect bidding.

Finally, we carry out a structural estimation exercise in order to estimate the primitives of the model: valuation distribution and bidder beliefs. There are two difficulties associated with such an exercise in our environment. The first is that it is hard to disentangle bidders beliefs from the true distribution of valuations in auctions. In private value auctions, under the assumption of a common prior, the two distribution are the same. Following the rich literature in empirical auctions, the

[3]Bidding is dynamically inconsistent if a bidder reverses her preference over sequences of bidding functions over time.

distributions of values can then typically be identified and non-parametrically estimated using the distribution of bids (Guerre et al. (2000)). However, with ambiguity, the distributions of valuations may not be same as the (worst-case) beliefs. For single unit auctions, Aryal et al. (2018) showed that generally the two distributions cannot be identified unless some source of exogenous variation, such as number of bidders, can be used to separate the two. Complimenting this paper we show that dynamic bidding data from sequential auctions can be used to identify both the distributions without any additional restrictions. Specifically, for simplicity consider a two round sSPA. Since it is weakly dominant to bid one's valuation in the final round, it is straightforward to identify the true distribution of values from final round bids. However, bids in the first round depend on the bidder's beliefs as well as her expected payoff, and therefore her bid, in the final round. Thus, we can identify bidders (worst-case) beliefs using bidders' bids from *both* rounds and the inter-temporal first-order condition from the first round, which gives a tight connection between the two bids and bidders' beliefs.

A second difficulty is that of dynamic inconsistency. As discussed previously, bidding in our model can be history-dependent as bidders worst-case beliefs can change from round to round depending on previous round prices. Such changing beliefs can cause reversal of bidder preferences over sequences of bidding functions, thus causing bidding to be dynamically inconsistent. Due to these changing beliefs, bidder's beliefs are a family of distributions which can potentially lead to non-identification. To overcome this issue we use monotonicity of the equilibrium bidding functions to back out all valuations and then treat previous round prices as 'state variables' when estimating beliefs. Specifically, consider a three round auction. Following the approach outlined in the previous paragraph we can estimate the true distribution of valuations from the final round bids. Then, matching percentiles of bids and valuations, we can recover the valuations of all bidders in the first and second rounds. Grouping close by valuations in the first round into 'bins' we then separately estimate bidders beliefs in the second round for each bin using the methodology we described in the previous paragraph. This gives us a family of worst-case conditional beliefs. Finally, using these beliefs, bidders valuations and bids in the first round we can estimate bidders beliefs in the first round, which correspond to bidders' unconditional worst-case beliefs. Estimating beliefs using this methodology also allows us to check for dynamic inconsistency in bidding behavior.

Following Aryal et al. (2018), for estimation we use a direct Bayesian approach based on Bernstein polynomials that are flexible enough to allow for the kinds of behavior that we observe in the data. This approach also allows us to estimate the worst-case beliefs for various previous round prices and compare them. Using data on the train ticket auctions we find that all worst-case beliefs stochastically dominate the true distribution of valuations providing structural evidence of ambiguity. We also find that bidder worst-case beliefs change with previous round prices, providing evidence for dynamic inconsistency. This result also suggests an added benefit of using ambiguity versus misspecification as a modeling choice as with the latter behavior is dynamically consistent.[4]

Using the recovered distributions we find that ambiguity greatly contributes to the seller's revenue: removing ambiguity or switching to a uniform price auction would have decreased the revenue by 18% to 21%. Thus we find that declining prices may be synonymous with higher revenues compared to the no ambiguity case in sequential auctions. Finally, we also carry out an exercise where we replace the selling mechanism with sequential First Price Auctions and find that this would have increased revenue by 11% to 15%. We also find that the level of ambiguity seems to reduce over time. Bidders' beliefs seem closer to the true distribution of values in the second half of our data set than the first half. We also perform various other robustness checks to check the validity of our approach.

1.1. **Literature.** This paper is related to a few strands of literature. As mentioned previously, there is considerable evidence in support of declining prices in sequential auctions. In addition Keser and Olson (1996) and Neugebauer and Pezanis-Christou (2007) document the existence of this phenomenon in experimental settings. Chanel et al. (1996) and Deltas and Kosmopoulou (2003) found evidence of increasing prices, all though the occurrence of declining prices seems to be more common (Ashenfelter and Graddy (2003), Ashenfelter and Graddy (2011)). While all of these studies document the evidence of price anomalies, to the best of our knowledge, the current paper is the first to empirically investigate the declining prices using a structural econometrics approach as well as empirically test the existing theories that can account for this anomaly.

Since the finding in Ashenfelter (1989), several explanations for puzzle have been offered. One set of explanations suggest that specifics of the sale mechanism, such as winner's option to buy remaining units at the same price (Ashenfelter (1989)), absentee bidding (Ginsburgh (1998)), participation fees (Menezes and Monteiro (1997)) and supply side uncertainty (Jeitschko (1999)), can account for the anomaly. Another set suggests that the specific features of the goods being sold can lead to declining prices. For example Bernhardt and Scoones (1994), Engelbrecht-Wiggans (1994), Gale and Hausch (1994) and Kittsteiner et al. (2004) found that heterogeneity between the goods can lead to declining prices. In a non-auction setting Sweeting (2012) showed that the selling price of perishable goods declines as one gets closer to the expiry date.[5] All these explanations, while

[4]Under belief misspecification, it is as if bidders are in a common prior world and do not update their 'incorrect' beliefs over time. See Remark 5.4 in Ghosh and Liu (2021).

[5]Given that we are studying perishable goods as well, this explanation may seem particularly relevant. However there are some important differences. The sequential auctions for train tickets end within one hour of each other (so less likelihood of entry and exit of bidders and the travel date is not that much closer to the final sequential auction compared to the first) and the bidders are strategic. Thus prices are set by the bidders by competing against each other and not by the seller who may have an incentive to lower prices as the travel date approaches. All these aspects differentiate our setting from the baseball tickets environment studied in Sweeting (2012).

important, are specific to particular settings. However, declining prices have been observed across a wide variety of formats, goods, and settings. Thus, while our setting may share some features with the aforementioned papers, we evaluate more 'general' explanations of the anomaly so that our structural methodology may be applicable to other data sets where this phenomenon has been observed. Furthermore, the suitability of one explanation over another is not obvious since few papers have compared the various approaches, as we do in our paper.

Our paper is also connected to the the literature on auctions with ambiguity. In various settings Salo and Weber (1995), Lo (1998), Levin and Ozdenoren (2004), Chen et al. (2007) and Laohakunakorn et al. (2019) study single-unit first and second price auctions with ambiguity. In the presence of ambiguity and dynamic inconsistency Bose and Daripa (2009) and Auster and Kellner (2022) show that a dutch auctions can perform better than static auctions in contrast to Karni and Safra (1989) who show that the two are revenue equivalent under dynamic consistency. Bose et al. (2006) and Bodoh-Creed (2012) study optimal auctions under ambiguity. Bougt et al. (2024) compares the auction revenue from several multi-unit auction formats including sequential auctions under ambiguity, under the assumption of dynamic consistency. To the best of our knowledge, our paper and Ghosh and Liu (2021) are the only papers that study sequential auctions where multiple units are sold in a general environment with ambiguity.

There is a rich literature on the estimation of variables of interest from auction data.[6] Most papers in this literature require the assumption of a common prior to identify and estimate the valuation distributions. A notable exception is Aryal et al. (2018) who identify and estimate the distribution of valuations in static auctions in the presence of ambiguity using variation in the number of bidders. Complimenting this work, our identification result shows that considering data from *dynamic auctions* can provide techniques to ascertain the presence of ambiguity and correctly estimate the variables of interest.[7]

Jofre-Bonet and Pesendorfer (2003), Donald et al. (2006), Groeger (2014), Donna and Espín-Sánchez (2018) and Kong (2021) study dynamic auctions using models with capacity constraints, multi-unit demand, learning by doing, complementarities and synergies and affiliation respectively. Much like our paper, in these papers bidding in different rounds is linked. However all these papers are in the common prior framework. Furthermore, to the best of our knowledge, ours is the only paper that tackles the declining price anomaly in sequential auctions using a structural approach.

[6]See Athey and Haile (2007) and Hickman et al. (2012) for surveys. See Hortaçsu and McAdams (2018) for a survey on multi-unit auctions.

[7]There are several papers that measure the effect of ambiguity on bidding in static auctions in experimental settings, as in Chen et al. (2007). Beyond auction settings, most papers that carry out estimation exercises in the presence of ambiguity consider experimental (laboratory or field) settings. See Cabantous (2007), Abdellaoui et al. (2011), and Ahn et al. (2014) among others.

Finally, our paper is also related to the literature on identification in games of incomplete information. While much of this literature assumes that players beliefs are correct in equilibrium, Aguirregabiria and Magesan (2020) study dynamic discrete games with rational players who may have incorrect beliefs. They show that an exclusion restriction, typically used to identify games of incomplete information, provides testable nonparametric restrictions of the null hypothesis of equilibrium beliefs as well as a function that only depends on a players beliefs which can be used to estimate players beliefs. Complimenting this approach we show that data on players actions at multiple points in time (bids in different rounds) can also be used to estimate biased (i.e. worst-case) beliefs without appealing to an exclusion restriction.

1.2. **Outline.** The rest of the paper is organized as follows. Section 2 describes the auction mechanism in our data set and provides summary statistics, including the pattern of declining prices. Section 3 introduces our model of sequential auctions with ambiguity and provides the main theoretical results regarding equilibrium existence and uniqueness and price path. In Section 4 we first discuss other theoretical explanations for the anomaly and then use the train ticket auction data to provide reduced form evidence that supports ambiguity aversion as a plausible explanation. Section 5 presents the empirical analysis, including our identification result and the methodology we employ in our estimation as well as the counterfactual results. Section 6 concludes. All proofs are contained in Appendix A. Appendix B contains formal descriptions of other explanations of the declining price anomaly. Appendix C contains additional data. Finally, additional analysis where indicated can be found in the Supplemental Appendices.

2. Train Ticket Auctions

2.1. **Auction Mechanism.** The data set we use consists of bids made in auctions for train tickets sold by the Swedish rail road company (SJ). The auctions were executed online using Tradera's website, at the time a subsidiary of eBay. There were often multiple tickets sold in a group of auctions, where the winner of one of the auctions within the group received one ticket. Within a group, all tickets are observationally identical. Specifically, tickets within a group are for the same route, same type of train, same class, and same departure time. All other features such as seat number, aisle vs. window seat, and direction of the seat were not known until a winner of an auction actually went to the train station and picked up the ticket on the day of departure.

Auctions within a group ran parallel for some time, before ending within a one hour time span. The exact closing time of each auction differed within that hour. The difference in closing times gave the auctions within a group a distinctive sequential nature. We therefore refer to a group of auctions selling identical tickets as a *sequence* from here on. The closing time of an auction was made public to the bidders at the same time as the auction was posted.

Figure 1 illustrates a sequence of three auctions. All auctions within a sequence start within one hour of each other. Then all auctions in the sequence are active in parallel for about two days. Lastly, all auctions within the sequence end, in a sequential manner, between 9 and 10 pm two days prior to the departure of the train. We index auctions in a sequence by the ending order, where auction 1 is the auction that ends first in a sequence, and so on. By this logic, the last auction is auction K, where K is the total number of tickets in a sequence. We will often refer to an auction within a sequence as a *round*.

FIGURE 1. Illustration of a sequence with 3 tickets

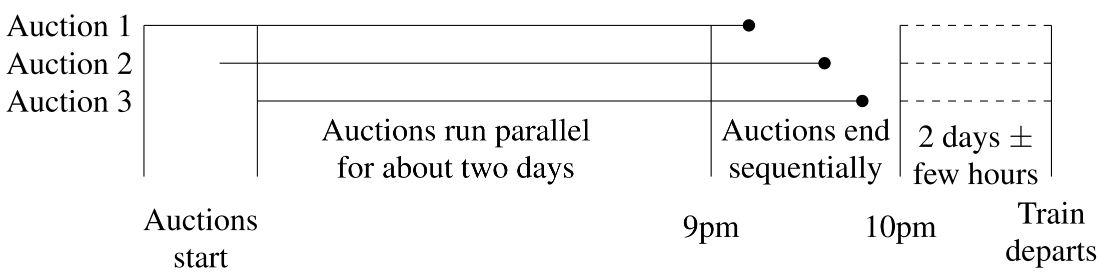
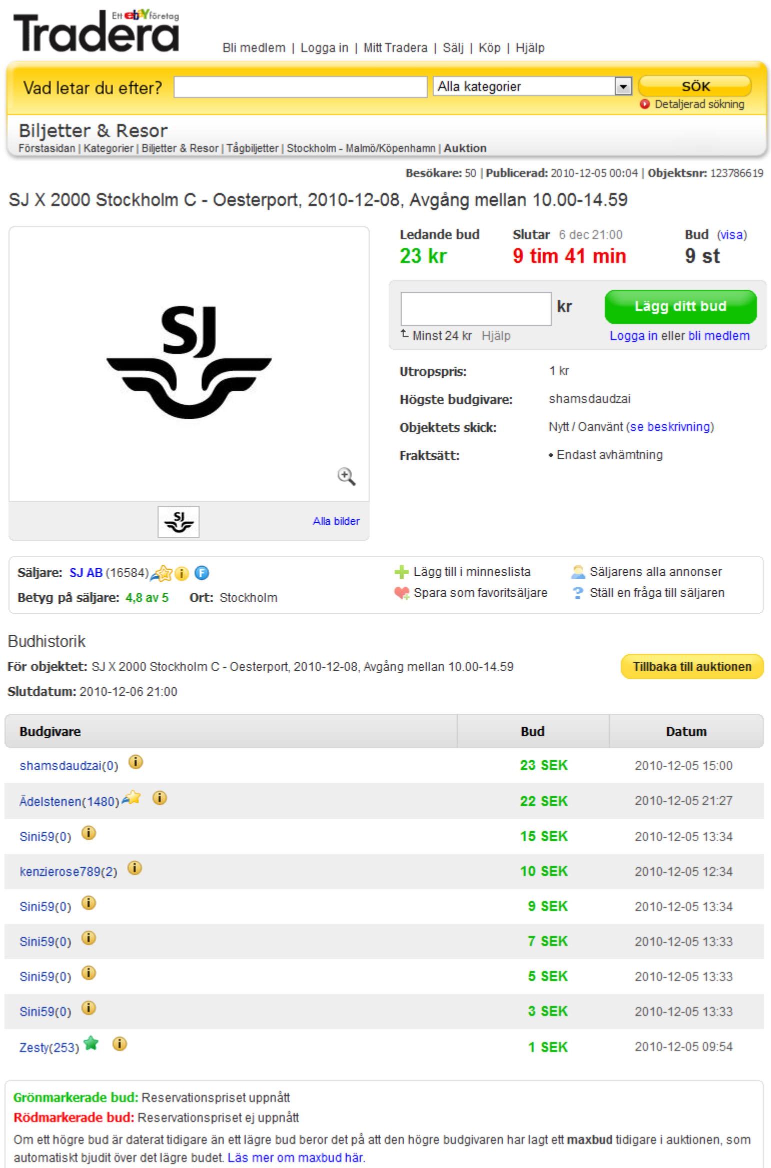


Figure 2 shows the screen faced by a bidder who is about to place a bid in an auction. Each auction within a sequence is executed using an incremental bidding mechanism, which is often referred to as the "proxy" bidding mechanism. In this mechanism, each auction is an ascending price auction, where bidders can choose to actively participate, or they can choose to record a maximum willingness to pay (MWTP). Active bidders bid the smallest amount necessary to become leaders of the auction, and can then raise their bid if a higher bid comes in. The smallest amount necessary is equal to the (current leading bid + an increment).[8] These bidders are often referred to as incremental bidders because they only raise the leading bid by one increment at a time. On the other hand, if a bidder records an MWTP, then the website will place bids in increments (so called " proxy" bids) on the bidder's behalf when new bids are placed by other bidders. The website will do so until another bidder records a bid that is higher than the first bidder's MWTP. As a result, a bidder who wins an auction having placed an MWTP only has to pay the second highest bid plus an increment, or, if the second highest bid is within an increment of the MWTP, her own bid. Thus this auction format is a hybrid between second price and first price auction (see Hickman (2010) for a formal analysis of this mechanism).

To illustrate how proxy bidding works consider the following example. If a bidder records an MWTP, call it bid_1, that is higher than the current highest recorded MWTP of some other bidder,

[8]The increments in these auctions were: 1 Swedish Krona (SEK) if the leading bid is in the interval 1–99 SEK, 5 SEK for 100–249 SEK, 10 SEK for 250–999 SEK, 25 SEK for 1000–2499, 50 SEK for 2500–4999 SEK and 100 SEK for 5000 SEK and up.

FIGURE 2. Auction screen shot



call it bid_2, then the new leading bid becomes $lead = min\{bid_2 + increment, bid_1\}$. Now, if a third bidder records a bid, call it bid_3, that is higher than $lead$, but lower (or equal) to bid_1. Then the website will keep the bidder who placed bid_1 as the leader of the auction, and raise the leading bid on behalf of this bidder to $lead' = min\{bid_3 + increment, bid_1\}$. The bidder who placed bid_1 will win the auction if no more bids are placed, and she will pay the price $lead'$.

2.2. **Declining prices.** There were 42,007 tickets (auctions) grouped into 7,202 sequences that were conducted between November 10, 2010 and June 6, 2011. In the reduced form analysis, we consider sequences of 15 tickets since this includes over ninety five percent of the data.[9] There are 35,157 tickets grouped into 6,874 sequences with 15 tickets or less. The ticket information contained in the data includes departure-destination pairs, departure time and date, and the type of train (i.e. fast train or regional train). We present more details about the data in Appendix C.[10]

To formally document that prices decline we estimate equation (1):

$$ln(price_{k,j}) = \beta_0 + \beta_1 k + \beta_2 k^2 + \beta_3 x_k + \theta_j + \varepsilon_{k,j} \quad (1)$$

where $price_{k,j}$ is the price of auction k in sequence j, x_k captures auction-specific covariates, and θ_j is a sequence fixed effect.

