FINN 6203 Homework 1 Solutions

1. The budget set is the set of all consumption processes satisfying

$$\begin{aligned}
-28\theta_1 - 59\theta_2 - 15\theta_3 &= c(0) - 400 \\
60\theta_1 + 110\theta_2 + 30\theta_3 &= c(T,1) - 500 \\
50\theta_1 + 120\theta_2 + 30\theta_3 &= c(T,2) - 1270 \\
53\theta_1 + 127\theta_2 + 30\theta_3 &= c(T,3) - 300
\end{aligned}$$

Now

$$\left[\begin{array}{ccc|c}
-28 & -59 & -15 & c(0) - 400 \\
60 & 110 & 30 & c(T,1) - 500 \\
50 & 120 & 30 & c(T,2) - 1270 \\
53 & 127 & 30 & c(T,3) - 300
\end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c}
-28 & -59 & -15 & c(0) - 400 \\
0 & -\frac{115}{7} & -\frac{15}{7} & c(T,1) - \frac{9500}{7} + \frac{15}{7}\,c(0) \\
0 & 0 & -\frac{9}{23} & c(T,3) - \frac{53425}{23} + \frac{179}{46}\,c(0) + \frac{429}{460}\,c(T,1) \\
0 & 0 & 0 & c(T,2) - \frac{32810}{3} + \frac{50}{3}\,c(0) + 4\,c(T,1) + \frac{10}{3}\,c(T,3)
\end{array}\right]$$

Thus, we must have

$$c(T,2) - \frac{32810}{3} + \frac{50}{3}\,c(0) + 4\,c(T,1) + \frac{10}{3}\,c(T,3) = 0.$$

This is the equation of a hyperplane in $\mathbb{R}^4$.

2. Fix $c = \{c(0), c(T)\} \in X = \mathbb{R}^{S+1}$. Since the market is complete, there is some θ^c such that $D\theta^c = c(T)$. Now suppose θ^a is an arbitrage strategy (of the second type). Choose w so that

$$c(0) = -\sum_{n=1}^{N} \theta_n^c p_n - w \sum_{n=1}^{N} \theta_n^a p_n.$$

This is possible since $-\sum_{n=1}^{N} \theta_n^a p_n > 0$. Then

$$c(T) = \sum_{n=1}^{N} \theta_n^c d_n \leq \sum_{n=1}^{N} \theta_n^c d_n + w \sum_{n=1}^{N} \theta_n^a d_n,$$

since $\sum_{n=1}^{N} \theta_n^a d_n \geq 0$. So the strategy $\theta^c + w\theta^a$ provides at least $c(T)$ consumption at time T. So $c \in B(0,p)$ and the result follows since $B(0,p) \subseteq B(e^i, p)$.

3.

i.

$$D = \begin{bmatrix} 24 & 44 & 12 \\ 20 & 44 & 12 \\ 48 & 36 & 12 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Since Rank(D) = 3, all contingent claims are attainable. As there are 3 states, the attainable set is $\mathbb{R}^3$. (Technically, we should say $\mathbb{R}^4$ since any consumption at time 0 is attainable. But since this is always the case, I prefer to just consider attainable terminal consumption.)

ii. Since this market is complete, this consumption process is attainable. To find a trading strategy that attains it, solve the system $D\theta = c(T)$ for θ. Since

$$\left[\begin{array}{ccc|c} 24 & 44 & 12 & 6 \\ 20 & 44 & 12 & 5 \\ 48 & 36 & 12 & 12 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1/4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

the solution is $\theta = [1/4, 0, 0]'$. That is, buy 1/4 of a unit of Security 1 and 0 units of Securities 2 and 3. The price of this trading strategy is $\theta \cdot p = 35/4$. Since $c(0) = 10$, the initial endowment is $e(0) = 10 + 35/4 = 18.75$.

iii. Since this market is complete, this consumption process is attainable. To find a trading strategy that attains it, solve the system $D\theta = c(T)$ for θ. Since

$$\left[\begin{array}{ccc|c} 24 & 44 & 12 & 9 \\ 20 & 44 & 12 & 1 \\ 48 & 36 & 12 & 17 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & -259/12 \end{array}\right]$$

the solution is $\theta = [2, 5, -259/12]'$. That is, buy 2 units of Security 1, buy 5 units of Security 2, and sell short 259/12 units of Security 3. The price of this trading strategy is $\theta \cdot p = 11$. Since $c(0) = 0$, the initial endowment is $e(0) = 11$.

iv. We simply need to check whether $D'\Psi = p$ has a positive solution. Since

$$\left[\begin{array}{ccc|c} 24 & 20 & 48 & 35 \\ 44 & 44 & 36 & 40 \\ 12 & 12 & 12 & 12 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1/4 \\ 0 & 1 & 0 & 1/4 \\ 0 & 0 & 1 & 1/2 \end{array}\right]$$

we have that $\Psi = [1/4, 1/4, 1/2]$, and thus a state price vector exists. Hence the given price system does not permit arbitrage.

4. Let P be the statement $(U(b^i) > U(c^i))$ and let $\neg P$ denote "not P". Then the statement in Definition 1.4 is

A feasible allocation $\{c^i\}$ is Pareto Efficient $\Leftrightarrow \forall$ feasible $\{b^i\}$, $\neg P$.

Now ($\forall$ feasible $\{b^i\}$, $\neg P$) $\Leftrightarrow$ ($\neg\exists$ feasible $\{b^i\}$, P) follows from an axiom of the predicate calculus (see http://en.wikipedia.org/wiki/First-order_logic

for more than you probably wanted to know about this).

So

A feasible allocation $\{c^i\}$ is Pareto Efficient $\Leftrightarrow \neg\exists$ feasible $\{b^i\}$, P

which is the statement you are asked to prove in Problem 4.