Finite-Length Patents and the Dynamics of Functional Differential Equations: What did Elhanan Helpman not Tell Three Decades Ago?

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Purpose and Significance

- My research extends the seminal North-South innovation/imitation model that Helpman (1993) developed three decades ago in two significant directions:
  - Introduce *finite-length patents* for product innovation to be empirically consistent.
  - Introduce Shumpeterian *creative destruction* to account for the externality of innovation when the world moves to optimize the North-South protection of intellectual property rights.
- My research therefore builds a more empirically relevant North-South model. Yet, this model's dynamic general-equilibrium system is found to be a computationally complicated one characterized by *mixed-type functional differential equations (FDEs)*.
- My research uses *homotopy continuation* plus the regular solver of boundary value problems (BVP) to solve the system of mixed-type FDEs.

Related Studies

Bellman & Cook (1963)
Differential-Difference Equations.
1963. New York: Academic Press.

Helpman, Elhanan. (1993)
Innovation, Imitation, and Intellectual Property Rights.
1993. *Econometrica* 61, 1247-1280.

Judd, Kenneth (1998)
Numerical Methods in Economics.
1998. MIT Press.

Lin, Hwan C. (2018)
Computing Transitional Cycles for a Deterministic Time-to-Build Growth Model.
2018. Computational Economics 51, 677–696.

Lin, Hwan C. & L. F. Shampine (2018)
R&D-based calibrated growth models with finite-length patents: A novel relaxation algorithm for solving an autonomous FDE system of mixed type.
2018. Computational Economics. 1–36.

The North-South Dynamical System of Helpman (1993)

System of ordinary differential equations (ODEs) of order two:

$$\dot{g}(t) = \left[\frac{L^N}{a} - g(t)\right]\left[\rho + m + g(t) - \frac{1-\alpha}{\alpha}\left(\frac{L^N}{a} - g(t)\right)\frac{1}{\zeta(t)}\right] \quad (1)$$

$$\dot{\zeta}(t) = g(t) - [g(t) + m]\zeta(t) \quad (2)$$

where $\dot{x}(t) \equiv \frac{dx(t)}{dt}$, $x \in \{g, \zeta\}$.

1. $g(t) =$ time-t innovation rate; $\zeta(t) =$time-t share of patented goods
2. $L^N =$Northern workers; $a =$ research productivity parameter
3. $\rho =$time preference; $m =$ imitation rate (a proxy of IP protection)
4. $\alpha \in [0,1] =$ product differentiation

What did Helpman (1993) not tell?

- In Helpman's seminal paper, the dynamical system presents a regular boundary value problem (BVP), and it can be solved numerically with ease. His research concludes that a tightening of IP protection always harm the imitating South, but not the innovating North.
- However, it has remained unclear whether Helpman's conclusions can still hold in a more empirically relevant economic environment.
- For instance:
  - In the real world, patent length is 20 years, but it is assumed to be infinity in Helpman(1993).
  - In the real world, innovation often comes with so-called "creative destruction," but it is assumed away in Helpman (1993).

Problems with Helpman's Assumptions

- Infinite patents:
  1. Monopolistic duration is extended unduly .
  2. The market value of patents is raised excessively.
- Lack of creative destruction:
  1. The risk-adjusted cost of capital is biased downwards, thereby further raising the market value of patents.
  2. The social cost (negative externality)of innovation is biased downwards.

The North-South Dynamical System of Lin (2023)