FIGURE 3. Visual - Declining Price

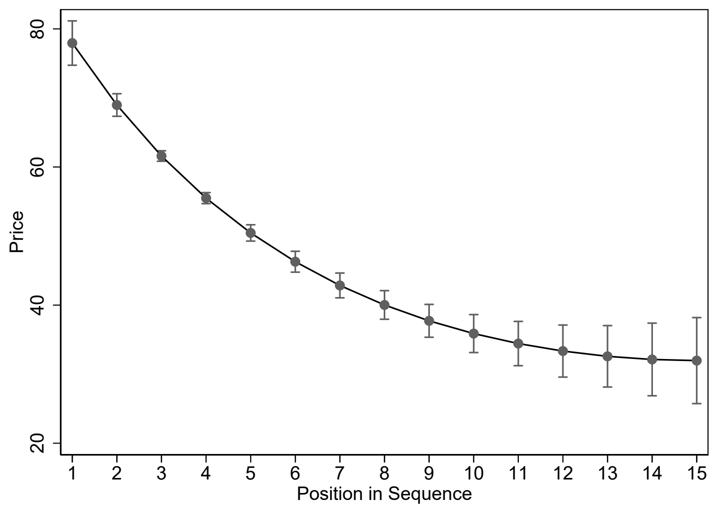


Figure 3 and Table 1 confirm the declining price path. The decline is stronger early on, and the price path evens out towards the end in sequences of many tickets. Our preferred specification for documenting declining prices is column (2) of table 1 where the sequence fixed effects are included to deal with unobserved sequence heterogeneity. The estimates in column (3), which uses sequence observable characteristics as controls instead of sequence fixed effects, are not statistically different from the estimates in column (2). This suggests that auction heterogeneity is captured well in the variables observed in the data.[11] In Appendix D we show that the documented decline in prices is robust to alternative specifications as well as holding sequence length fixed.

[9]In the entire data set there were sequences of up to 30 tickets. Even when the the entire data set is considered the declining price pattern can still be observed.

[10]See Andersson et al. (2012) and Andersson and Andersson (2017) for additional information about the data.

[11]An important assumption in our estimation in section 5 is that auction heterogeneity is observed. The fact that the estimates in column 2 and 3 of table 1 are very similar is consistent with this assumption.

TABLE 1. Declining Price - $K \leq 15$

VARIABLES	(1) ln Price	(2) ln Price	(3) ln Price
k	-0.141***	-0.136***	-0.170***
	(0.00816)	(0.0124)	(0.0160)
k squared	0.00546***	0.00451***	0.00560***
	(0.000784)	(0.000939)	(0.00129)
Observations	34,982	34,982	34,955
Sequence FE	YES	YES	
Sequence Controls			YES
Auction Controls		YES	YES

Note: *** p<0.01, ** p<0.05, * p<0.1. Robust standard errors clustered at the sequence level. In column (3), θ_j is replaced by sequence specific observable variables x_j. The variables in x_j are the same variables that we use to homogenize bids in section **??**.

3. MODEL

The aim of our theoretical model is to capture salient features of the auction mechanism used to sell train tickets while also maintaining minimal assumptions to get robust theoretical predictions. To this end our model is a second price version of the model studied in Ghosh and Liu (2021). We begin by describing the benchmark setting for a representative sequential auction. Suppose $K \geq 2$ identical tickets (units of a good) are sold, one in each round, sequentially in sealed-bid second-price auctions with no reserve price to $N \geq K+1$ bidders.[12] Each bidder i has a unit demand. In each round the bidder who submits the highest bid wins the unit and pays the second highest bid. The winning bidder leaves the auction and the winning bid is announced. The remaining bidders compete in the next round using the same procedure until all units are sold. Let p_k be the winning bid in round k. A public history in round k is a sequence of winning bids $\tilde{p}_{k-1} = (p_1, \dots, p_{k-1})$. Ties are broken via a fair coin-flip.[13]

3.1. **Bidder valuations.** Bidders draw their values, $v_i \in [\underline{v}, \overline{v}]$, independently, from a common distribution function $\tilde{F}$ with a strictly positive density function $\tilde{f}$. The distribution $\tilde{F}$ belongs to a compact and convex set of atom-less and piecewise smooth distributions Δ. Formalizing the notion of ambiguity, bidders do not know the distribution $\tilde{F}$ but are aware of the parent set Δ. For any

[12]Many previous studies have used second price auctions to model eBay style auctions. (examples). We also follow this approach and also note that for over 90% of the auctions the second price rule was used to determine price. See table 5.

[13]The tie-breaking rule will be irrelevant since we consider monotone equilibria.

$y \in [\underline{v}, \overline{v}]$, the set of conditional distributions, denoted by Δ_y, is

$$\left\{ F_y(\cdot) = \frac{F(\cdot)}{F(y)} : [\underline{v}, y] \to [0,1],\ \forall F \in \Delta \right\}. \tag{2}$$

As we will show, in each round, bidders 'beliefs' about their probability of winning belong to the above set, conditional on the price history. Thus, to ensure the existence of a monotone equilibrium in pure strategies in our set-up, we make the following 'richness' assumption. We will further eloborate on the precise role played by this assumption after we present our equilibrium result.

Assumption 3.1. For each $y \in [\underline{v}, \overline{v}], \Delta_y$ is a complete join semi-lattice.[14]

An implication of this assumption is that the lower envelope of Δ_y is always contained with the set. Additionally, let $\bar{F} \in \Delta$ be such that $\bar{F} \geq_{FOSD} F$ for all $F \in \Delta$. Furthermore, let $\bar{F}(\cdot|y) \in \Delta_y$ be such that $\bar{F}(\cdot|y) \geq_{FOSD} F(\cdot|y)$ for all $F \in \Delta$.[16] Finally, for some results, we will need the following stochastic order. Let $\geq_{rh}$ be the reverse-hazard rate stochastic dominance relation. From Shaked and Shanthikumar (2007) 1.B.41, $F_1 \geq_{rh} F_2$ if and only if $F_1(x)F_2(y) \leq F_1(y)F_2(x)$ for all $x \leq y$. Note that if $F_1 \geq_{rh} F_2$ then $F_1 \geq_{FOSD} F_2$.

3.2. **Payoffs, strategies and equilibrium concept.** As mentioned previously, prior beliefs for bidders are described by the set Δ following the multiple-priors approach of Gilboa and Schmeidler (1989). Accordingly, we assume bidders have *maxmin expected utility* (MEU): in each round they maximize the minimum expected utility over the set of priors Δ, conditional on the available information. The distribution that is used to evaluate one's payoff in each round is called the *worst-case belief*. In accordance with the classical approach, we assume that bidders follow prior-by-prior Bayesian updating.

A strategy for a player i, $\beta_i = \{\beta_{i,1}, \ldots, \beta_{i,K}\}$, is a sequence of bid functions, where $\beta_{i,k}(v_i, \tilde{p}_{k-1})$ is bidder i's bid in auction k given the history of winning bids. A strategy β_i is *monotone* if for each $k = 1, \ldots, K$, $\beta_{i,k}$ is increasing in v_i for all $\tilde{p}_{k-1}$. Denote $\beta_k = \{\beta_{i,k}\}_{i=1}^N$ and $\boldsymbol{\beta} = \{\beta_k\}_{k=1}^K$. A strategy profile $\boldsymbol{\beta}$ is symmetric if $\beta_i = \beta_j \triangleq \beta$, for all $i, j = 1, \ldots, N$. As is common in auctions, we focus on monotone and symmetric strategies and drop the subscript with respect to bidders. To slightly abuse notations, we use β to denote a monotone and symmetric strategy profile.

Observe that, since strategies are monotone, a bidder with the k-th highest value will win the k-th round. Therefore, the previous winning bids, $p_1, \ldots, p_{k-1}$, can be mapped back to the realized values, $y_1 \geq \ldots \geq y_{k-1}$, of the winners before round k. By induction backwards, the bidding

[14]The following definitions are slight variations of those in Topkis (2011). Let $\geq_{FOSD}$ be the first-order stochastic dominance partial order on Δ, i.e., $F_1 \geq_{FOSD} F_2$ if and only if $F_1(x) \leq F_2(x)$ for all $x \in [\underline{v}, \overline{v}]$. For any $F_1, F_2 \in \Delta$, let $F_1 \vee F_2$ be the *join* of F_1 and F_2.[15] Note that $F_1 \vee F_2 \geq_{FOSD} F_i$ for $i = 1, 2$. If the join of every pair of distributions in Δ belongs to Δ, then Δ is a *semi-lattice*. A semi-lattice Δ is *complete* if every nonempty subset of Δ has a join in Δ.

[16]See example 4.3 in GL to see cases where $\bar{F}$ and $\bar{F}(\cdot|y)$ are different. That is $\bar{F}(\cdot|y) \neq \bar{F}(\cdot)/\bar{F}(y)$ for some y.

functions can be rewritten as $\beta_k(v, y_{k-1})$, as in round k a bidder believes that all other remaining bidders' values are bounded above by y_{k-1}.

It is well known that in dynamic decision problems under ambiguity optimal (sequentially rational) choices made by agents can violate dynamic consistency (Siniscalchi (2011)). In order to accommodate possible time inconsistency that can occur for maximin expected utility maximizing bidders, we follow the multiple-selves approach introduced by Strotz (1955) and use *consistent planning equilibrium* as our solution concept. Essentially the idea behind this approach is that each active bidder in a each round is treated as different 'self' (of the bidder) who has different information than her previous self, which can influence her worst-case beliefs. Thus, this 'self' of the bidder may choose optimal actions differently than her previous version. That is a bidder's optimal bid in a round of the auction may be different that what she would have chosen in that round with beliefs she held in some previous round. Consistent planning allows for such discrepancy by allowing a bidder to maximize her optimal action in a round while being cognizant of the optimal actions of her future self.

Formally, a bidder's payoff is given by

$$\Pi_K(v,z,y_{K-1}) = \min_{F\in\Delta}\left(\frac{F(z)}{F(y_{K-1})}\right)^{N-K}\int_{\underline{v}}^{z}(v-\beta_K(x,y_{K-1}))\,d\left(\frac{F(x)}{F(z)}\right)^{N-K},$$

and, for $k = 1, \dots, K-1$,

$$\Pi_k(v,z,y_{k-1}) = \min_{F\in\Delta}\left(\frac{F(z)}{F(y_{k-1})}\right)^{N-k}\int_{\underline{v}}^{z}(v-\beta_k(x,y_{k-1}))\,d\left(\frac{F(x)}{F(z)}\right)^{N-k} + \int_{z}^{y_{k-1}}\Gamma_{k+1}(v,x,F)\,d\left(\frac{F(x)}{F(y_{k-1})}\right)^{N-k}. \tag{3}$$

In any around the 'probability' of winning is calculated conditional on the previous round winner's valuation, and the payment conditional on winning is the second highest bid. For the final round, there is no future round so the payoff is the standard payoff function for a second price auction amended to include ambiguity and maxmin expected utility. For all other rounds, the first term in equation (3) is the bidder's current round payoff and the second term is her *continuation payoff* if she bids as if her valuation was z in the current round. The term Γ_{k+1} is defined recursively as

$$\Gamma_{k+1}(v,x,F) = \left(\frac{F(v)}{F(x)}\right)^{N-k-1}\int_{\underline{v}}^{v}(v-\beta_{k+1}(z,v))\,d\left(\frac{F(z)}{F(v)}\right)^{N-k-1} + \int_{v}^{x}\Gamma_{k+2}(v,w,F)\,d\left(\frac{F(w)}{F(x)}\right)^{N-k-1} \tag{4}$$

and

$$\Gamma_K(v,x,F) = \left(\frac{F(v)}{F(x)}\right)^{N-K} \int_{\underline{v}}^{v} (v - \beta_K(z,x))\, d\left(\frac{F(z)}{F(v)}\right)^{N-K}.$$

In words, $\Gamma_{k+1}(v,x,F)$ is the payoff in round $k+1$ to a bidder with value v who bids according to the strategy $\boldsymbol{\beta}$ in $k+1$ and all future rounds, given the value of round k's winner, x, evaluated using some belief F. Importantly, the belief F also enters the bidder's continuation payoff. Then we have the following definition.

Definition 3.2. A strategy profile $\beta = (\beta_k)_{k=1}^{K}$ is a *consistent planning equilibrium* if for each $k = 1,\dots,K$, v, and y_{k-1}, we have

$$v \in \arg\max_{z} \Pi_k(v,z,y_{k-1}).$$

3.3. **Equilibrium and prices.** In single unit auctions, a bidders maximization problem of choosing the optimal bid can be de-coupled from her minimization problem of finding a worst-case belief which will be used to evaluate various bids. This can be seen from the payoff function in the final round of the sequential auction, where the distribution that minimizes the payoff can be found by a point-wise minimization. However, in all other rounds such a technique is not directly applicable as the bidder's beliefs also impact her continuation payoff. In face of this challenge, our approach is to first establish certain monotonicity properties of the continuation payoff. Specifically, we show that Γ is weakly declining in the second argument. Furthermore, it turns out that $\Gamma_{k+1}(v,x,F)$ is less than $v - \beta_k(v,y_{k-1})$ for $x \geq v$. As a result, the integrand in the objective function of the minimization problem is monotone decreasing, which implies that the worst-case belief–the minimizer–is the lower envelope of the set of conditional distributions. We can now state the equilibrium of the auction in the following proposition.

Proposition 3.3. *In the unique symmetric equilibrium bidders follow the strategy*

$$\beta_k(v,y_{k-1}) = \int_{\underline{v}}^{v} \beta_{k+1}(x,v)\, d\frac{\bar{F}(x|\,y_{k-1})^{N-k-1}}{\bar{F}(v|\,y_{k-1})^{N-k-1}},$$

for $k = 1,\dots,K-1$ *and* $\beta_K(v,y_{K-1}) = v$.

In the final round of a sSPA, it is weakly dominant for a bidder to bid her valuation. This is independent of what beliefs a bidder has about the valuations of others in this round. To understand the optimal bidding function in the other rounds, note that at the margin, a bidder with valuation v calculates her optimal bid conditional on the event that she is the highest valued bidder in the current round. Now, suppose the bidder contemplates bidding a little less. This deviation would only matter if there was another bidder with a valuation equal to hers. For this deviation to not be profitable, her current round payoff in this event must equal her next round expected payoff. That is her current round payment $\beta_k(v, y_{k-1})$ (if she won the second highest bid would be approximately equal to her bid since there is another bidder with valuation v) must equal what she would pay in expectation in the next round, the expected second highest bid amongst bidders with valuations lower than hers. Importantly the expectation is taken with respect to the bidder's worst case belief in the current round which is given by $\bar{F}(x|y_{k-1})$. And due to assumption 3.1 we know that this distribution is contained in the set of conditional distributions and that all remaining bidders in a round share the same worst-case belief conditional on observing a price a history. Thus our assumption ensures the existence of a symmetric equilibrium.

As a comparison, the unique symmetric equilibrium in sequential second price auctions with a common prior is

$$\tilde{\beta}_k(v) = \int_{\underline{v}}^{v} \tilde{\beta}_{k+1}(x) d\frac{\tilde{F}(x)^{N-k-1}}{\tilde{F}(v)^{N-k-1}}$$

for $k < K$ with final rounds bids being the same.[17] Note that this equilibrium can be derived from the equilibrium bidding functions we derived under ambiguity, if Δ is a degenerate and equal to the true valuation distribution. From the closed form of the equilibrium strategies in the two models we can see that bidding in the model with ambiguity can be history-dependent which is a consequence of the dependence of worst-case conditional beliefs on the price history.[18] In a model with a common prior, prices in previous rounds only scale payoffs in a current round and thus do not affect bidding behavior.

With regard to prices note that in sSPAs the price in a round is the second highest bid. Thus, in equilibrium the price in round k will be the bid placed by the bidder with the second highest valuation in that round. That is,

$$p_k = \beta_k(y_{k+1}, y_{k-1}).$$

Bidders in the next round observe the winning bid and hence the valuation of the winner, y_k. Thus, the expected price in the next round calculated at some $F \in \Delta$ is given by the expected bid of the second highest valued bidder out of $N-k$ bidders conditional on the value being less than y_{k-1}.

$$\mathbb{E}\left[P_{k+1}|p_k\right] = \int_{\underline{v}}^{y_{k+1}} \beta_{k+1}(x, y_k) dF_2^{(N-k)}(x|x \leq y_{k+1}),$$

where $F_2^{(N-k)}$ is the distribution of the second highest value out of $N-k$ draws.

Example 3.4. Before stating our results on prices it may be useful to consider a simple example to demonstrate how ambiguity aversion can lead to declining prices in contrast to the standard model. For simplicity consider a case where $N = 3$ and $K = 2$. Since in the final round bidders bid their

[17] See Milgrom and Weber (2000) or Krishna (2009) chapter fifteen for a derivation.

[18] $\bar{F}(\cdot|y) = \min_{F\in\Delta} F(\cdot)/F(y)$ depends on y.

values, in this example bidding will be history independent. However, this allows us to focus on prices. Valuations are distributed over the interval $[0,1]$ with $\Delta = \{F|F(v) = v^a, a \in [0.5, 2]\}$ and $\tilde{F}(x) = x$. Note that $\bar{F}(x) = x^2$. Then, $\beta_1(x) = 2x/3$ and $\beta_2(x) = x$, where as the equilibrium bid functions in the common prior model are $\tilde{\beta}_1(x) = x/2$ and $\tilde{\beta}_2(x) = x$. Using these closed forms,

$$p_1 = \int_0^1 \beta_1(x) d\tilde{F}_2^{(3)}(x) = \frac{1}{3}; \quad p_2 \int_0^1 \beta_2(x) d\tilde{F}_3^{(3)}(x) = \frac{1}{4}$$

are the prices in the model with ambiguity and

$$\tilde{p}_1 = \int_0^1 \tilde{\beta}_1(x) d\tilde{F}_2^{(3)}(x) = \frac{1}{4}; \quad \tilde{p}_2 = \int_0^1 \tilde{\beta}_2(x) d\tilde{F}_3^{(3)}(x) = \frac{1}{4}$$

are the prices in which the prior $\tilde{F}$ is known. As we can see that that the expected prices in the ambiguity model shows a declining trend.

Why do prices in the above example show a declining trend? Intuitively, ambiguity causes an overestimation of future competition: since bidders believe that the distribution of values is stronger they will underestimate their chances of winning in the future. This causes them to underestimate their option value of moving to the next round which makes them bid relatively more aggressively in the current around. This can cause a declining price trend.

However, as Ghosh and Liu (2021) showed in the context of sequential FPAs, with ambiguity, average prices can also rise for some sets of priors. The reason for this possibility is that with history dependence, bidders beliefs can change in radical ways that makes prices rise. Thus, we provide two sufficient conditions under which prices will decline.

Proposition 3.5. *If for any $y \in [\underline{v}, \bar{v}], \bar{F}(\cdot|y) \geq_{rh} F(\cdot)/F(y)$ then $p_{K-1} > \mathbb{E}_F[P_K|p_{K-1}]$. Furthermore there exists $\varepsilon > 0$ such that for $y_{k+1} \in (y_k - \varepsilon, y_k]$, $p_k > \mathbb{E}_F[P_{k+1}|p_k]$.*

The above condition state that if the bidder's beliefs do not change too much from round to round then continuity of bid functions ensures that prices will decline due to the intuition we mentioned before. Importantly, the above result shows that prices can decline in our framework, under dynamic inconsistency as well as positive or negative history dependence. However, from the above result, a sufficient condition for prices to decline on average is that the winning valuations are not too far apart in previous rounds. This is not necessary as we can see from the proof which only uses continuity to establish the result. The ε can be as high as y_k. We also provide another sufficient condition for declining prices that relies on a stronger form of stochastic relationship between worst-case beliefs.

Proposition 3.6. *If for any* $y \in [\underline{v}, \bar{v}], \bar{F}(\cdot|y) \geq_{rh} F(\cdot)/F(y)$ *then* $p_{K-1} \geq \mathbb{E}_F[P_K|p_{K-1}]$. *Furthermore if* $\bar{F}(\cdot|y) \geq_{rh} \bar{F}(\cdot|y')/\bar{F}(y|y')$ *for* $y \leq y'$ *then* $p_k > \mathbb{E}_F[P_{k+1}|p_k]$.

Remark 3.7. Both the above results provide sufficient conditions for declining prices. Due to continuity of bid functions this also implies that the results are generically true in the neighborhoods of beliefs for which the results hold with strict inequality as is the case in the two propositions. We also think that the above two conditions are reasonable in many settings. Especially in ours as we discuss in the empirical portions of the paper.

4. Case for Ambiguity

The standard sequential auction model studied in Milgrom and Weber (2000) predicts constant prices in the i.p.v. setting.[19] The price prediction of the standard model is clearly refuted by the data. Many previous studies have provided theoretical explanations for the declining price anomaly. Among these, ambiguity aversion, risk aversion (Mezzetti et al. (2008)) and loss aversion (Rosato (2023)) are based on generalizations of bidder preferences beyond risk neutrality and standard expected utility maximization, and thus are not specific to particular auction mechanisms.

Another aspect of equilibrium bidding in the standard auction model is that of *history independence*: prices in previous rounds do not affect a bidders bid in a round. Under i.p.v. a bidder will bid more aggressively as the rounds progress due to shrinking relative supply but will not be affected by previous round prices, as prices only scale a bidders current round payoff. The three theoretical models that predict declining prices however have different predictions regarding history dependence of bids in an i.p.v. framework compared to each other. To understand these differences, we briefly describe the other two models.

4.1. **Risk aversion.** Informally, Ashenfelter (1989) argued that risk aversion could possibly explain the anomaly since, theoretically, prices in later rounds are more variable than the current round and hence risk-averse bidders may bid more in the current round to avoid future price variations.[20] This logic was formalized in McAfee and Vincent (1993) but the results required that the bidder's utility function satisfy non-decreasing absolute risk aversion in wealth which is an unconventional assumption.[21] Alternatively, using an additively separable utility function with a convex cost function, which captures 'aversion to price risk', Mezzetti (2011) proves the existence

[19] In an affiliated values setting prices increase. See pages 219-220 of Krishna (2009) for a full and concise treatment.
[20] For example, in Krishna (2009) (page 226):"[Even] though prices are expected to decline in the future, his greater aversion to risk offset the incentive to wait for a random future price, which is lower on average."
[21] McAfee and Vincent (1993) write that "Ashenfelter suggests that declining prices is consistent with risk averse bidders....We show that this intuition is not likely to be satisfied in practice."

of a monotone equilibrium that generates declining prices. Importantly, the equilibrium is characterized by history-independent bidding in the i.p.v. paradigm. In appendix B.1 we reproduce these results from Mezzetti (2011) formally using an example.

4.2. **Loss aversion.** Rosato (2023) studied a sequential auction model where bidders have expectations based *reference dependent preferences* ((Köszegi and Rabin, 2006)), within an i.p.v. paradigm. The winning bid in a round is part of forming reference points for the bidders who remain in the next round. The equilibrium in this setting is also characterized by declining prices due to a 'discouragement effect': a higher winning bid in a round leads to less aggressive bidding in the next round, and since bidders choose bids conditional on being pivotal, they underestimate the discouragement effect. This model also predicts a negative relationship between winning bids in the a round and bids in the next, which can be tested empirically. A more formal example is presented in appendix B.2

Thus, to sum up, while all three models predict declining prices, risk aversion predicts history-independence in bids, loss-aversion predicts a negative relationship and ambiguity aversion allows for any kind of history independence as we can see from the closed form of the bidding functions in 3.3. In the latter, the price history affects the worst-case beliefs but does not put any restriction on how these beliefs will specifically affect bidding. This observation is illustrated by our two sufficient conditions for declining prices in Propositions 3.5 and 3.6. The former result allows for positive history dependence whereas the latter leads to negative history dependence. Thus, the ambiguity model is flexible enough to generate declining prices under positive or negative history dependence.

4.3. **Effect of price history in train ticket auctions.** We now document the relationship between various observable variables and bidding within a sequence and make a case for ambiguity aversion as a plausible explanation for declining prices. Some of the variables we consider include the number of bidders (demand), the number of items left (supply), and the price in the previous auction (history). The main comparative static on equilibrium bidding that distinguishes the preference based explanations from each other is the relationship between bids in a round and the price-history in the sequence. We estimate equation (5) to evaluate the relationships between bidding and observables:

$$\ln(bid_{i,k,j}) = \beta_0 + \beta_1 \ln(price_{k-1,j}) + \beta_2 n_{k,j} + \beta_3 (K_j - k) + \theta_{i,j} + \varepsilon_{i,k,j}. \tag{5}$$

Here $bid_{i,k,j}$ is the bid placed by bidder i in auction k of sequence j. $n_{k,j}$ is the number of bidders in auction k of sequence j, and K_j is the number of tickets for sale in sequence j. $\theta_{i,j}$ captures bidder-sequence fixed effects. This implies that identification comes from bidders who have participated in at least two of the auctions in sequence j. Furthermore, this implies that valuation for the object is held constant, given that we have included bidder specific fixed effects, when evaluating how the covariates of interest impact bidding behavior.

The main parameter of interest is β_1, which captures the potential history dependence of round k bids on round $k-1$ prices. A complication is that auctions are not purely sequential but run parallel before ending sequentially. Hence, bids made while auctions run parallel are made when the history is unknown. For this reason, equation (5) will be estimated using bids made after auction $k-1$ has ended.

TABLE 2. Bidder Behavior - $K \leq 15$

VARIABLES	(1) ln bid	(2) ln bid
ln price(k-1)	0.0759***	0.0487***
	(0.0143)	(0.0101)
Bidders when Bidding	0.0600***	0.0671***
	(0.00416)	(0.00636)
(K-k)	-0.0776***	-0.0422***
	(0.00377)	(0.00266)
Bidders in current auction		-0.0636***
		(0.00585)
ln Current Lead when Bidding		0.379***
		(0.0121)
Observations	25,963	25,947
BidderXSequence FE	YES	YES
BidderXRound Controls	YES	YES

Note: *** p<0.01, ** p<0.05, * p<0.1. Robust standard errors that are clustered at bidder X sequence level. *Current Lead when Bidding* is the bid that must be surpassed when the bidder places the bid.

Table 2 reports the results from estimating equation (5). All estimates indicate that bids are history dependent. It can further be rejected the coefficient on $\ln price(k-1)$ is less than or equal to 0 ($p < 0.01$ when testing $H_0 : \beta_{\text{ln price(k-1)}} \leq 0$). Among the preference based explanations that can account for the declining price pattern, only the ambiguity aversion model can account for the documented positive relationship between bids in a round and the price in the previous round in an i.p.v. paradigm. This evidence suggests that neither risk nor loss aversion can *alone* account for the declining price anomaly as these preferences structures predict zero and negative history dependence respectively.

While being positive, the magnitude of the history dependence is small. A one percent increase in the price in round $k-1$ corresponds to 0.04-0.08 percent increase in the bids in round k. It also appears as if bidders bid more aggressively if there are many bidders present in the auction when they make their bid. Bidders also bid more aggressive when there are few tickets left. This captures the supply effect.

While the number of bidders at the time of bidding has a positive effect on bids, the number of bidders at the end of an auction seems to be negatively related to the bid. The source of this relationship could be the open bidding nature of the auction as late arriving low valued bidder do not submit bids if the current leading bid at the time is too high.

5. Estimation

5.1. **Identification.** Having made a case for ambiguity aversion we now turn to estimation of primitives of the model. Our objects of interest are bidder's valuations, the true valuation generating process, $\tilde{F}$, and the beliefs bidders use to calculate their bids. The latter include, the worst-case unconditional belief, $\bar{F}$ and worst-case conditional beliefs $\bar{F}(\cdot|y)$.

Due to second price nature of the sequential auctions, we can estimate valuations and the true value distribution simply from the final round bids since bidders bid their values from the final rounds. Then, by matching percentiles in each round, we can estimate all valuations in all rounds by only appealing to monotonicity of the equilibrium.

Identification of $\bar{F}(\cdot|y)$ follows from bidders optimality conditions. For simplicity, suppose $K = 2$. Then using bidders first order conditions in the first round, we can express a bidders bid as a function of her valuation, number of competitors and her beliefs, which in two round auction would be given by $\bar{F}$. Specifically, the optimal bid in the first round is

$$b_1 = \beta_1(v) = \int_{\underline{v}}^{v} x d\frac{\bar{F}(x)^{N-2}}{\bar{F}(v)^{N-2}}.$$

Thus, $\bar{F}$ can be identified. Now, for $K > 2$ the beliefs that will be identified using bids in the penultimate round are bidder worst-case conditional beliefs $\bar{F}(\cdot|y)$. Essentially, we can treat previous round prices/valuations as a state variable and then identify the conditional beliefs. Thus, we have the following result.

Proposition 5.1. *Under assumption 3.1 if $K \geq 3$ then $\tilde{F}, \bar{F}$ and $\bar{F}(\cdot|y)$ are identified.*

5.2. **Data set.** We consider sequences where there are at least 3 items being sold ($K_j \geq 3$).We observe J sequences and from each sequence we have $\mathbf{b^j} = \{b^{(2)}_{j,k}|y_{k-1}, N_j, K_j\}_{k=1}^{K_j}$, where $b^{(2)}_{j,k}|N_j, K_j$ is the second highest (homogenized) bid in auction k of sequence j out of $(N_j - k + 1)$ potential bids when the winner of auction $k-1$ had valuation y_{k-1}, and the total number of items to be auctioned off is K_j. In essence we observe the second highest order statistic of bids in the first auction, third highest order statistic of bids in the second auction, fourth highest order statistic of bids in the third auction, and so on, where the order statistic is relative to valuations. We collect the observations from the first auction of each sequence in $\mathbf{b_1} = \{b^{(2)}_{j,1}|N_j, K_j\}_{j=1}^{J}$, the second auction in each sequence in $\mathbf{b_2} = \{b^{(2)}_{j,1}|y_1, N_j, K_j\}_{j=1}^{J}$, and the last auction in each sequence in $\mathbf{b_K} = \{b^{(2)}_{j,K}|y_{K-1}, N_j, K_j\}_{j=1}^{J}$.

In the first auction of a sequence, bidders use the *unconditional* worst-case beliefs,$\bar{F}(\cdot)$, when placing their bid:

$$b_1(v|N,K) = b_1(v|N) = \int_{\underline{v}}^{v} b_2(x|v,N) \frac{d\bar{F}(x)^{N-2}}{\bar{F}(v)^{N-2}}.$$

In the second auction of a sequence, bidders use the *conditional* worst-case belief, $\bar{F}(\cdot|y_1)$, when placing their bid:

$$b_2(v|y_1,N,K) = b_2(v|y_1,N) = \int_{\underline{v}}^{v} x \frac{d\bar{F}(x|y_1)^{N-3}}{\bar{F}(v|y_1)^{N-3}}.$$

In the last auction of a sequence, it is a weakly dominant strategy for bidders to bid their valuation. Bids are therefore independent of any beliefs:

$$b_K(v|y_{K-1},N,K) = b_K(v) = v.$$

Note that $\bar{F}(\cdot|y_1) = \bar{F}(\cdot|y_k)$ for $k = 3, ..., K-1$ if $y_1 = y_k$. As a result, with variation in y_1 across sequences, bids from the second round carry all the necessary information about worst-case conditional beliefs. While using bids from all intermediate auctions in a sequence would increase the power in the estimation, it is not necessary and it comes at a cost of increased computational time. We therefore restrict the data used to recover worst-case conditional beliefs to bids placed in the second auction of a sequence.

5.3. **Statistical model.** The goal is to estimate the distribution of valuations, $F(v)$, the unconditional worst-case beliefs, $\bar{F}(v)$, and the conditional worst-case beliefs $\bar{F}(v|y_{k-1})$ along with their density functions. To do this, we implement a Bayesian estimation technique that relies on a flexible (almost non-parametric) specification that utilizes Bernstein polynomials.

We collect the observable bids in $\mathbf{b} = (\mathbf{b_1}, \mathbf{b_2}, \mathbf{b_K})$, the observed number of bidders in each sequence, and we collect the latent structural parameters to be estimated in $\Theta = (\theta_F, \theta_{\bar{F}}, \theta_{\bar{F}|y_{k-1}})$.

Bidder i in sequence j has valuation $v_{i,j}$. Assume that

$$v_{i,j} \overset{iid}{\sim} F(v_{i,j}|\theta_F)$$

with an associated density $f(v_{i,j}|\theta_F)$, which is strictly positive on its bounded support $[\underline{v}, \overline{v}] \subset R_+$.

The conditional density of observing the second highest bid in auction K of sequence with N bidders, $b_K^{(2)}(v)$, is then[22]

$$g'_K(b_K^{(2)}(v)|\theta_F) = \left(\frac{N!}{(N-K-1)!K!}\right) f(b_K^{(2)}(v)|\theta_F)F(b_3^{(2)}(v)|\theta_F)^{N-K-1}[1-F(b_3^{(2)}(v)|\theta_F)]^K$$

Given a prior distribution $P_{0,F}(\theta_F)$ for the latent variable θ_F, and an associated density $p_{0,F}(\theta_F)$, the posterior density of θ_F is

$$\begin{aligned} p_F(\theta_F|\mathbf{b_K}) = &\quad p_{0,F}(\theta_F)\prod_{j=1}^{J} g'_K(b_{K,j}^{(2)}|y_{K-1},N_j,K_j,\theta_F)ML(\mathbf{b_K})^{-1} \\ \propto &\quad p_{0,F}(\theta_F)\prod_{j=1}^{J} g'_K(b_{K,j}^{(2)}|y_{K-1},N_j,K_j,\theta_F). \end{aligned}$$

$ML(\mathbf{b_K})$ is the marginal likelihood of the data and is defined by

$$ML(\mathbf{b_3}) = \int p_0(\theta_F)p(\theta_F|\mathbf{b_K})d\theta_F.$$

Moving to estimation of $\theta_{\bar{F}}$. Given an estimates, $\hat{f}(v)$ and $\hat{F}(v)$, of $f(v)$ and $F(v)$, let $\theta_{\bar{F}}$ be the parameter capturing the worst-case beliefs in auction 1. To estimate the worst-case belief, we use a change of variable {cite GPV}. Let $\phi_{1,N,\theta_{\bar{F}}} = \phi_1(b_1(v|N,\theta_{\bar{F}}))$ be the inverse bidding function in auction 1 when N bidders participate in the sequence. Then we have that the distribution of bids in the first auction is

$$G(b_1(v|N)|\hat{F}(v),\theta_{\bar{F}}) = \hat{F}(\phi_{1,N,\theta_{\bar{F}}})$$

$$g(b_1(v|N)|\hat{f}(v),\theta_{\bar{F}}) = \hat{f}(\phi_{1,N,\theta_{\bar{F}}})\phi'_{1,N,\theta_{\bar{F}}}$$

Bidding in auction one can also be re-written in terms of the inverse bidding strategy.

$$b_1(v|N,\theta_{\bar{F}}) = b_2(v|N,\theta_{\bar{F}}) - \frac{\bar{F}(v|\theta_{\bar{F}})}{(N-2)\bar{f}(v|\theta_{\bar{F}})\phi'_{1,N,\theta_{\bar{F}}}}$$

The density of bids in auction 1 can now be expressed in terms of parameters to be estimated

$$\begin{aligned} & g(b_1(v|N)|\hat{f}(v),\theta_{\bar{F}}) \\ & = \frac{\hat{f}(v)\bar{F}(v|\theta_{\bar{F}})}{(N-2)\bar{f}(v|\theta_{\bar{F}})[b_2(v|N,\theta_{\bar{F}})-b_1(v|N,\theta_{\bar{F}})]}\mathbf{1}(b_1(\underline{v}|N,\theta_{\bar{F}}) \leq b_1(v|N,\theta_{\bar{F}}) \leq b_1(\overline{v}|N,\theta_{\bar{F}}) \end{aligned}$$

[22]The Kth + 1 highest order statistic (of valuations) out of N.

The conditional density of observing the second highest bid, $b_1^{(2)}(v)|N$, in auction 1 of sequence with N bidders is then[23]

$$g'(b_1^{(2)}(v|N)|\hat{f}(v),\hat{F}(v),\theta_{\bar{F}}) = \left(\frac{N!}{(N-2)!}\right) g(b_1(v|N)|\hat{f}(v),\theta_{\bar{F}})\hat{F}(v)^{N-2}[1-\hat{F}(v)]$$

Given a prior distribution $P_{0,\bar{F}}(\theta_{\bar{F}})$ for the latent variable $\theta_{\bar{F}}$, and an associated density $p_{0,\bar{F}}(\theta_{\bar{F}})$, the posterior density of $\theta_{\bar{F}}$ is

$$\begin{aligned} p_{\bar{F}}(\theta_{\bar{F}}|\mathbf{b_1}) = &\ p_{0,\bar{F}}(\theta_{\bar{F}}) \prod_{j=1}^{J} g'(b_{1,j}^{(2)}(v|N_j)|\hat{\theta}_v,\theta_{\bar{F}}) ML(\mathbf{b_1})^{-1} \\ \propto &\ p_{0,\bar{F}}(\theta_{\bar{F}}) \prod_{j=1}^{J} g(b_{1,j}^{(2)}(v|N_j)|\hat{\theta}_v,\theta_{\bar{F}}). \end{aligned}$$

$ML(\mathbf{b_1})$ is defined analogously to $ML(\mathbf{b_K})$.