System of mixed-type FDEs of order three

$$\dot{g}(t) = \left[\frac{L^N}{a(1+\psi)} - g(t)\right]\left[\rho + m + (1+\psi)g(t) - \frac{1-\alpha}{\alpha}\left(\frac{L^N}{a} - (1+\psi)g(t)\right)\frac{1}{\zeta(t)}\right]$$

$$+\left[\frac{L^N}{a(1+\psi)} - g(t)\right]\left[\frac{1-\alpha}{\alpha}\left(\frac{L^N}{a} - (1+\psi)g(t)\right)\frac{1}{\zeta(t+T)}e^{-T[\rho+m+(1+\psi)\bar{g}(t)]}\right] \tag{3a}$$

$$\dot{\zeta}(t) = g(t) - [g(t)+m]\zeta(t) - (1+\psi)g(t-T)e^{-T[(1+\psi)\bar{g}(t-T)]} \tag{3b}$$

$$\dot{\bar{g}}^N(t) = \frac{1}{T}[g(t+T) - g(t)] \tag{3c}$$

1. T = patent length; ψ = coefficient of creative destruction
2. $\bar{g}^N(t) \equiv \frac{1}{T}\int_t^{t+T} g(t)dt$ = T-period forward average of innovation rates between t and $t+T$

Types of Functional Differential Equations (FDEs)

- A functional differential equation takes the general form,

$$u'(t) = f(u(t), u(t-\tau), u(t+\tau)) \quad (4)$$

  where time t is an independent variable, $\tau > 0$ is a constant span (or retardation), $u(t)$ is an unknown function, and $f(.)$ is a functional.
- A mixed-type FDE is present if its arguments include both delays and advances $[u(t-\tau), u(t+\tau)]$.
- A delay differential equation is present if the delay term is included, but the advance term is not.
- An advance differential equation is present if the advance term is included, but the delay term is not.
- A neutral-type FDE is present if $u'(t-\tau)$ enters as an argument:

$$u'(t) = f(u(t), u(t-\tau), u(t+\tau), u'(t-\tau)) \quad (5)$$

- See Bellman, R., & Cooke, K. L. (1963). *Differential-difference equations*. New York: Academic Press .

The Dynamical System of Lin (2023) if T goes to infinity

By setting $T \to \infty$, the system of FDEs (3a)-(3b) reduces to a system of ODEs (4a)-(4b)

$$\dot{g}(t) = \left[\frac{L^N}{a(1+\psi)} - g(t)\right]\left[\rho + m + (1+\psi)g(t) - \frac{1-\alpha}{\alpha}\left(\frac{L^N}{a} - (1+\psi)g(t)\right)\frac{1}{\zeta(t)}\right] \tag{6a}$$

$$\dot{\zeta}(t) = g(t) - [g(t) + m]\zeta(t) \tag{6b}$$

$$\dot{g}^N(t) = 0 \tag{6c}$$

1. Note that equation (6c) is actually a redundant one in the dynamical system with $T \to \infty$.
2. Also note that if the coefficient of creative destruction ψ is set to zero, the system of ODEs (6a)-(6c) returns to Helpman (1993); see eqs. (1)-(2).

Homotopy Continuation and Deformation

- Set up a homotopy system

$$\dot{\mathbf{X}}(t,\theta) = \mathbf{H}(\mathbf{X}(t), \mathbf{X}(t-T), \mathbf{X}(t+T), \theta) \in \mathbb{R}^3, \quad \theta \in [0,1] \tag{7}$$

- The homotopy $\mathbf{H}(.)$ is formed by

$$\mathbf{H}(\mathbf{X}(t), \mathbf{X}(t-T), \mathbf{X}(t+T), \theta) = (1-\theta)\mathbf{G}(\mathrm{X}(t)) + \theta\mathbf{F}(\mathbf{X}(t), \mathbf{X}(t-T), \mathbf{X}(t+T)) \tag{8}$$

- Deformation:
  1. As $\theta = 0$, $\mathbf{H}(.)$ reduces to $\mathbf{G}(.)$, the R.H.S. of the system of ODEs [(6a)-(6c)];
  2. As $\theta = 1$, $\mathbf{H}(.)$ reduces to $\mathbf{F}(.)$, the R.H.S. of the system of FDEs [(3a)-(3c)];
  3. As θ is set to increase from 0 with a small increment till $\theta = 1$, the Homotopy can deform $\mathbf{G}(.)$ continuously into $\mathbf{F}(.)$.