Note that we can back out y_1 once we have an estimate of $F(v)$.[24] To estimate $\theta_{\bar{F}|y_1}$, we group observations of bids in the second auctions into bins based on y_1 and assume that the worst case beliefs within a bin is the same. Suppose we divide $[\underline{v},\bar{v}]$ into M many equally spaced bins, with each bin being denoted Y_m with $m = 1,...,M$, and let

$$y^m = max\{y|y \in Y_m\}.$$

Then we have $\theta_{\bar{F}|y_1} = \{\theta_{\bar{F}|Y_m}\}_{m=1}^{M}$ to estimate. Use the same change of variable as with bids in the first auction, $\phi_{2,N,\theta_{\bar{F}|Y_m}} = \phi_2(b_2(v|N,y_1,\theta_{\bar{F}|Y_m}))$, which is the inverse bidding function in auction 2 when N bidders participate in the sequence and the winners valuation in auction 1 was in bin Y_m ($y_1 \in Y_m$). We have a similar mapping from distribution of bids to the distributions of estimated valuations

$$G_2(b_2(v|N,y_1)|\hat{F}(v),\theta_{\bar{F},Y_m})) = \frac{\hat{F}(\phi_{2,N,\theta_{\bar{F}|Y_m}})}{\hat{F}(y_1)}$$

$$g_2(b_2(v|N,y_1)|,\hat{f}(v),\hat{F}(v)\theta_{\bar{F},Y_m})) = \frac{\hat{f}(\phi_{2,N,\theta_{\bar{F}|Y_m}})\phi'_{2,N,\theta_{\bar{F}|Y_m}}}{\hat{F}(y_1)}$$

Similarly, we also have that

$$b_2(v|N,y_1,\theta_{\bar{F}|Y_m}) = v - \frac{\bar{F}(v|\theta_{\bar{F}|Y_m})}{(N-3)\bar{f}(v|\theta_{\bar{F}|Y_m})\phi'_{2,N,\theta_{\bar{F}|Y_m}}}.$$

[23] The second highest order statistic (with respect to valuations) out of N.

[24] This is under the assumption that you observe also the highest bid in the first auction. We have to make an assumption on what we use as the "history-relevant" variable".

The density of bids in auction 2 can now be expressed in terms of parameters to be estimated

$$g_2(b_2(v|N,y_1)|\hat{f}(v),\hat{F}(v),\theta_{\bar{F},Y_m})) = \frac{\hat{f}(v)\bar{F}(v|\theta_{\bar{F}|Y_m})}{(N-3)\bar{f}(v|\theta_{\bar{F}|Y_m})[v-b_2(v|N,\theta_{\bar{F}|Y_m})]\hat{F}(y_1)}$$
$$\times \mathbf{1}(b_2(\underline{v}|N,y_1,\theta_{\bar{F}|Y_m}) \leq b_2(v|N,y_1,\theta_{\bar{F}}) \leq b_2(y^m|N,\theta_{\bar{F}|Y_m})$$

The conditional density of observing the second highest bid, $b_2^{(2)}(v|N,y_1)$, in auction 2 of a sequence with N bidders is then[25]

$$g_2'(b_2^{(2)}(v|N,y_1)|\hat{f}(v),\hat{F}(v),\theta_{\bar{F},Y_m}) = \left(\frac{(N-1)!}{(N-3)!}\right)\frac{g_2(b_2^{(2)}(v|N,y_1)|\hat{f}(v),\hat{F}(v),\theta_{\bar{F},Y_m})\hat{F}(v)^{N-3}[\hat{F}(y_1)-\hat{F}(v)]}{\hat{F}(y_1)^{N-1}}$$

Let J_m denote the subset of observed sequences of auctions where $y_{1,j}$ is in Y_m. Given a prior distribution $P_{0,\bar{F}|y_1}(\theta_{\bar{F}|y_1})$ for the latent variable $\theta_{\bar{F}|y_1}$, and an associated density $p_{0,\bar{F}|y_1}(\theta_{\bar{F}|y_1})$, the posterior density of $\theta_{\bar{F}|y_1}$ is

$$\begin{aligned} p_{\bar{F}|y_1}(\theta_{\bar{F}|y_1}|\mathbf{b_2}) = \;& p_{0,\bar{F}|y_1}(\theta_{\bar{F}|y_1})\prod_{m=1}^{M}\prod_{j\in J_m} g_2'(b_{2,j}^{(2)}(v|N,y_{1,j})|\hat{f}(v),\hat{F}(v),\theta_{\bar{F}|Y_m})ML(\mathbf{b_2}) \\ \propto \;& p_{0,\bar{F}|y_1}(\theta_{\bar{F}|y_1})\prod_{m=1}^{M}\prod_{j_m\in J_m} g_2'(b_{2,j_m}^{(2)}(v|N,y_1)|\hat{f}(v),\hat{F}(v),\theta_{\bar{F}|Y_m}) \end{aligned}$$

$ML(\mathbf{b_2})$ is also defined analogously to $ML(\mathbf{b_3})$.

The marginal likelihood is useful for comparing different models (different parameter spaces). However, given the parameter space, the marginal likelihood is just a constant that does not affect the comparison of likelihoods of different parameters from that space.

To summarize, the estimation will be done in the following steps:

1 Use $\mathbf{b_K}$ to estimate θ_F.
2 Use $\mathbf{b_1}$ along with the estimated $\hat{F}(v)$ to calculate $\hat{y}_1$.
3 Use $\mathbf{b_1}$ and $\mathbf{b_2}$ along with the estimated $\hat{F}(v)$ and $\hat{y}_1$ to estimate $\theta_{\bar{F}}$ and $\theta_{\bar{F}|y_1}$.

5.4. **Estimation.** While tickets within a sequence are for the same train, tickets in different sequences vary across dimensions such as type of train used, route, departure day of the week, and so on. Thus, in order to carry out a structural estimation using the above procedure, we need to homogenize the bids for different sequences. Using the sequence specific covariates we homogenize bids by removing the effect of these variables that are common to all bidders in a sequence. In Appendix E we discuss this procedure which is based on a similar exercise in Shneyerov (2006).

[25] Observing the third order-statistic conditional on the first order-statistic being equal to y_1.

We model the latent distributions using Bernstein polynomials (Petrone (1999)). Bernstein polynomials have previously been used in the structural auctions literature Aryal et al. (2018). A Bernstein polynomial of order K of a function $F(x)$ on $x \in [0,1]$ is defined by

$$B(x|K,F) = \sum_{k=0}^{K} F\left(\frac{k}{K}\right)\binom{K}{k} x^k (1-x)^{K-k}. \tag{6}$$

The derivative of equation (6) is

$$b(x|K,F) = \sum_{k=1}^{K} w_k beta(x|k, K-k+1) \tag{7}$$

where

$$w_k = F\left(\frac{k}{K}\right) - F\left(\frac{k-1}{K}\right) \text{ for } k = 1, \ldots, K$$

and $beta(\cdot|k, K-k+1)$ is the standard beta distribution with parameters k and $K-k+1$. From Lorentz (1953) we have that

Theorem 5.2. *For a function $F(x)$ bounded on $[0,1]$, the relation*

$$\lim_{K\to\infty} B(x|K,F) = F(x)$$

holds at each point of continuity x of F. The relation holds uniformly on $[0,1]$ if F is continuous on this interval.

In that sense, Bernstein polynomials are flexible enough that they can approximate any continuous distribution function on a bounded interval. From an estimation point of view, Bernstein polynomials are attractive as they are just a mixture of beta distributions, which are relatively easy to compute.

Given K, $f(v)$ is given by

$$f(v|\theta_F, K) = \sum_{k=1}^{K} w_k beta(v|k, K-k+1)$$

and given $F(v)$ is given by

$$F(v|\theta_F, K) = \sum_{k=1}^{K}\left(\left[\sum_{j=1}^{k} w_k\right]\binom{K}{k} v^k (1-v)^{K-k}\right).$$

$\bar{f}(v|\theta_{\bar{F}}, \bar{K})$, $\bar{F}(v|\theta_{\bar{F}}, \bar{K})$, $\{\bar{f}_m(v|\theta_{\bar{F}|Y_m}, \bar{K}_Y, v \le Y_m)\}_{m=1}^M$, $\{\bar{F}_m(v|\theta_{\bar{F}|Y_m}, \bar{K}_Y, v \le y_m)\}_{m=1}^M$ are analogously defined.

When estimating the density, we will be using Kth + 1 order statistic. A complication following from this is that there will be few observations in the upper range of the support. Inferring what the upper bound is from observations is therefore difficult. To deal with this, we will treat the upper bound, $\bar{v}_K$, as a parameter to be estimated.

The parameters to be estimated are $\theta_F|K = (w_1,...,w_K,\bar{v}_K)$, $\theta_{\bar{F}}|\bar{K} = (\bar{w}_1,...,\bar{w}_{\bar{K}})$, and $\theta_{\bar{F},y_1}|\bar{K}_Y = \{(\bar{w}_1^m,...,\bar{w}_{\bar{K}_Y}^m)\}_{m=1}^M$.

In principle, K, $\bar{K}$, and $\bar{K}_Y$[26] can be treated as parameters to be estimated. We choose the alternative to first estimate $\theta_F|K$ for different Ks and treat them as different models. Then we use Bayes model selection to choose K. Given the chosen K, we then get the estimates for the distribution and density of values given the chosen K. We do the same for $\bar{K}$, and $\bar{K}_Y$. The procedure for selecting K is as follows. If $p_{0,K}(K)$ is the prior for $K \in \mathscr{K}$ and $p_{0,F,K}(\theta_F|K)$ is the prior given K, and $p_{F,K}(\theta_F|\mathbf{b_3},K)$ is the posterior of θ_F given K and the data. Then we choose the model, K, for which the posterior of K is maximized, where the posterior is

$$p_K(K|\mathbf{b_3}) \propto p_{0,K}(K)ML(\mathbf{b_3}|K).$$

As priors over the parameter-space we use

$$p_{0,F,K}(\theta_F|K) = Dirichlet(\{1\}_{k=1}^K) \times U[0,1]$$

$$p_{0,\bar{F},\bar{K}}(\theta_{\bar{F}}|\bar{K}) = Dirichlet(\{1\}_{k=1}^{\bar{K}})$$

$$p_{p,\bar{F}|y_1,\bar{K}_Y}(\theta_{\bar{F}|y_1}|\bar{K}_Y) = \prod_{m=1}^M p_{0,\bar{F}_m,\bar{K}_Y}(\theta_{\bar{F}|Y_m}|\bar{K}_Y)$$

$$p_{0,\bar{F}_m,\bar{K}_Y}(\theta_{\bar{F}|Y_m}|\bar{K}_Y) = Dirichlet(\{1\}_{k=1}^{\bar{K}_Y})\ m = 1,...M$$

$$p(K) = pois(\lambda = 10).$$

5.4.1. *Posterior inference:* We use posterior moments for inference. For a a measurable function $h(\theta)$, the posterior mean is defined as

$$\mathbb{E}[h(\theta|x)] = \int h(\theta)p(\theta|x)d\theta. \tag{8}$$

We are interested in the density of valuations, $f(v)$, and the distribution bidders use when they place their bids in the first round, $\bar{F}(v)$.[27]

[26] $\bar{K}_Y$ could be different across ms, and the parameters could be estimated separately for each bin. We choose to estimate these parameters jointly. Our structure still allows as to select different Ks for different bins after we have estimated them jointly.

[27] The prior over parameters gives that $\mathbb{E}_0[f(v|K)] = U[0,1]$ for any K.

Given an ergodic sample of the parameters $\{\theta^s_F, \theta^s_{\bar{F}}\}_{s=1}^S$, the estimate for $f(v)$ is then given by the point-wise mean

$$\hat{f}(v) = \frac{1}{S}\sum_{s=1}^{S} f(v|\theta^s_F) \tag{9}$$

along with its 95% confidence interval which we get by taking the 2.5 and 97.5 percentile of $\{f(v|\theta^s_F)\}_{s=1}^S$; and the estimate for $\bar{F}(v)$ is given by the point-wise mean

$$\hat{\bar{F}}(v) = \frac{1}{S}\sum_{s=1}^{S} \bar{F}(v|\theta^s_{\bar{F}}) \tag{10}$$

along with its 95% confidence interval. In order to estimate $\hat{f}(v)$ and $\hat{\bar{F}}(v)$ we must ensure that the sample $\{\theta^s_F, \theta^s_{\bar{F}}\}_{s=1}^S$ is drawn from the respective posterior $p_{F,K}(\theta_F|\mathbf{b_3})$ and $p_{\bar{F},\bar{K}}(\theta_{\bar{F}}|\mathbf{b_1})$. To this end we employ the Gaussian Metropolis Hastings algorithm, which can be implemented without having to compute the marginal likelihood of the data. In Supplemental Appendix F we describe the algorithm as well as illustrate the performance of our estimator using an example.

5.5. **Train-ticket estimation.** We now apply our outlined estimation to the train-ticket auction data. We restrict the sample to sequences where three tickets were sold. We also remove sequences where the same bidder was the second highest bidder more than once. This leaves us with 265 sequences, which translates into 265 bids in each auction round. The composition of unique bidders within a sequence can be found in Figure 4

The estimated density of valuations, density of *unconditional* worst-case beliefs, along with their associated probability distributions can be found in Figure 5. The probability distributions reveal a first order stochastic dominance relationship that is consistent with declining prices.

In the train-ticket data, we do not observe y_1, which is an input to the estimation of *conditional* worst-case beliefs. However, given estimates of $F(v)$ and the *unconditional* worst-case beliefs, $\bar{F}(v)$, we can back out the implied expected value of y_1 conditional on the second highest bid in the first auction. Call this backed out expected value $\tilde{y}_1$.

$$\tilde{y}_1 = \mathbb{E}_{\hat{F}}\left(Y_1|\phi_1\left(b_1^{(2)}(v|N,\hat{\bar{F}})\right) \leq Y_1\right)$$

To estimate *conditional* worst-case beliefs, we bin bids using $\tilde{y}_1$. The counts of $\tilde{y}_1$ in each bin is visualized in the histogram in Figure 6.

In figure 7 we report our estimates of bidders worst-case conditional beliefs. As we can see that bidders beliefs show a shift with changes in previous round prices, thus indicating dynamically

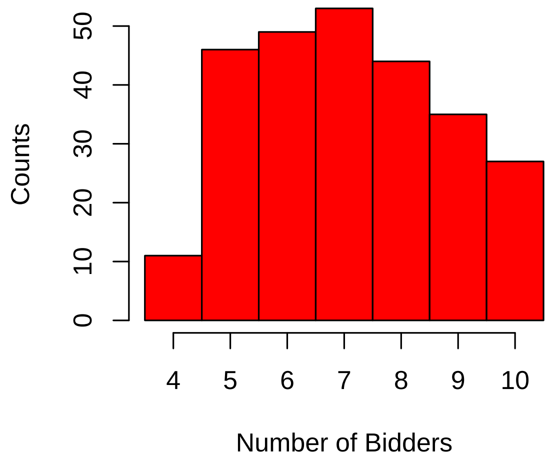


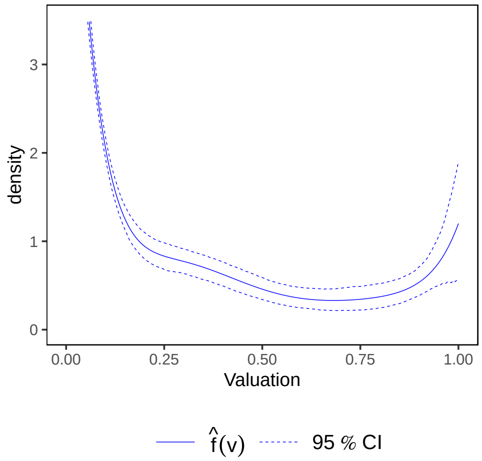
FIGURE 4. Histogram of number of bidders per sequence

inconsistent behavior. And as we can see from figure 5 we can see that bidders beliefs stochastically dominate the true distribution of valuations indicating the presence of ambiguity.

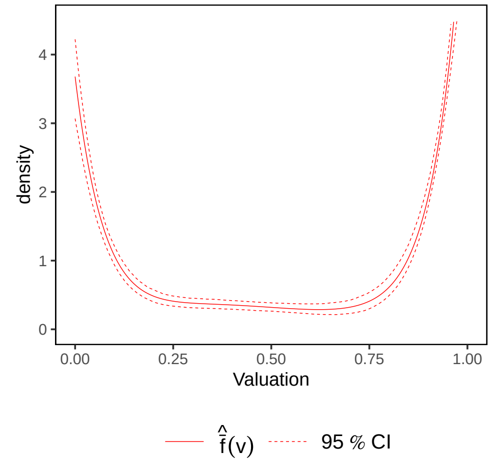
5.6. **Counter factual analysis.** We perform three counter factual experiments based on our recovered distributions of valuations and beliefs. First, is a robustness check of our method. We use our estimated model primitives to generate auction outcomes and compare them against the results from the reduced form analysis. Second, we check the revenue implications of ambiguity by calculating the revenue generated in a counterfactual model where the bidders beliefs were the same as the true (recovered) distribution of valuations. And finally, we calculate the revenue that would be generated in alternate auction formats to see if the seller could have done better/worse by using some other commonly used auction formats.

To offer evidence on the internal consistency of our approach we use the mean distributions, $\hat{F}(v)$ and $\hat{\hat{F}}(v)$, that we have estimated to simulate prices in sequences of auctions. For simplicity, we only use the unconditional worst-case distributions.[28] In this exercise we hold K fixed to be able to compare the price pattern in the simulated data to the price pattern estimated using equation (1) in the non-homogenized raw data. By holding K fixed we can normalize both patterns by the average price in auction K. Hence, we can compare the price pattern in terms of price ratios, where the normalizing price for an object k is the hypothetical average price that would have realized if no ambiguity were present. The result of this exercise for $K = 5$, which is the average length of a

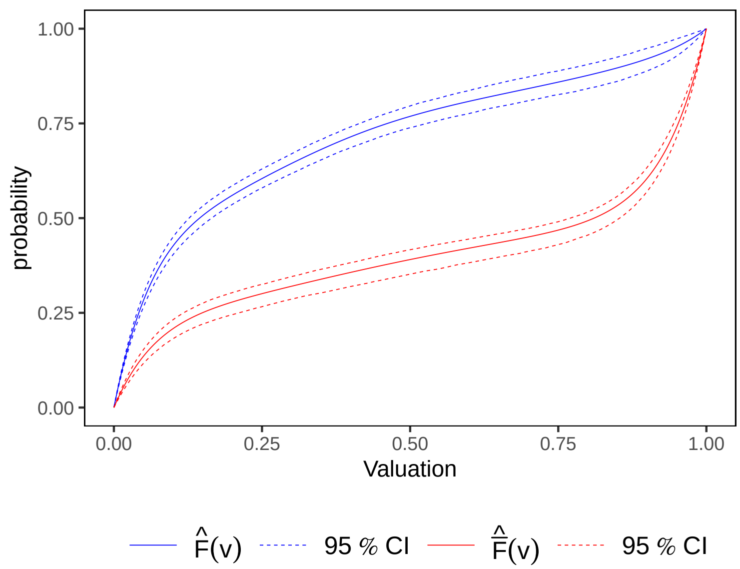
[28]This adds an additional bias in our estimates since we are ignoring the dynamic inconsistency.



(A) Density - valuations



(B) Density - *unconditional* worst case belief



(C) CDFs - valuation and *unconditional* worst case belief

FIGURE 5. Plotted distributions from estimation using train-ticket auction data. Data restrictions: 3 tickets for sale and different second highest bidder in each auction within each sequence ($J = 265$).

sequence in the data, can be seen in figure 8. The result suggests that the simulated price pattern falls within the 95% confidence interval of the price pattern estimated in the raw data.[29]

In the next part of the counter factual analysis we change some aspect of the environment to see what implications the change has for revenues. We first ask how much revenue would SJ lose if

[29]A joint test fails to reject the null hypothesis that the price trend in the raw data is the same as for the simulated price trend (p-value of an F-test of the dummy regression is 0.22, and the p-value of and F-test of the linear quadratic regression is 0.15).