Algorithm based on Homtopoy Continuation

Steps for homotopy continuation:

1. Prepare a grid for $\theta = \{\theta_0 = 0, \theta_1, \theta_2, ..., \theta_n = 1\}$.
2. Set $\theta = \theta_0 = 0$ and solve $\dot{\mathbf{X}}(t, \theta = 0) = \mathbf{G}(\mathbf{X}(t))$ using a regular BVP solver. Denote the solution by $\mathbf{X}(t)^{(k=0)}$.
3. Recursive loop for $k = 1, 2, ..., n$:
   - Use solution $\mathbf{X}(t)^{k-1}$ to initialize the lags and leads,

$$\dot{\mathbf{X}}(t, \theta_k) = \mathbf{H}(\mathbf{X}(t)^{(k)}, \mathbf{X}(t - T)^{(k-1)}, \mathbf{X}(t + T)^{(k-1)}, \theta) \qquad (9)$$

   - Solve (9) recursively with a workable regular BVP solver to obtain solution $\mathbf{X}(t)^k$.
     - For instance, Matlab's bvp4c or bvp5c; Python has a BVP solver too.
   - The k-loop ends when $k = n$ and solution $\mathbf{X}(t)^{k=n}$ is the solution obtained for the system of FDEs (3a)-(3c).

Numerical Exercise: Solving FDEs (6a)-(6c) using Homotopy continuation

Benchmark Model Parameters

- $L^N = 1$: Labor force - North
- $L^S = 6$: Labor force - South
- $a = 5$: Productivety parameter in innovation function
- $\alpha = 0.75$: Demand function parameter, alpha in $[0, 1]$
- $\rho = 0.025$: Rate of time preference
- $m = 0.025$: Imitation rate
- $\psi = 0$: Creative destruction
- $T = 20$ (years): Patent length

Tightening North-South IP protection: This is done by decreasing m the benchmark rate 2.5% to 1.5% at time $t = 0$.

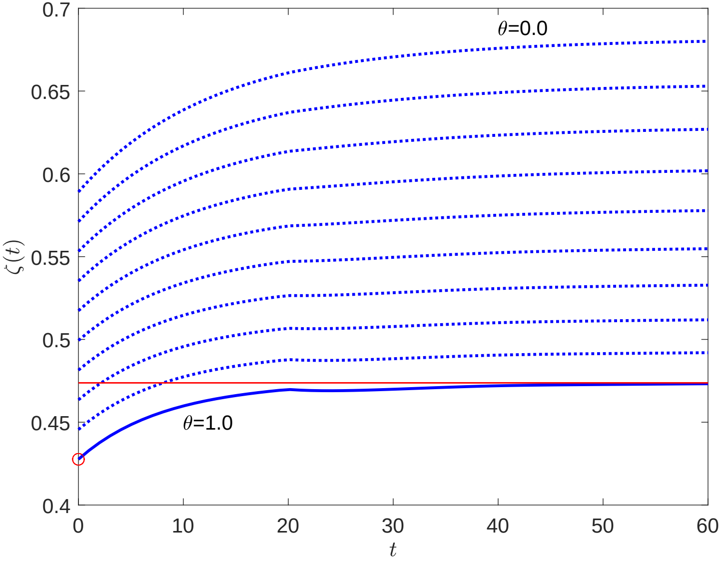

Figure 1: **The transition dynamics of $\zeta(t)$ after IP protection is tightened at $t = 0$ (i.e. a decrease in m from 0.025 to 0.15)**