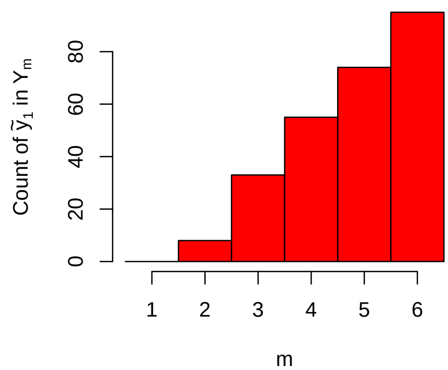


FIGURE 6. Observations per bin, Y_m

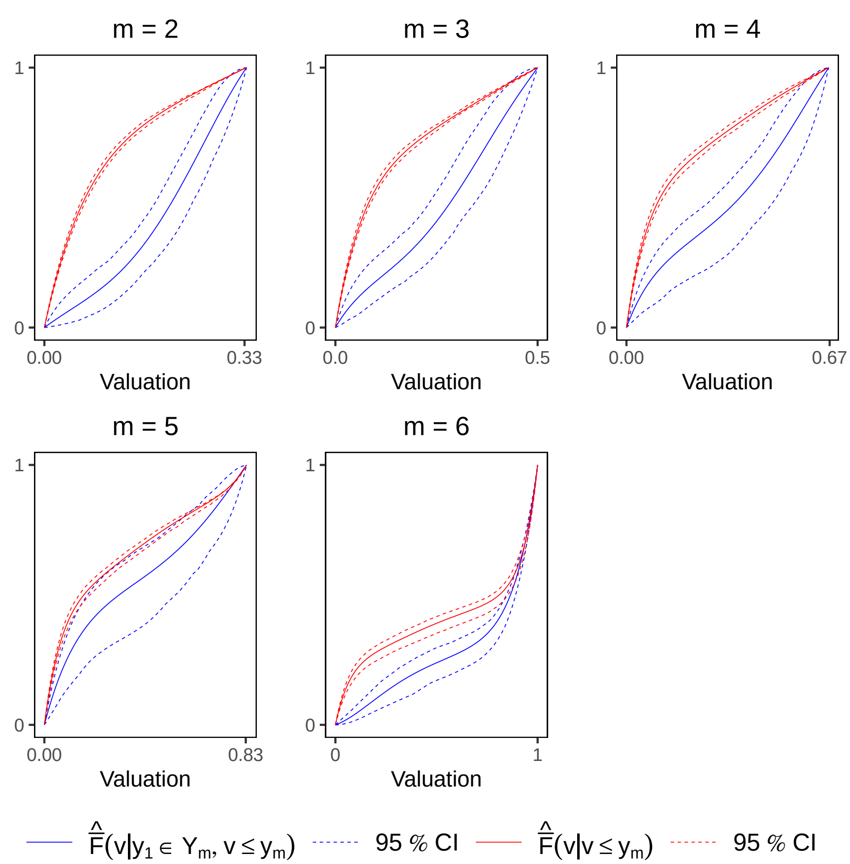
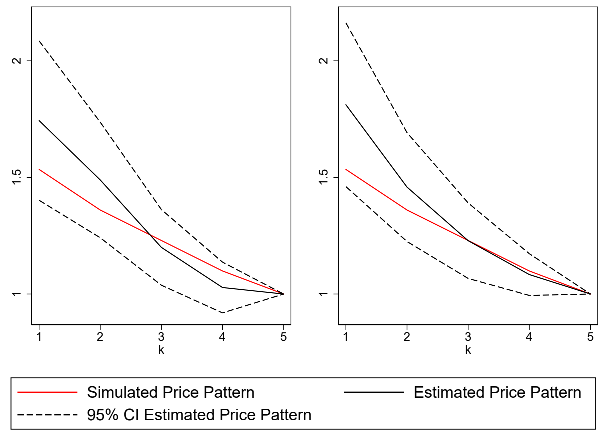


FIGURE 7. History dependent beliefs vs. beliefs without history

there was no uncertainty regarding $F(v)$? To do this exercise we compute bidders' optimal bids as if they knew the true distribution of (pseudo) values, $\hat{F}$.

FIGURE 8. Price Pattern - Simulated vs. Reduced Form



Note: The simulated price pattern is generated using the mean distributions, $\hat{F}(v)$ and $\hat{\hat{F}}(v)$. The sequences in the simulated sample holds length fixed at $K = 5$, and uses $N \in \{6,...,21\}$. The fraction of the sample with a particular N matches the fraction in the true data for $K = 5$. 87% of sequences where $K = 5$ have $N \in \{6,...,21\}$, 10% of sequences have $N \leq 5$, and the remainder has $N > 21$. The estimated price pattern is the normalized predictions from estimating equation (1) on the sub-sample where $K = 5$ and $N \in \{6,...,21\}$. The estimates are normalized by the predicted price in the last auction. The left panel predictions are based on a dummy regression: $ln(price_{kj}) = \sum_{i=1}^{K_j} \gamma_i D_{i,k} + \theta_j + \beta x_k + \varepsilon_{kj}$. The right panel is based on a prediction where a linear quadratic relationship is imposed: $ln(price_{kj}) = \beta_0 + \beta_1 k + \beta_2 k^2 + \beta_3 x_k + \theta_j + \varepsilon_{kj}$.

The second question we ask is how much revenue would SJ have gained by selling the tickets through sequences of first price auctions while maintaining the uncertainty regarding $F(v)$. From the estimation, since we know $\hat{F}, \hat{\hat{F}}$ and bidders' pseudo valuations we can do the above exercise by using the optimal bidding function for sequential FPAs from Ghosh and Liu (2021). Further, Bougt et al. (2024) show that sequential FPAs generate more revenue than sequential SPAs under some conditions. The results can be found in Table 3. The findings suggest that SJ's revenues would decrease by something in the range of 18.6 and 21.2% if there was no uncertainty regarding $F(v)$.[30] On the other hand, if SJ had been able to change the selling mechanism to a first price auction instead, then their revenues would increase by something in the range of 11.5 to 15.4%.

TABLE 3. Revenue Changes

	(1) Mean %Δ in Revenues	(2) CI
Remove Uncertainty	−19.7%	[−21.2%,−18.6%]
Change to first price	13.5%	[11.5%,15.4%]

Note: The changes in revenues are calculated using the revenue made in sequences of second price auctions when uncertainty is present as the baseline.

[30] The mean is given by $\frac{1-1.245}{1.245}$, while the CI is given by $[\frac{1-1.229}{1.229}, \frac{1-1.269}{1.269}]$. These numbers can be found in table 4.

TABLE 4. Revenue Comparisons

	(1) Average	(2) Lower Bound	(3) Upper Bound	(4) CI
No Ambiguity (distributions used)	$\hat{F}$	$\hat{F}_{.975}$	$\hat{F}_{.025}$	
first price=second price=uniform price	1.000	0.971	1.031	
Ambiguity (distributions used)	$\hat{F}$ $\hat{\hat{F}}$	$\hat{F}_{.975}$ $\hat{\hat{F}}_{.975}$	$\hat{F}_{.025}$ $\hat{\hat{F}}_{.025}$	
first price	1.412	1.329	1.509	[1.370, 1.464]
second price	1.245	1.192	1.307	[1.229, 1.269]
uniform price	1.000	0.971	1.031	

Note: The revenues are from sequences with (N,K) pairs that match the SJ data, where $1 \leq K \leq 10$ and $K < N \leq 20$. There are 5,279 sequences in the SJ data that satisfy this criteria. 1,000 sequences have been simulated for each (N,K) pair, after which the average revenue has been calculated. A weighted average of these averages have then been used to get total revenues. The weights correspond to the frequency in the real data of the (N,K) pair relative to the total number of sequences that satisfy the restriction above. Lastly, the revenues have been normalized by the revenues made in the hypothetical case of no ambiguity where valuations have been drawn from $\hat{F}(v)$ (column 1).

6. CONCLUSION

Using a data set comprised of bids placed in sequential train ticket auctions in Sweden, we studied various aspects of bidder behavior. Prices in these auctions show a declining trend, confirming the presence of the declining price anomaly. Using a similar model as Ghosh and Liu (2021) we showed that a model of sequential SPAs with ambiguity averse bidders can generate declining prices as well as positive history dependence (dynamic inconsistency) in bids in an ipv setting. The latter prediction distinguished our model from other theoretical explanations of the anomaly. Based on the theoretical model we carried out a structural estimation exercise that provided further evidence for the presence of ambiguity and dynamic inconsistency.

The declining price anomaly has been a long standing feature of sequential auctions for identical units of a good. It has been observed for many types of goods and auction mechanisms. In future work we would like to apply our methodology to other data sets and further understand the role of ambiguity and and experience over time.

APPENDIX A. PROOFS

A.1. **Proof of Proposition 3.3.** We solve for the equilibrium strategies backward starting from the final round. In the final round bidders have a weakly dominant strategy to bid their valuation. Given this, their payoff in the final round is given by

$$\min_{F_{y_{K-1}} \in \Delta_{y_{K-1}}} F_{y_{K-1}}(v)^{N-K} v - \int_{\underline{v}}^{v} x dF_{y_{K-1}}(x)^{N-K} = \min_{F_{y_{K-1}} \in \Delta_{y_{K-1}}} \int_{\underline{v}}^{v} (v-x) dF_{y_{K-1}}(x)^{N-K}$$

where $F_{y_{K-1}} = F(\cdot)/F(y_{K-1})$. Since $v-x$ is decreasing in x, the above payoff function is minimized by the lower envelope of $\Delta_{y_{K-1}}, \bar{F}(\cdot|y_{K-1})$. Now, define the following payoff function for round K.

$$\Gamma_K\left(v, y, F_{y_{K-2}}\right) = \int_{\underline{v}}^{v} (v-x) d\frac{F_{y_{K-2}}(x)^{N-K}}{F_{y_{K-2}}(y)^{N-K}}$$

This is the consistent planning bidder's evaluation of her round K payoff in round $K-1$ given some conditional belief about the distribution of values in round $K-1$, $F_{y_{K-2}} \in \Delta_{y_{K-2}}$ and the round $K-1$ winner's valuation y. Clearly Γ_K is increasing in v and decreasing in y. Now consider the payoff in round $K-1$. Suppose bidders are following some strategy β_{K-1} in this round that possibly depends on previous round prices and a bidder bids as if her valuation is $z \geq v$. Then,

$$\begin{aligned}\Pi_{K-1}(v,z,y_{K-2}) = &\min_{F_{y_{K-2}} \in \Delta_{y_{K-2}}} \int_{\underline{v}}^{z} \left(v - \beta_{K-1}(x, y_{K-2})\right) dF_{y_{K-2}}(x)^{N-K+1} \\ &+ \int_{z}^{y_{K-2}} \Gamma_K(v, x, F_{y_{K-2}}) dF_{y_{K-2}}(x)^{N-K+1}\end{aligned} \tag{11}$$

Let $\hat{F}_{y_{K-2}}$ minimize the above payoff. Taking first order conditions with respect z and setting $z = v$, we get

$$v - \beta_{K-1}(v, y_{K-2}) = \Gamma_K(v, v, \hat{F}_{Y_{K-2}}) = v - \int_{\underline{v}}^{v} x d\frac{\hat{F}_{y_{K-2}}(x)^{N-K}}{\hat{F}_{y_{K-2}}(v)^{N-K}} \implies \beta_{K-1}(v, y_{K-2}) = \int_{\underline{v}}^{v} \beta_K(x) d\frac{\hat{F}_{Y_{K-2}}(x)^{N-K}}{\hat{F}_{Y_{K-2}}(v)^{N-K}}.$$

Note that for $x \leq v \leq y$,

$$v - \beta_{K-1}(x, y_{K-2}) \geq v - \beta_{K-1}(v, y_{K-2}) = \Gamma_K(v, v, \hat{F}_{Y_{K-2}}) \geq \Gamma_K(v, y, \hat{F}_{Y_{K-2}})$$

since $\beta_{K-1}(\cdot, y_{K-2})$ is increasing and Γ_K is non-increasing in the second argument. Thus, inspecting (11) it is clear that in $\hat{F}_{Y_{K-2}}(\cdot) = \bar{F}(\cdot|y_{K-2})$. Thus

$$\beta_{K-1}(v, y_{K-2}) = \int_{\underline{v}}^{v} \beta_K(x) d\frac{\bar{F}(x|y_{K-2})^{N-K}}{\bar{F}(v|y_{K-2})^{N-K}}.$$

Next, we consider the consistent planning continuation payoff function Γ_{K-1}.

$$\Gamma_{K-1}(v,y,F_{y_{K-3}}) = \int_{\underline{v}}^{v} (v-\beta_{K-1}(x,y))\, d\frac{F_{y_{K-3}}(x)^{N-K+1}}{F_{y_{K-3}}(y)^{N-K+1}} + \int_{v}^{y} \Gamma_K(v,x,F_{y_{K-3}}) d\frac{F_{y_{K-3}}(x)^{N-K+1}}{F_{y_{K-3}}(y)^{N-K+1}}$$
$$= \int_{\underline{v}}^{v} \left(v - x + \int_{\underline{v}}^{x} \frac{\bar{F}(z|y)^{N-K}}{\bar{F}(x|y)^{N-K}} dz \right) d\frac{F_{y_{K-3}}(x)^{N-K+1}}{F_{y_{K-3}}(y)^{N-K+1}}$$
$$+ \int_{v}^{y} \Gamma_K(v,x,F_{y_{K-3}}) d\frac{F_{y_{K-3}}(x)^{N-K+1}}{F_{y_{K-3}}(y)^{N-K+1}}.$$

Note that the first term in the above is non-increasing in y due to the Envelope Theorem. For the second term, note that the derivative with respect to y,

$$(N-K+1)\frac{f_{y_{K-3}}(y)}{F_{y_{K-3}}(y)} \left(\Gamma_K(v,y,F_{y_{K-3}}) - \int_{v}^{y} \Gamma_K(v,x,F_{y_{K-3}}) d\frac{F_{y_{K-3}}(x)^{N-K+1}}{F_{y_{K-3}}(y)^{N-K+1}} \right) \leq 0$$

since $\Gamma_K(v,y,F)$ is non-increasing in y as we established in step 1 of the proof. Thus, $\Gamma_{K-1}(v,y,F_{y_{K-3}})$ is non increasing in y.

A bidder's payoff function in round $K-2$ is given by

$$\Pi_{K-2}(v,z,y_{K-3}) = \min_{F_{y_{K-3}} \in \Delta_{y_{K-3}}} \int_{\underline{v}}^{z} (v-\beta_{K-2}(x,y_{K-3}))\, F_{y_{K-3}}(x)^{N-K+2}$$
$$+ \int_{z}^{y_{K-3}} \Gamma_{K-1}(v,x,F_{y_{K-3}}) dF_{y_{K-3}}(x)^{N-K+2}.$$

Let $\hat{F}_{K-3}$ minimize the above payoff function. Then, first order conditions and $z=v$ imply

$$v-\beta_{K-2}(v,y_{K-3}) = \Gamma_{K-1}(v,v,\hat{F}_{y_{K-3}}) = \int_{\underline{v}}^{v} (v-\beta_{K-1}(x,v))\, d\frac{\hat{F}_{y_{K-3}}(x)^{N-K+1}}{\hat{F}_{y_{K-3}}(v)^{N-K+1}} \implies$$
$$\beta_{K-2}(v,y_{K-3}) = \int_{\underline{v}}^{v} \beta_{K-1}(x,v) d\frac{\hat{F}_{y_{K-3}}(x)^{N-K+1}}{\hat{F}_{y_{K-3}}(v)^{N-K+1}}$$

Note that

$$\int_{\underline{v}}^{v} \beta_{K-1}(x,v) d\frac{\hat{F}_{y_{K-3}}(x)^{N-K+1}}{\hat{F}_{y_{K-3}}(v)^{N-K+1}} = \int_{\underline{v}}^{v} \left(x - \int_{\underline{v}}^{x} \frac{\bar{F}(z|v)^{N-K}}{\bar{F}(x|v)^{N-K}} dz \right) d\frac{\hat{F}_{y_{K-3}}(x)^{N-K+1}}{\hat{F}_{y_{K-3}}(v)^{N-K+1}}$$

Again, it is straightforward to show that the above is increasing in v using the Envelope Theorem. Finally, note that

$$v-\beta_{K-2}(x,y_{K-3})\geq v-\beta_{K-2}(v,y_{K-3})=\Gamma_{K-1}(v,v,\hat{F}_{y_{K-3}})\geq\Gamma_{K-1}(v,y,\hat{F}_{y_{K-3}})$$

where the second inequality follows monotonicity of $\beta_{K-2}(\cdot,y_{K-3})$ and the fourth inequality follows from the monotonicity of $\Gamma_{K-1}(v,\cdot,F)$. Thus $\hat{F}_{y_{K-3}}=\bar{F}(\cdot|y_{K-3})$ and

$$\beta_{K-2}(v,y_{K-3})=\int_{\underline{v}}^{v}\beta_{K-1}(x,v)d\frac{\bar{F}(x|y_{K-3})^{N-K+1}}{\bar{F}(v|y_{K-3})^{N-K+1}}$$

The proof for the remaining rounds follows exactly the same procedure. The independence of $\bar{F}(\cdot|y)$ from the bidders valuation follows from assumption 3.1. For a proof see Step 5 of the proof of Proposition 4.1 of Ghosh and Liu (2021).

A.2. **Proof of Proposition 3.5.** Note that

$$\begin{aligned}\mathbb{E}[P_{k+1}|p_k]&=\int_{\underline{v}}^{y_{k+1}}\beta_{k+1}(x,y_k)dF_2^{(N-k)}(x|x\leq y_{k+1})\\&=(N-k)\int_{\underline{v}}^{y_{k+1}}\beta_{k+1}(x,y_k)d\frac{F(x)^{N-k-1}}{F(y_{k+1})^{N-k-1}}-(N-k-1)\int_{\underline{v}}^{y_{k+1}}\beta_{k+1}(x,y_k)d\frac{F(x)^{N-k}}{F(y_{k+1})^{N-k}}\end{aligned}$$

Now, for $k=K-1$,

$$\begin{aligned}p_{K-1}&=\beta_{K-1}(y_K,y_{K-2})\\&=\int_{\underline{v}}^{y_K}\beta_K(x)d\left(\frac{\bar{F}(x|y_{K-2})}{\bar{F}(y_{k+1}|y_{K-2})}\right)^{N-K}\\&\geq\int_{\underline{v}}^{y_K}\beta_K(x)d\left(\frac{F(x|y_{K-2})}{F(y_{k+1}|y_{K-2})}\right)^{N-K}\\&=(N-K+1)\int_{\underline{v}}^{y_K}\beta_K(x)d\frac{F(x)^{N-K}}{F(y_K)^{N-K}}-(N-K)\int_{\underline{v}}^{y_K}\beta_K(x)d\frac{F(x)^{N-K}}{F(y_K)^{N-K}}\\&>(N-K+1)\int_{\underline{v}}^{y_K}\beta_K(x)d\frac{F(x)^{N-K}}{F(y_K)^{N-K}}-(N-K)\int_{\underline{v}}^{y_K}\beta_K(x)d\frac{F(x)^{N-K+1}}{F(y_K)^{N-K+1}}\\&=\mathbb{E}[P_K|p_{K-1}]\end{aligned}$$

where the third equality follows from the supposition in the statement of the proposition.

In second price auctions, since prices are determined by the second highest values, y_k and y_{k+1} are valuations that determine prices in rounds $k-1$ and k. Thus, the supposition in the second part of the Proposition does restrict the valuation that determines prices in round $k+1$. Now, for the second part of the result, suppose $y_k = y_{k+1}$. Then,

$$\mathbb{E}\left[P_{k+1}|p_k\right] = (N-k)\int_{\underline{v}}^{y_k} \beta_{k+1}(x,y_k)d\frac{F(x)^{N-k-1}}{F(y_k)^{N-k-1}} - (N-k-1)\int_{\underline{v}}^{y_k} \beta_{k+1}(x,y_k)d\frac{F(x)^{N-k}}{F(y_k)^{N-k}}.$$

Now, consider

$$\begin{aligned}
p_k &= \beta_k(y_{k+1},y_{k-1}) \\
&= \beta_k(y_k,y_{k-1}) \\
&= \int_{\underline{v}}^{y_k} \beta_{k+1}(x,y_k)d\frac{\bar{F}(x|y_{k-1})^{N-k-1}}{\bar{F}(y_k|y_{k-1})^{N-k-1}} \\
&\geq \int_{\underline{v}}^{y_k} \beta_{k+1}(x,y_k)d\frac{F(x|y_{k-1})^{N-k-1}}{F(y_k|y_{k-1})^{N-k-1}} \\
&= (N-k)\int_{\underline{v}}^{y_k} \beta_{k+1}(x,y_k)d\frac{F(x)^{N-k-1}}{F(y_{k+1})^{N-k-1}} - (N-k-1)\int_{\underline{v}}^{y_k} \beta_{k+1}(x,y_k)d\frac{F(x)^{N-k-1}}{F(y_{k+1})^{N-k-1}} \\
&> \mathbb{E}\left[P_{k+1}|p_k\right]
\end{aligned}$$

Then due to continuity of the bid functions, the second part of the result follows from the final inequality.

A.3. **Proof of Proposition 3.6.** The first part of the proof as the same as in the previous result. For the second part note that for any k and $y \leq y'$,

$$\begin{aligned}
\beta_k(x,y) &= \int_{\underline{v}}^{x} \beta_{k+1}(z,x)d\frac{\bar{F}(z|y)^{N-k-1}}{\bar{F}(x|y)^{N-k-1}} \\
&\geq \int_{\underline{v}}^{x} \beta_{k+1}(z,x)d\frac{\bar{F}(z|y')^{N-k-1}}{\bar{F}(x|y')^{N-k-1}} \\
&= \beta_k(x,y').
\end{aligned} \tag{12}$$

Note,

$$\begin{aligned}
p_k &= \beta_k(y_{k+1}, y_{k-1}) \\
&= \int_{\underline{v}}^{y_{k+1}} \beta_{k+1}(z, y_{k+1}) d\frac{\bar{F}(z|y_{k-1})^{N-k-1}}{\bar{F}(y_{k+1}|y_{k-1})^{N-k-1}} \\
&\geq \int_{\underline{v}}^{y_{k+1}} \beta_{k+1}(z, y_k) d\frac{\bar{F}(z|y_{k-1})^{N-k-1}}{\bar{F}(y_{k+1}|y_{k-1})^{N-k-1}} \\
&\geq \int_{\underline{v}}^{y_{k+1}} \beta_{k+1}(z, y_k) d\frac{F(z)^{N-k-1}}{F(y_{k+1})^{N-k-1}} \\
&= (N-k) \int_{\underline{v}}^{y_{k+1}} \beta_{k+1}(x, y_k) d\frac{F(x)^{N-k-1}}{F(y_{k+1})^{N-k-1}} - (N-k-1) \int_{\underline{v}}^{y_{k+1}} \beta_{k+1}(x, y_k) d\frac{F(x)^{N-k-1}}{F(y_{k+1})^{N-k-1}} \\
&> (N-k) \int_{\underline{v}}^{y_{k+1}} \beta_{k+1}(x, y_k) d\frac{F(x)^{N-k-1}}{F(y_{k+1})^{N-k-1}} - (N-k-1) \int_{\underline{v}}^{y_{k+1}} \beta_{k+1}(x, y_k) d\frac{F(x)^{N-k}}{F(y_{k+1})^{N-k}} \\
&= \mathbb{E}[P_{k+1}|p_k]
\end{aligned}$$

where the second inequality follows from (12) and the third inequality follows from the first supposition in the statement of the proposition.

For the final result, continuity also implies that we prices will decline for 'close-by' distributions as well which may not satisfy the reverse hazard rate condition.

A.4. **Proof of Proposition 5.1. Identification of v and $\hat{F}$:** In sequential SPAs bidders bid their valuations in the final round. Thus, from bids in the final round, we can immediately identify the valuations of bidders remaining in the final round. Thus we know the $N-K+1$-th, $N-K$-th and so on order statistics from $\hat{F}$. Thus, the true data generating process is identified from bids in the final round. Given $\hat{F}$, note that due to monotonicity of equilibrium bidding in valuations, all valuations from each round can be identified by matching percentiles bid distributions with their corresponding percentiles in $\hat{F}$.

Identification of $\bar{F}(\cdot|y)$: For notational parsimony, assume that N and K are fixed. Then, let $G_k(b_k|\tilde{y}_{k-1})$ represent the bid distribution in the k-th round conditional on the price history, and $g_k(b_k|\tilde{y}_{k-1})$ the corresponding bid densities. Abusing notation we will write $\tilde{y}_{k-1}$ as just y_{k-1}. Conditional on previous round price, let $\phi_k(\cdot|y_{k-1}) = \beta_k^{-1}(\cdot|y_{k-1})$ be the inverse bidding strategy. In equilibrium, bidding strategies are monotone and increasing. Thus,

(13) $$G_k(b_k|y_{k-1}) = F(\phi_k(b_k|y_{k-1})) \text{ and } g_k(b_k|y_{k-1}) = f(\phi_k(b_k|y_{k-1}))\phi_k'(b_k|y_{k-1}).$$

Now, consider round $K-1$. First order conditions for an optimal bid imply that conditional on winning round $K-1$, bidders payoff on round $K-1$ must equal her expected payoff from the next round. That is

$$v - b_{K-1} = \Pi_K(v, v, \phi_{K-1}(b_{K-1})) = \int_{\underline{v}}^{v} (v-x) d\left(\frac{\bar{F}(x|y_{K-2})}{\bar{F}(v|y_{K-2})}\right)^{N-K}.$$

Now, note that the above can also be written as

$$b_{K-1}\bar{F}(\phi_{K-1}(b_{K-1}|y_{K-2}))^{N-K} = \int_{\underline{v}}^{\phi_{K-1}(b_{K-1}|y_{K-2})} x d\bar{F}(x|y_{K-1})^{N-K}.$$

Taking derivative of the above with respect to b_{K-1} we get

$$\begin{aligned}
&\bar{F}(\phi_{K-1}(b_{K-1}|y_{K-2}))^{N-K} + \\
&b_{K-1}(N-K)\bar{F}(\phi_{K-1}(b_{K-1}|y_{K-2}))^{N-K-1}\bar{f}(\phi_{K-1}(b_{K-1}|y_{K-2}))\phi'_{K-1}(b_{K-1}|y_{K-2}) = \\
&\phi_{K-1}(b_{K-1}|y_{K-2})(N-K)\bar{F}(\phi_{K-1}(b_{K-1}|y_{K-2}))^{N-K-1}\bar{f}(\phi_{K-1}(b_{K-1}|y_{K-2}))\phi'_{K-1}(b_{K-1}|y_{K-2}).
\end{aligned}$$

Rearranging and using the fact that $\phi_K(b_K|y_{K-2}) = v$ for any y_{K-2}, we have

$$\begin{aligned}
v &= b_{K-1} + \frac{\bar{F}(\phi_{K-1}(b_{K-1}|y_{K-2}))}{(N-K)\bar{f}(\phi_{K-1}(b_{K-1}|y_{K-2}))\phi'_{K-1}(b_{K-1}|y_{K-2})} \\
&= b_{K-1} + \frac{\bar{F}(v|y_{K-2})}{(N-K)\bar{f}(v|y_{K-2})\phi'_{K-1}(b_{K-1}|y_{K-2})}.
\end{aligned} \tag{14}$$

Substituting from (13) we get

$$\frac{\bar{f}(v|y_{K-2})}{\bar{F}(v|y_{K-2})} = \frac{f(v)}{(N-K)(b_K - b_{K-1})g_{K-1}(b_{K-1}|y_{K-2})}. \tag{15}$$

where f is estimated using final round bids. Thus, with variation in $y_{K-2}, \bar{F}(\cdot|y)$ is identified from the above equation along with the condition $\bar{F}(y|y) = 1$. From this part of the proof we can also see that in order to identify worst-case conditional beliefs, data from the final two rounds is sufficient. Using a similar approach we can add to the power of our tests by estimating worst-case conditional beliefs from other rounds, as they only depend on the previous round y.

Identification of $\bar{F}$: Once $\bar{F}(\cdot|y)$ have been identified, note that we can estimate $\beta_k(x,y)$ for any $K > k > 1$, where $y_{k-1} = y$. Specifically, since β_2 can be estimated, note from the closed form of the equilibrium

$$b_1 = \int_{\min b_{i,1}}^{\phi_1(b_1)} \beta_2(x, \phi_1(b_1)) d\frac{\bar{F}(x)^{N-2}}{\bar{F}(\phi_1(b_1))^{N-2}}$$

Now, since all v have already been estimated for each bidder in each round, we know the value of $\phi_1(b_1)$. Thus, $\bar{F}$ can be estimated from the above equality. Alternatively, note that $\bar{F}(\cdot|\bar{v}) = \bar{F}(\cdot)$, thus we can also use estimates of $\bar{F}(\cdot|y)$.

APPENDIX B. OTHER MODELS

B.1. **Aversion to price risk.** Bidder's preferences are defined as

$$\text{Payoff from winning} = v - l(payment); \quad l''(\cdot) \geq 0$$

where l is increasing and convex. Now consider $K = 3$, and N bidders. Much like bidding in a second price auction, the final round with aversion to price risk has dominant strategies where bidders bid

$$\beta_3^{risk}(v) = l^{-1}(v).$$

Then first order conditions in the second round imply

$$v - l\left(\beta_2^{risk}(v, y_1)\right) = \int_{\underline{v}}^{v} (v - l(l^{-1}(x))d\frac{(F(x)/F(y_1))^{N-3}}{(F(v)/F(y_1))^{N-3}} \implies \beta_2(v) = l^{-1}\left(\int_{\underline{v}}^{v} xd\frac{F(x)^{N-3}}{F(v)^{N-3}}\right)$$

where the first term in the left hand is the payoff in the second round conditional on being a pivotal bidder and the term after the first equality is the expected payoff from winning the final round. Note that bidding in the second round is *history-independent*. With these bid functions,

$$p_2 = \beta_2^{risk}(y_3) = l^{-1}\left(\int_{\underline{v}}^{y_3} xd\frac{F(x)^{N-3}}{F(y_3)^{N-3}}\right) \geq \int_{\underline{v}}^{y_3} l^{-1}(x)d\frac{F(x)^{N-3}}{F(y_3)^{N-3}} = \mathbb{E}[P_3|p_2],$$

where the inequality is due to Jensen's inequality. Similarly, we can show that prices between round 1 and 2 are also a super-martingale.

B.2. **Loss aversion.** Under these preferences,

$$\text{Payoff in round } k = \begin{cases} v - payment; \text{ if } b_k > \max b_{j,k} \\ -\Lambda v \frac{F(v)^{N-k}}{F(y_{k-1})^{N-k}}; \text{ otherwise} \end{cases}$$

where Λ is a parameter of loss-aversion. Consider $K = 3$ and N bidders. Using the standard argument, bidding in the final round is

$$\beta_3^{loss}(v, y_2) = v + \Lambda v \frac{F(v)^{N-3}}{F(y_2)^{N-3}}$$

Note that bidding in the final round is negatively related to the price/valuation of the winner in the second.

Now, using first order conditions

$$v-\beta_2^{loss}(v,y_1)+\Lambda v\frac{F(v)^{N-2}}{F(y_1)^{N-2}}=\int_{\underline{v}}^{v} v-\beta_3^{loss}(x,v)d\frac{F(x)^{N-3}}{F(v)^{N-3}}$$

$$\implies \beta_2^{loss}(v,y_1)=\int_{\underline{v}}^{v}\beta_3^{loss}(x,v)d\frac{F(x)^{N-3}}{F(v)^{N-3}}+\Lambda v\frac{F(v)^{N-2}}{F(y_1)^{N-2}}$$

Again, bidding in the second round is inversely related to first round winning bid. Calculating prices we get that

$$p_2=\beta_2^{loss}(y_3,y_1)=\int_{\underline{v}}^{y_3}\beta_3^{loss}(x,y_3)d\frac{F(x)^{N-3}}{F(y_3)^{N-3}}+\Lambda v\frac{F(y_3)^{N-2}}{F(y_1)^{N-2}}>\int_{\underline{v}}^{y_3}\beta_3^{loss}(x,y_3)d\frac{F(x)^{N-3}}{F(y_3)^{N-3}}$$

$$>\int_{\underline{v}}^{y_3}\beta_3^{loss}(x,y_2)d\frac{F(x)^{N-3}}{F(y_3)^{N-3}}=\mathbb{E}[P_3|p_2]$$

Thus the model generates prices that are a super-martingale and bidding that is negatively history dependent.

APPENDIX C. DATA

C.1. **Summary statistics.** The data set contains all bids made in the auctions for train tickets. A bidder who engaged in incremental bidding could have recorded multiple bids in the same auction. We therefore treat the highest bid that a bidder records in an auction as the bidder's revealed bidding strategy. A further complication is that we cannot treat most winning bids as revealed strategies. That is due to proxy bidding and the fact that most auctions are settled using the second price rule (see table 5). Thus, we treat the winning bids as prices only, and treat non-winning bids as revealed bidding strategies. In addition to the bids, the data also contains bidder identifiers and the date, hour, and minute that the bid was placed.

In the following sections we describe some features of the data. In many cases we divide the data into two categories. One is the set of all bids that were submitted. The other is a subset of bids that were placed in an auction *after* the previous round of the sequence ended. These bids capture the sequential nature of the auctions.

As we discussed before, the bidding mechanism implies that winners of an auction on occasion have to pay their own bid. This happens about 9 percent of the time (see table (5)). If one considers auctions where the winning price was greater than 99 SEK, the share increases to 14 percent.

Another feature of the proxy bidding mechanism is that the increment that a bidder must raise the current leading bid by increases as the current leading bid increases. This has implications

TABLE 5. Pricing Rule

	All Auctions		Price> 99	
	Number	Percent	Number	Percent
price=2nd bid + increment	31,941	91	14,995	86
price=1st bid	3,216	9	2,364	14
Total	35,157	100	17,379	100

for the available bidding range to a bidder. As can be seen in table 6, at the time of placing their highest bid, 72 percent of bidders were free to place any bid as long as it was higher than the current leading bid. 20 percent of bidders were restricted to place a bid that surpassed the current leading bid by at least 5 SEK.

TABLE 6. Highest Bid by Increment Category

Increment	Number	Percent
1 SEK	107,254	72
5 SEK	29,926	20
10 SEK	12,226	8
25 SEK	6	0
missing	72	0
Total	149,484	100

TABLE 7. Summary Statistics: Auctions

	Auctions=35157	
	Mean	SD
Price	131.90	128.43
Bidders in auction	4.25	2.31
Bids	11.02	8.79

On average 4 bidders participated in each auction and 11 in a sequence. The average sequence consisted of 5 tickets. Almost all tickets were for either intercity or fast train trips. There was a fairly even distribution of the weekday of train departure. The auctions were conducted in close succession with less than ten minutes between closing times. This suggests that there was no discounting for tickets sold in later rounds of a sequence. The average price of tickets was 131.90 SEK, not conditioning on train types. The average price decline between two auctions within a sequence was about 9%.

Turning to all the bids, the average bid in the auctions was about 90.69 SEK. Importantly this was below the price at which the bid increments changed. In addition, on average, bidders led the auction only once. That is bidders typically did not lead the auction multiple times which suggests that bidders were bidding as if they were in a sealed bid mechanism.

TABLE 8. Summary Statistics: Sequences

	Sequences=6874	
	Mean	SD
Bidders in sequence	10.75	6.48
Tickets in sequence	5.11	2.82
Traintype: Intercity	0.48	0.50
Traintype: Regional	0.01	0.08
Traintype: X2000	0.52	0.50
Sunday	0.07	0.26
Monday	0.15	0.36
Tuesday	0.16	0.36
Wednesday	0.18	0.38
Thursday	0.16	0.37
Friday	0.14	0.34
Saturday	0.14	0.35
Minutes between auctions	8.65	5.96

TABLE 9. Summary Statistics: Bids

	All Bids Bids=387579		Highest Bids Bids=149484		Non-Winning Bids Bids=114327	
	Mean	SD	Mean	SD	Mean	SD
Bid	90.69	100.76	106.20	114.43	98.30	108.56
Share incremental bid	0.28	0.45	0.24	0.43	0.23	0.42
Share auction elapsed	89.36	22.43	87.58	24.51	85.94	25.74
Share leading bid	0.42	0.49	0.70	0.46	0.61	0.49
Bids by bidder	2.59	3.02	.	.	.	
Leading bids by bidder	1.08	0.95	.	.	.	
Share ever leading	0.78	0.41	.	.	.	
Share leading and returning	0.25	0.43	.	.	.	