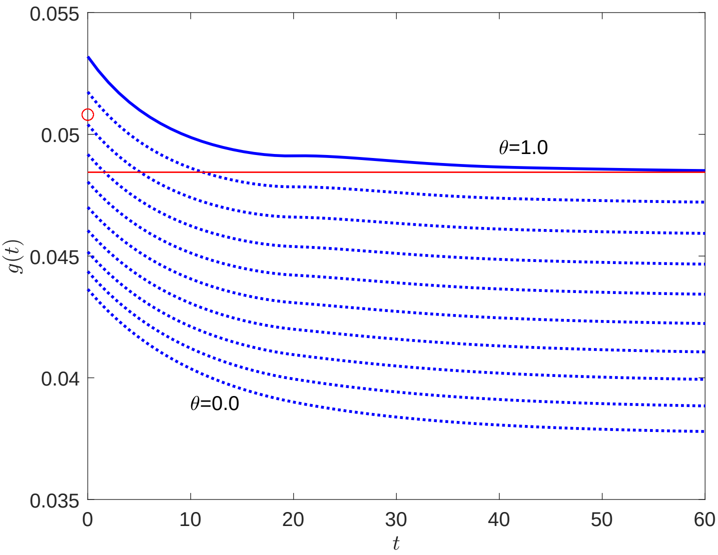

Figure 2: **The transition dynamics of $g(t)$ after IP protection is tightened at $t = 0$ (i.e. a decrease in m from 0.025 to 0.15)**

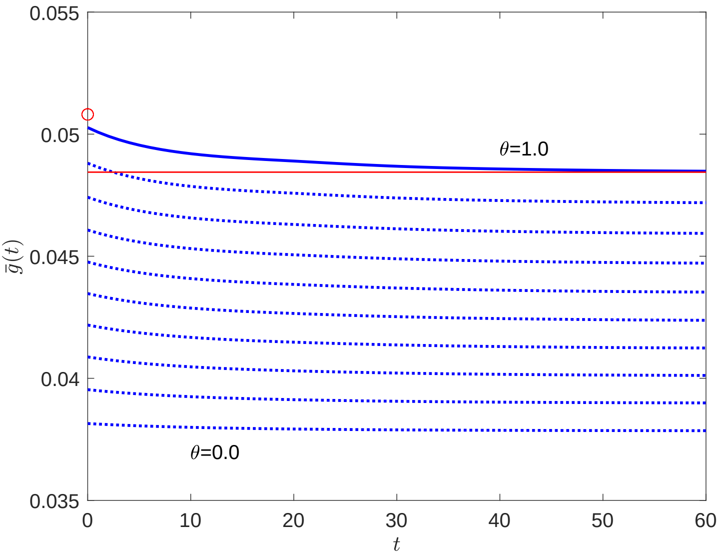

Figure 3: **The transition dynamics of $\bar{g}(t)$ after IP protection is tightened at $t = 0$ (i.e. a decrease in m from 0.025 to 0.15)**

How Fast is Homotopy Continuation for solving FDEs?

Homtopy Continuation Index, k	Homotopy Continuation Parameter, θ	Number of Iterations for each k
$k = 1$	$\theta = 0.0$	0
$k = 2$	$\theta = 0.1$	6
$k = 3$	$\theta = 0.2$	7
$k = 4$	$\theta = 0.3$	8
$k = 5$	$\theta = 0.4$	9
$k = 6$	$\theta = 0.5$	11
$k = 7$	$\theta = 0.6$	14
$k = 8$	$\theta = 0.7$	18
$k = 9$	$\theta = 0.8$	25
$k = 10$	$\theta = 0.9$	41
$k = 11$	$\theta = 1.0$	83

Figure 4: Homotopy Continuation from $k = 1$ to $k = 11$: Relative Error Tolerance is set equal to 1e-4 and Absolute Error Tolerance is set at 1e-6.

Concluding Remarks

- This paper has demonstrated that the homotopy continuation method along with some BVP solvers is able to solve the North-South dynamical system that includes mixed-type functional differential equations.
- Note that the way we form the homotopy system is based on *linear homotopy*. We believe that the same hometopy continuation method should also work if the homotopy system is formed using either *fixed-point homotopy* or *Newton homotopy*.
- The next step for this research project is to quantify the intertemporal effects of a tightening of IP protection using the North-South economy's transition dynamics we have obtained.
- It is also important to see whether the incidence of creative destruction can change such intertemporal welfare effects.

Merci!
Thank you!