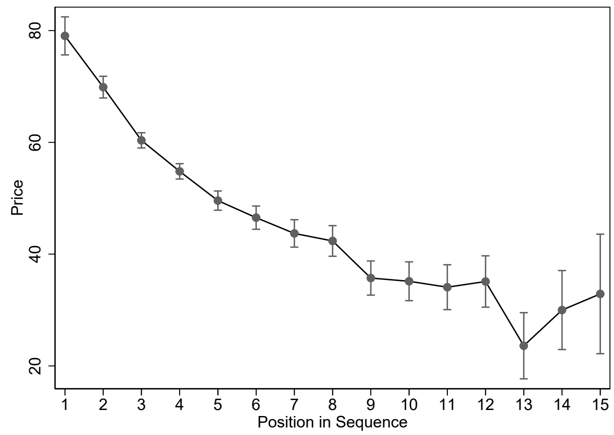
APPENDIX D. ROBUSTNESS - DECLINING PRICES

In section 2.2 we document declining prices by assuming a linear-quadratic relationship between log prices and the position in a sequence. This result is robust to alternative specifications as well. In particular, consider the event study specification

$$ln(price_{kj}) = \sum_{i=1}^{15} \gamma_i D_{i,k} + \theta_j + \beta x_k + \varepsilon_{kj}. \tag{16}$$

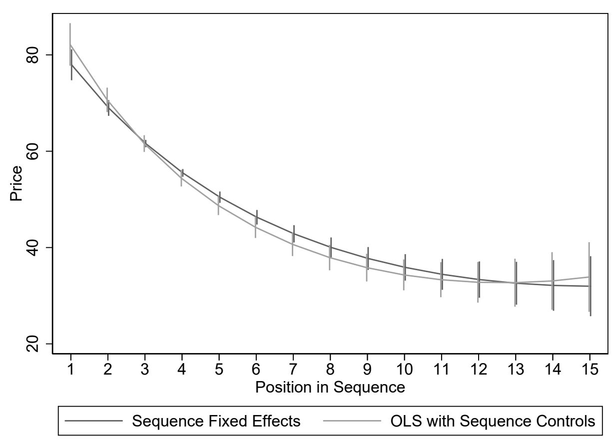
Here $D_{i,k} = 1$ if $i = k$ and 0 otherwise. Figure 9 plots the estimated effects of such a specification. Again, the declining price path is clear also when no particular functional form is assumed.

FIGURE 9. Visual - Declining Price Dummy



In table 1 in section 2.2 we also include the results from an estimation where the the sequence fixed effects, θ_j, are replaced with a set of sequence covariates x_j. To complement the coefficients reported in column (3), here we include a graph where the estimated effect at the averages of sequence covariates overlay the estimated effect from the fixed effect regression. The results is captured in figure 10.

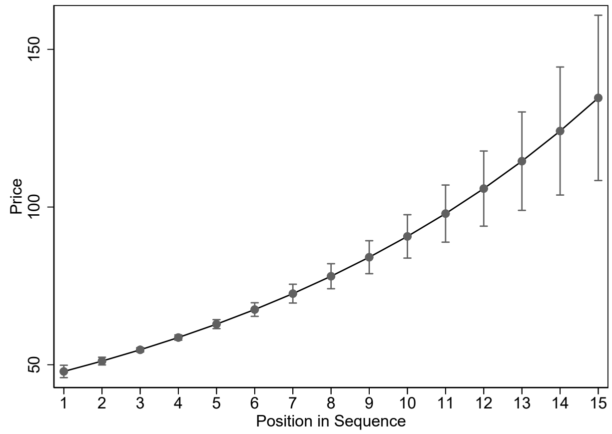
FIGURE 10. Visual - Fixed effect vs. sequence controls



Note: Estimated marginal effects at each position in the sequence. For OLS, the marginal effects have been computed at the average of sequence covariates.

When documenting the declining price anomaly, we pool sequences of various length. One concern then is that this gives rise to some spurious relationships that are misinterpreted as declining prices. To address this, we first reverse the assignment of a position within a sequence. In the original estimation, we assign $k = 1$ to the auction ending first within a sequence, $k = 2$ to the auction ending second, and so on. We know re-estimate equation (1) but assign $k = 1$ to the auction ending last in a sequence, $k = 2$ to the auction ending second to last, and so on. We should then expect a figure with the same concavity as in figure 3, but trending upwards. The result is displayed in figure 11.

FIGURE 11. Visual - Auction order reversed



Note: Estimated marginal effects for each position in a sequence based on estimating equation (1), but assigning $k = 1$ to the auction ending last, $k = 2$ to the auction ending second to last, and so on.

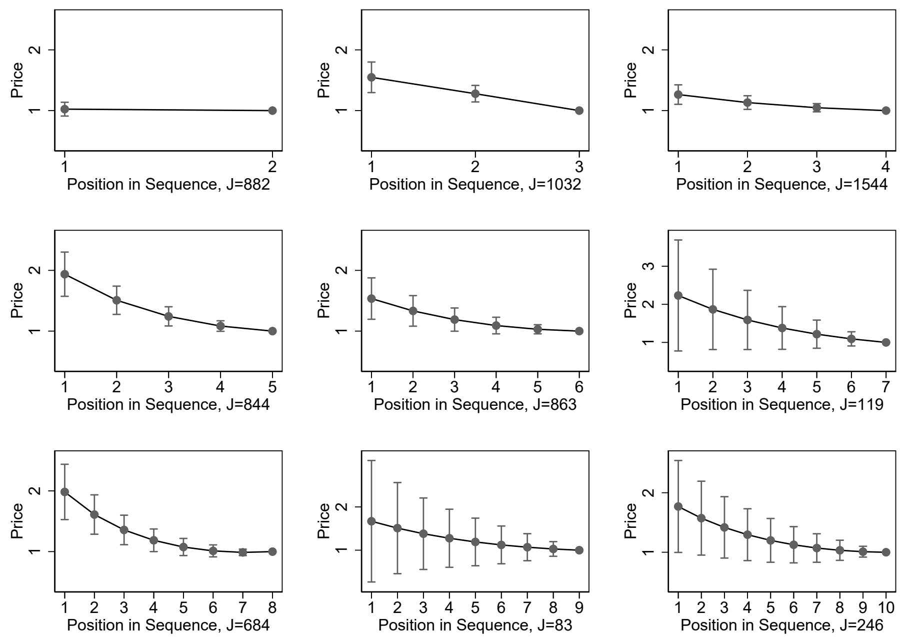
In addition to reversing the order of auctions, we can also document the pattern holding the number of items fixed. That is estimating equation (1) separately for each K. Figure 12 displays the results for $K = 2, ..., 10$. We have here, for comparability chosen to normalize the price pattern with the price in auction K. As can be seen, declining prices is present in for all K apart from $K = 2$ (although not significant for some K), which displays a flat price trend.

In appendix E, we derive the bidding functions when sequence heterogeneity enters multiplicatively. We show that $b_{i,j,k} = a_j \beta_{k,N_j}(v_i)$ when $v_{ij} = a_j v_i$. This means that $p_{k,j} = a_j \beta_{k,N_j}(v^{k:N_j})$, where $v^{k:N_j}$ is the kth highest valuation out of N_j valuations. If the assumption that auction heterogeneity enters multiplicatively is correct, then we should be able to construct similar estimates of price ratios $P_{k,j} = \frac{p_{k+1,j}}{p_{k,j}}$ by estimating equation 16 as we do by estimating a log differences equation without fixed effects.

APPENDIX E. HOMOGENIZING BIDS

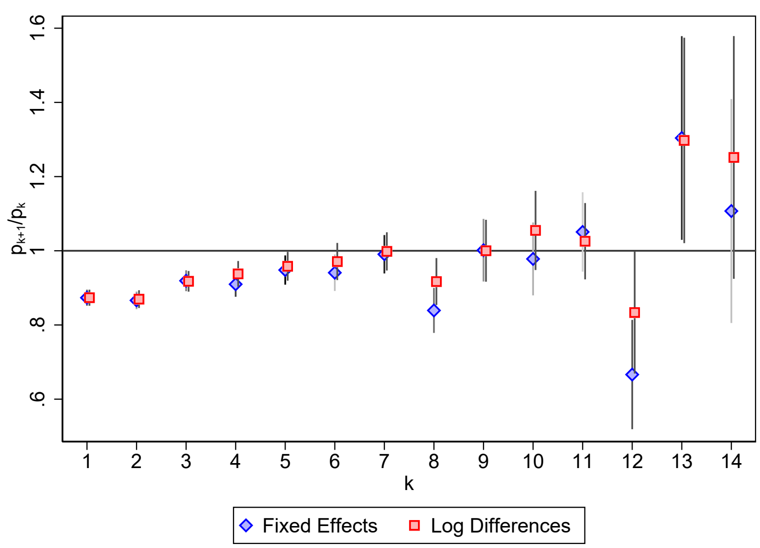
E.1. **Sequence heterogeneity.** In order to homogenize bids across sequences we follow the following procedure. Let there be J sequences. Tickets sold within a sequence are identical. Let N_j and K_j represent the number of bidders and the number of tickets in sequence j. Bidder i's valuation for a ticket sold in sequence j is given by $v_{i,j}$. A bidder's valuation depends on a privately drawn signal, v_i, as in the benchmark theoretical model. In addition, there is a sequence specific

FIGURE 12. Visual - Fixed number of items



Note: The predicted price pattern from estimating equation (1) holding K fixed. All estimates are normalized by the predicted price in auction K, and confidence intervals are based on standard errors calculated using the delta method.

FIGURE 13. Price ratio



Note: Plot of estimated price ratios from a fixed effects regression and a first differences regression. The Fixed Effects coefficients are from the following regression $ln(price_{kj}) = \sum_{i=1}^{15} \gamma_i D_{i,k} + \theta_j + \varepsilon_{kj}$. The Log Differences coefficients are from the following regression $\Delta ln(price_{k,j}) = \sum_{i=1}^{14} \gamma_i' D_{i,k} + \Delta\varepsilon_{k,j}$, where $\Delta ln(price_{k,j}) = ln(price_{k+1,j}) - ln(price_{k,j})$ and $\Delta\varepsilon_{k,j} = \varepsilon_{k+1,j} - \varepsilon_{k,j}$.

signal, common to all bidders, a_j. This signal is commonly known to all bidders participating in the sequence j.

Multiplicative: One way of defining bidder's valuation is to assume that sequence heterogeneity enters the valuation multiplicatively. That is

$$v_{ij} = v_i a_j$$

With such a formulation note that bidders believe that $v_{ij} \sim \bar{F}_j\left[a_j\underline{v}, a_j\bar{v}\right]$ where $\bar{F}_j(x) = \bar{F}(x/a_j)$. Then, from the definition of equilibrium bidding functions it is straightforward to show that

$$b_{i,j,K_j} = \beta_{j,K_j,N_j}(v_{ij}) = v_{ij} = a_j v_i = a_j \beta_K(v_i)$$

$$\begin{aligned} b_{i,j,k} &= \beta_{j,k,N_j}(v_{ij}) = \int_{a_j\underline{v}}^{v_{ij}} \beta_{j,k+1,N_j}(x) d\left(\frac{\bar{F}_j(x)}{\bar{F}_j(v_{ij})}\right)^{N_j-k-1} \\ &= a_j \int_{\underline{v}}^{v_i} \beta_{k+1,N_j}(x) d\left(\frac{\bar{F}(x)}{\bar{F}(v_i)}\right)^{N_j-k-1} \\ &= a_j \beta_{k,N_j}(v_i) \end{aligned}$$

where β_{k,N_j} are bid functions as defined in the Proposition 3.3. From here, we can see that

$$\mathbb{E}\left[b_{i,j,k}\right] = a_j \mathbb{E}\left[\beta_{k,N_j}(v_i)\right]$$

where the expectation is taken with respect to the true distribution F. Similarly,

$$\text{var}\left[b_{i,j,k}\right] = a_j^2 \text{var}\left[\beta_{k,N_j}(v_i)\right]$$

From the above two equations, note that

$$\mathbb{E}\left[b_{i,j,k}\right] = \frac{\mathbb{E}\left[\beta_{k,N_j}(v_i)\right]}{\text{var}\left[\beta_{k,N_j}(v_i)\right]^{1/2}} \text{var}\left[b_{i,j,k}\right]^{1/2} \tag{17}$$

Addititive: Another way of defining bidder's valuation is to assume that sequence heterogeneity enters the valuation additively. That is

$$v_{ij} = v_i + a_j$$

Following the same procedure as above we note that in this case

$$\mathbb{E}\left[b_{i,j,k}\right] = a_j + \mathbb{E}\left[\beta_{k,N_j}(v_i)\right]$$

and

$$\text{var}\left[b_{i,j,k}\right] = \text{var}\left[\beta_{k,N_j}(v_i)\right]$$

Thus in this case there is no relationship between the first and second moments of the bids.

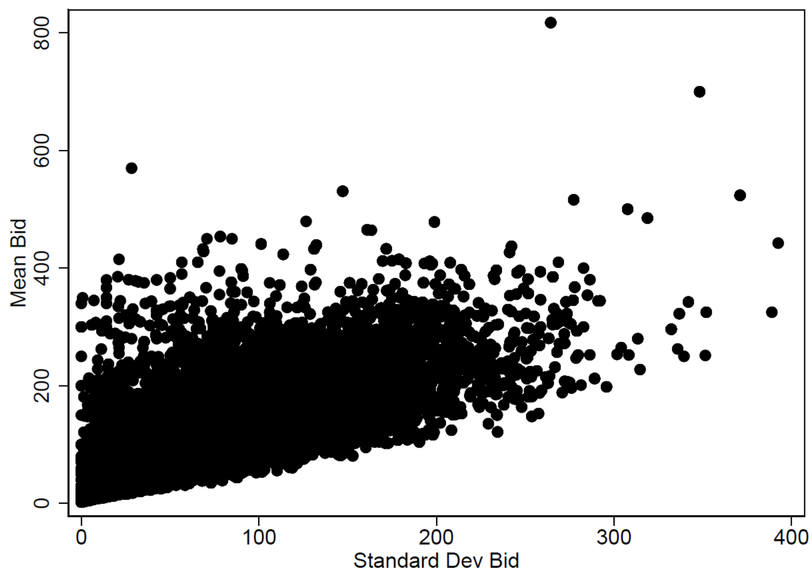
In order to test whether the sequence heterogeneity enters multiplicatively or additively, we can run the following regression on bids.

$$\bar{b}_{jk} = \gamma_0 + \gamma_1 sd(b_{jk}) + \varepsilon_{jk} \tag{18}$$

where $\bar{b}_{jk}$ is the mean bid placed in auction k in sequence j. The coefficient of interest is β_1, which captures the correlation between mean bids and standard deviation of bids within an auction. If $\beta_1 \neq 0$, then one can infer that a_j enters multiplicatively.

Figure 14 and table 10 display the result of analyzing equation (18). It shows a significant covariance between the mean bid and the standard deviation within an auction. This would suggest that auction heterogeneity is multiplicative rather than additive.

FIGURE 14. Mean and Standard Deviation of Bids by Auction



E.2. **Washing bids.** As the previous section suggests, auction heterogeneity seems to enter multiplicatively. Hence given a vector of observable covariates, z_j, in sequence j, the valuation can be written as $v_{ij} = \Gamma(z_j)v_i$, where v_i is bidder i's private information. This implies

$$b_{i,j,k} = \beta_{j,k,N_j}(v_{ij}) = \Gamma(z_j)\beta_{k,N_j}(v_i) \tag{19}$$

To account for the observed heterogeneity, the procedure outlined in Haile and Tamer (2003) and implemented in Shneyerov (2006) will be applied. Our goal is to obtain homogenized bids, $b_{i,k}$, that bidder i would have submitted in a generic auction. A generic auction is defined as one

TABLE 10. Auction Heterogeneity - Model Evaluation

VARIABLES	(1) Mean Bid	(2) Mean Bid
s.d. Bid	1.049***	0.990***
	(0.00718)	(0.00815)
Tickets Left		-1.036***
		(0.109)
Bidders		2.836***
		(0.173)
Constant	27.32***	20.45***
	(0.465)	(0.910)
Observations	24,862	24,862
Controls		YES

Robust standard errors in parentheses
*** p<0.01, ** p<0.05, * p<0.1

where the effect of covariates is given by the average covariates. Thus, let

$$b_0(n,k) = \mathbb{E}(\beta_{k,n}(v_i)) \text{ and } \Gamma_0 = \mathbb{E}_Z(\Gamma(z))$$

be the mean bid placed in the generic auction k and the mean sequence covariates. Then, rewriting equation (19) we get

$$b_{ijk} = \Gamma_0 \Gamma_1(z_j) b_0(n,k) b_1(v_i,n,k) \tag{20}$$

Using the above equation, our aim is to obtain homogenized bids, $b_{i,k}$ that a bidder would place in a generic auction k, where $\Gamma_1(z_j) = 1$. From the above equation

$$b_{ik} = \Gamma_0 b_0(n,k) b_1(v_i,n,k) = \frac{b_{ijk}}{\Gamma_1(z_j)}$$

Since b_{ijk} are the observed bids, we need to have estimates of $\Gamma_1(z_j)$ in order to obtain homogenized bids. Taking logs in (20) we get

$$\log(b_{ijk}) - \log(\Gamma_1(z_j)) = \log(\Gamma_0) + \log(b_0(n,k)) + \log(b_1(v_i,n,k))$$

To estimate $\hat{b}_{ij}$, it is further assumed that $\Gamma_1(z_j) = z_j^{\gamma_j}$, and that $\log(b_1(v_i,n,k))$ has mean zero conditional on n,k,z_j. Given this, the following regression is estimated to homogenize bids:

$$\log(b_{ijk}) = \alpha_0 + \gamma_j \log(z_j) + D(n,k) + \varepsilon_{ijk} \tag{21}$$

where $D(n,k)$ is a set of dummies for the number of a participants in an auction, and the number of tickets left after the auction. An issue with this is that most of the observable ticket characteristics are dummy variables, such as route and train type. An alternative way of specifying these variables would be to find continuous variables that can describe these tickets. To wit, we use the following variables are considered for this purpose.

TABLE 11. Continues Proxy Variables for Ticket Characteristics

Dummy Variable	Continuous Variable
Departure/Destination	Population Household disposable income Latitude Longitude Distance
Train Type	Est. Travel Time Distance (Distance) X (Travel Time)
Week Day	Treat as continuous
Month	Average Temperature Avg precipitation Sun Hours

To evaluate how well the continuous variables capture auction heterogeneity, the estimated R^2 from a regression of the categorical variables onto the continuous proxy variables are reported in table (12), along with the number of variables used for the regression. As can be seen, the captured variance ranges from .45 for the month variables, to .68 for the departure-destination variables.

TABLE 12. Continuous Variable Evaluation

	(1)	(2)	(3)
VARIABLES	Departure Destination	Departure Month	Train
R-squared	0.682	0.446	0.633
Observations	104,137	104,137	104,137
# of Variables	9	6	3

Note: Month variables used include both long run averages and the recordings of the 2010/2011 values.

To further evaluate how well the proxy variables capture ticket heterogeneity, the estimated bid correction $\hat{\Gamma}_1(z_j)$ is regressed on the categorical variables. The goodness of fit is reported in table 13. The first column is without weekday as it is also included as a continuous variable. The adjusted R^2 is over .9 in both estimations, suggesting that the fit is good overall.

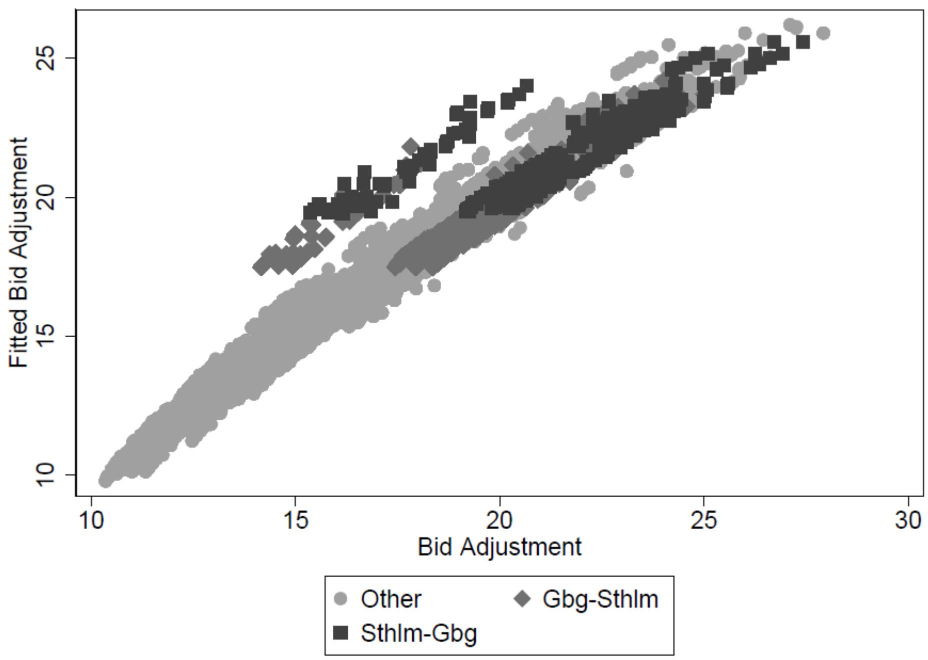
Going a step further and looking at the prediction from the regression against the estimated $\hat{\Gamma}_1(z_j)$ (figure 15), one can see that there are a few observations where the fitted value from the

TABLE 13. Bid Correction Evaluation

VARIABLES	(1) Without Weekday	(2) With Weekday
Adjusted R-squared	0.926	0.943
Observations	104,137	104,137

regressions in table (13) are above the estimated $\hat{\Gamma}_1(z_j)$. Common for these are that the tickets are for either Stockholm to Gothenburg and the other way around.

FIGURE 15. Continuous Proxy Variable Evaluation



Note: Fitted values of a regression of the estimated bid correction from using the continuous variables on the categorical variables. R^2=.94

APPENDIX F. ESTIMATOR PERFORMANCE

F.1. **Gibbs Metropolis Hastings Algorithm:** We use a Markov Chain Monte Carlo algorithm to estimate the posteriors of $\theta_F|K$ and $\theta_{\bar{F}}|\bar{K}$. The algorithm is the same for both parameters, but since the estimated $\hat{f}(v)$ and $\hat{F}(v)$ are inputs to the estimation of $\theta_{\bar{F}}|\bar{K}$, we start by estimating $\theta_F|K$, for $K = 4,6,8,10,12$.

Dropping, K, for exposition. We start by picking a θ_F such that

$$L(\theta_F^{(0)}) = p_0(\theta_F^{(0)}) \prod_{j=1}^{J} g'(b_{3,j}^{(2)}|N_j, y_2, \theta_F^{(0)})$$

is not zero. Then, for each iteration (s), generate a candidate, θ_F^*, from density $q(\cdot|\theta_F^{(s-1)})$. Let

$$z = \min\left\{ \frac{L(\theta_F^*)q(\theta_F^{(s-1)}|\theta_F^*)}{L(\theta_F^{(s)})q(\theta_F^*|\theta_F^{(s-1)})}, 1 \right\} \tag{22}$$

and set $\theta_F^{(s)} = \theta_F^*$ with probability z, and set $\theta_F^{(s)} = \theta_F^{(s-1)}$ with probability $1-z$. {cite Tierney 1994} shows that under mild conditions, the sample $\{\theta_F^{(s)}\}_{s=1}^S$ converges to the posterior, such that for any measurable function $h(\cdot)$

$$\lim_{S\to\infty} \frac{1}{S}\sum_{(s)=1}^{S} h(\theta_F^{(s)}) \overset{a.s.}{\to} \int h(\theta_F)p(\theta_F|\models)d\theta_F = \mathbb{E}(h(\theta_F)|\mathbf{b_1})$$

We further simplify calculation of 22 by using the Gaussian kernel. That is, we draw the candidate parameter from $N(\{\theta_F^{s-1}\}_{k=1}^{K-1}, \Omega_F)$. The proposal densities then cancel out in equation 22 by the symmetry of the Gaussian kernel.

To estimate $\theta_{\bar{F}|y_1}$ we partition the parameters according to the bins. Let $\theta_{\bar{F}|-Y_m} = (\theta_{\bar{F}|Y_1}, ..., \theta_{\bar{F}|Y_{m-1}})$ for $m \geq 2$ and $\theta_{\bar{F}|+Y_m} = (\theta_{\bar{F}|Y_{m+1}}, ..., \theta_{\bar{F}|Y_M})$ for $m < M$. Then we have that $\theta_{\bar{F}|y_1} = (\theta_{\bar{F}|-Y_m}, \theta_{\bar{F}|Y_m}, \theta_{\bar{F}|+Y_m})$. For each $m = 1, ..., M$, generate a candidate $\theta^*_{\bar{F}|Y_m}$ from $q(\cdot|\theta^{(s)}_{\bar{F}|-Y_m}, \theta^{(s-1)}_{\bar{F}|Y_m}, \theta^{(s-1)}_{\bar{F}|+Y_m})$. Note that given the structure of the model

$$\frac{L(\theta^{(s)}_{\bar{F}|-Y_m}, \theta^*_{\bar{F}|Y_m}, \theta^{(s-1)}_{\bar{F}|+Y_m})}{L(\theta^{(s)}_{\bar{F}|-Y_m}, \theta^{(s-1)}_{\bar{F}|Y_m}, \theta^{(s-1)}_{\bar{F}|+Y_m})} = \frac{p_{0,\bar{F}|Y_m}(\theta^*_{\bar{F}|Y_m}) \prod_{j_m \in J_m} g_2'(b^{(2)}_{2,j_m}(v|N,y_1)|\hat{f}(v), \hat{F}(v), \theta^*_{\bar{F}|Y_m})}{p_{0,\bar{F}|Y_m}(\theta^{(s-1)}_{\bar{F}|Y_m}) \prod_{j_m \in J_m} g_2'(b^{(2)}_{2,j_m}(v|N,y_1)|\hat{f}(v), \hat{F}(v), \theta^{(s-1)}_{\bar{F}|Y_m})} := \frac{L_m(\theta^*_{\bar{F}|Y_m})}{L_m(\theta^{(s-1)}_{\bar{F}|Y_m})}.$$

Hence, while we do estimate the parameters jointly, this is equivalent as to estimate them one-by-one, and we can still perform model selection individually for each bin m.

F.2. **Illustration with history dependent bids.** We here illustrate our estimator. Valuations are drawn from a log normal distribution with mean 0 and variance 1 truncated at .55 and 2.5, and transformed to fall within 0 and 1. For worst case beliefs we use the point-wise minimum of two distributions, $\bar{F}_1(v)$ and $\bar{F}_2(v)$. $\bar{F}_1 = U(0,1)$ and

$$\bar{F}_2(v) = \begin{cases} 0 & \text{if } v < 0 \\ 8^{1/2}v^{3/2} & \text{if } 0 \leq v \leq 1/2 \\ 1 & \text{if } 1/2 < v \end{cases}$$

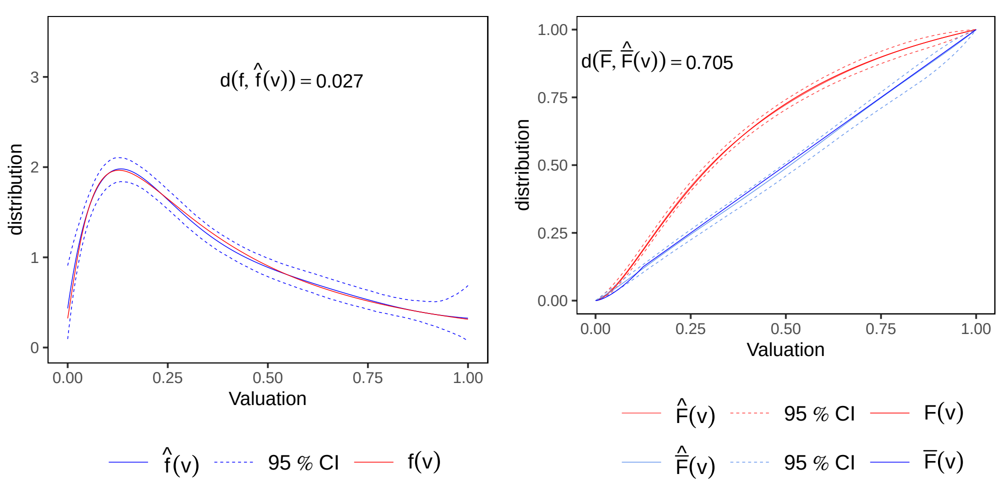
The chosen distributions introduces history dependence in bidding. They pose a slight challenge to estimation as they also introduces a kink in the distribution function of the worst case beliefs, which might be difficult to approximate without letting $\bar{K}$ be very large. However, we still restrict our estimation in this dimension to $\bar{K}$=4,...,15.

We use a sample that resembles what is available to us in the actual estimation when 3 train tickets are for sale by using 100 sequences with 4, 5, 6 and 7 bidders each, 80 sequences with 8 bidders, 70 sequences with 9 bidders, and 50 sequences with 10 bidders. This means that we have 600 observations that can be used to recover the distribution of valuations and 600 observations that can be used to recover worst case beliefs without any history. We also place bids in the second auction in 10 equally spaced bins based on the winner's valuation.

We estimate $(\theta_F|K)$ for $K = 4, \ldots, 15$, and then select $\hat{K}$ with the Bayes model selection described above. We then estimate $(\theta_{\bar{F}}|\bar{K})$ for $\bar{K} = 4, \ldots, 15$ using $\hat{f}(v)$ and $\hat{F}(v)$ as inputs, and again choose $\hat{\bar{K}}$ with Bayes model selection.

Lastly, we estimate $(\theta_{\bar{F}|y_1}|\bar{K}_Y)$ for $\bar{K}_Y = 4, \ldots, 15$ using $\hat{f}(v)$ and $\hat{F}(v)$ as inputs. Our assumed structure then allows as to select different $\bar{K}_Y$ for different bins. While the estimation procedure is done jointly for all $\theta_{\bar{F}|Y_m}|\bar{K}_Y$, observations not in bin Y_m are independent of $\theta_{\bar{F}|Y_m}|\bar{K}_Y$. Thus, after estimating $\theta_{\bar{F}|Y_m}|\bar{K}_Y$, we use Bayes model selection for each bin to select $\bar{K}_{Y_m}$.

We first show the estimated density if valuations along with estimated *unconditional* worst case belief, which can be found in Figure 16. The density of valuations is precisely estimated. The selected K is 10. The distribution of *unconditional* worst case beliefs falls within the estimated 95% confidence interval, but by inspection one can see that the estimate does not fully capture the kink where the worst case switches from F_2 to F_1.



(A) Valuation density (B) Worst case belief density

FIGURE 16. $F(v)$: lognormal($\mu = 0, \sigma = 1$); $\bar{F}(v)$: point wise min of $\bar{F}_1(v)$ and $\bar{F}_2(v)$.

Before looking at the estimated *conditional* worst case beliefs, we give a sense of how many observations there are per bin. This can be found in Figure 17. As the observations are grouped based on a high order statistic, "higher" bins have more observations.

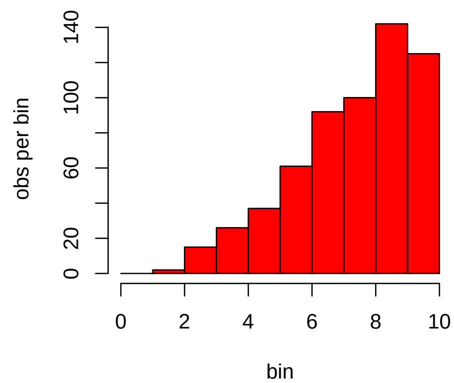


FIGURE 17. Observations per bin, Y_m

We now show the performance of the estimated *conditional* worst case beliefs. We contrast the estimates to (1) the true *conditional* worst case beliefs, which can be found in Figure 18; and (2) the estimated *unconditional* worst case beliefs, which can be found in Figure 19.

The second comparison is relevant as it will be the basis to evaluate the history dependence. Without history dependence, the estimated *conditional* worst case belief should be the same as the estimated *unconditional* worst case belief, conditional on common support.

Figure 18 shows that all the true *conditional* worst case beliefs are within the estimated confidence intervals. Figure 19 shows that the history dependence is recovered where it is most prevalent. That is when $y_1 \in Y_5$. Beliefs are changing in all bins, but for $m \leq 5$, $\bar{F}_2$ is the only relevant distribution for $\bar{F}(v|y_1 \in Y_m)$. For $m > 5$ over some part of the domain of valuations, $\bar{F}_1$ is the distribution relevant for $\bar{F}(v|y_1 \in Y_m)$, while over another part of the domain, $\bar{F}_2$ is the relevant distribution. For the *unconditional* worst case belief, $\bar{F}_2$ is the relevant distribution for only a very small part of the domain of valuations.

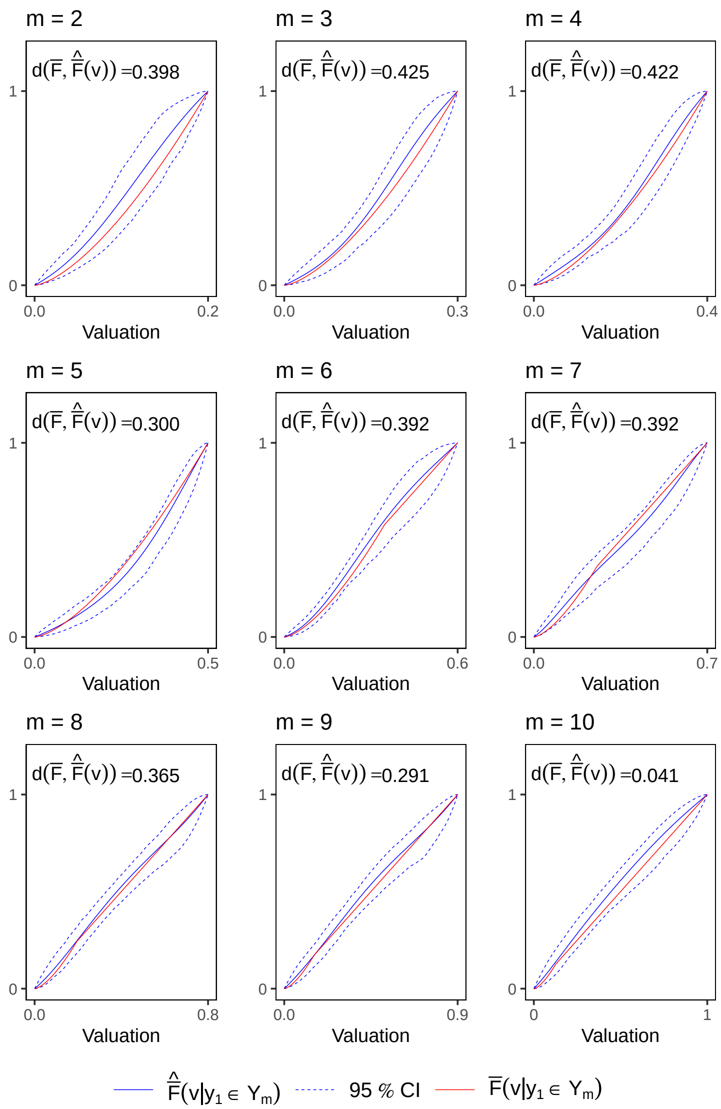


FIGURE 18. History dependent beliefs for bins 2,...,10 ($Y_m = (\frac{m-1}{10}, \frac{m}{10}]$).

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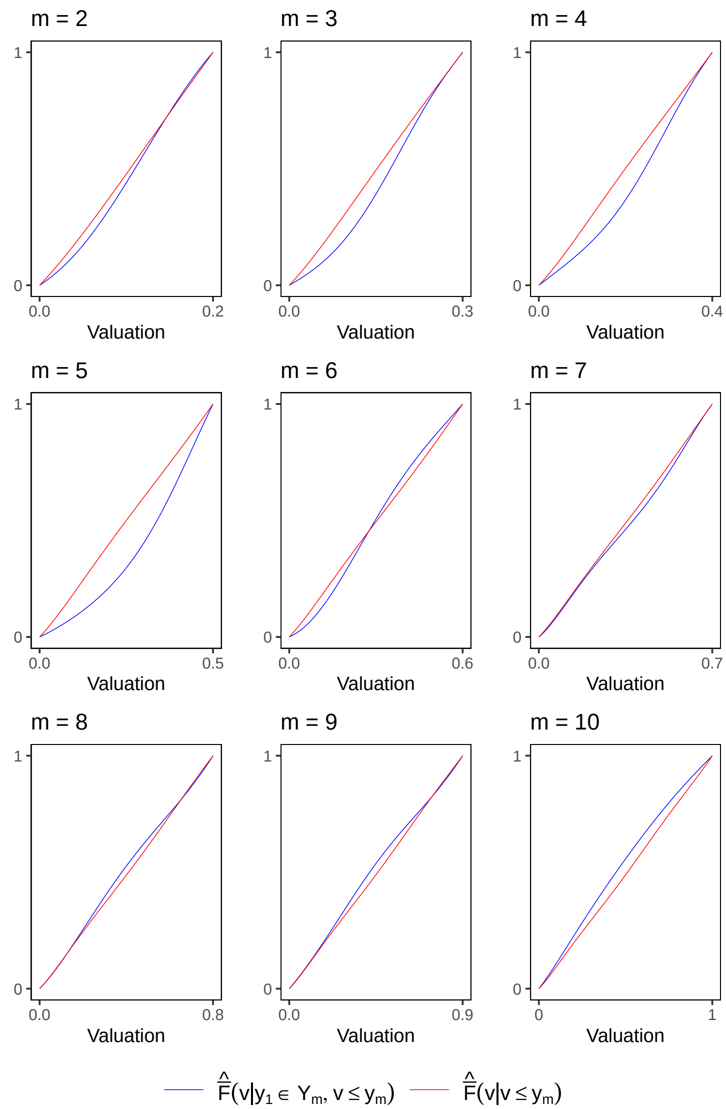

FIGURE 19. History dependent beliefs vs. beliefs without history

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