FINN 6203 Homework 2 Solutions

1. Consider a one-period model with $N = 3$ securities and $S = 3$ states. The payoffs of the securities are given by

$$D = \begin{bmatrix} 24 & 44 & 12 \\ 20 & 44 & 12 \\ 48 & 36 & 12 \end{bmatrix}$$

The prices of these securities are $p_1 = 35$, $p_2 = 40$, and $p_3 = 12$.

i. Do equilibrium price measures exist? Find them all.
Solution. We simply need to check whether $D'\Psi = p$ has a positive solution. Since

$$\left[\begin{array}{ccc|c} 24 & 20 & 48 & 35 \\ 44 & 44 & 36 & 40 \\ 12 & 12 & 12 & 12 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1/4 \\ 0 & 1 & 0 & 1/4 \\ 0 & 0 & 1 & 1/2 \end{array}\right]$$

we have that $\Psi = [1/4, 1/4, 1/2]$, and thus a state price vector exists. As for the equilibrium price measures, there is one unique EPM and since $r_f = 0$, the EPM is equal to Ψ.

ii. Does the given price system permit arbitrage?
Solution. Since an EPM exists, the given price system does not permit arbitrage.

iii. Is this market complete? Find the set M of all the attainable consumption processes.
Solution.

$$D = \begin{bmatrix} 24 & 44 & 12 \\ 20 & 44 & 12 \\ 48 & 36 & 12 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Since $\text{Rank}(D) = 3$, all contingent claims are attainable, so the market is complete. As there are 3 states, the attainable set is $\mathbb{R}^3$. (Technically, we should say $\mathbb{R}^4$ since any consumption at time 0 is attainable. But since this is always the case, I prefer to just consider attainable terminal consumption.)

iv. Find the equilibrium prices of the following securities:

(a) A call option on security 1 with an exercise price of 25.
Solution. The payoffs of this option are given by $X = [0, 0, 23]$. Using the state price vector Ψ found above, the price of X is $P(X) = X \cdot \Psi = 23/2$.

(b) A call option on security 2 with an exercise price of 40.
Solution. The payoffs of this option are given by $X = [4, 4, 0]$. Using the state price vector Ψ found above, the price of X is $P(X) = X \cdot \Psi = 2$.

2. There are $S = 3$ states and $N = 3$ securities with payoffs

$$D = \begin{pmatrix} 600 & 1100 & 300 \\ 500 & 1200 & 300 \\ 530 & 1270 & 300 \end{pmatrix}$$

and prices $p = [276, 594, 150]'$.

i. Is this market complete? Find the set M of all the attainable consumption processes.
Solution. Since $Rank(D) = 3 = S$, the market is complete, and $M = \mathbb{R}^3$ (or $\mathbb{R}^4$ if you count time 0 consumption).

ii. Is the consumption process

$$c(0) = 0,\ c(T, \omega_1) = 6,\ c(T, \omega_2) = 5,\ c(T, \omega_3) = 12$$

attainable? If so, find the initial endowment and trading strategy that attain it.
Solution. Since the market is complete, all consumption processes are attainable. The trading strategy $\theta = [77/1000, 67/1000, -1139/3000]$ attains it.

iii. Is the consumption process

$$c(0) = 100,000,\ c(T, \omega_1) = 12,\ c(T, \omega_2) = 2,\ c(T, \omega_3) = 5$$

attainable? If so, find the initial endowment and trading strategy that attain it.
Solution. Since the market is complete, all consumption processes are attainable. The trading strategy $\theta = [1/10, 0, -4/25]$ attains it.

iv. Do equilibrium price measures exist? If so, find all of them.
Solution. Since

$$\left[\begin{array}{ccc|c} 600 & 500 & 530 & 276 \\ 1100 & 1200 & 1270 & 594 \\ 300 & 300 & 300 & 150 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1/5 \\ 0 & 1 & 0 & 1/10 \\ 0 & 0 & 1 & 1/5 \end{array}\right]$$

we have that $\Psi = [1/5, 1/10, 1/5]$ is the unique state price vector. Thus $Q = [2/5, 1/5, 2/5]$ is the unique equilibrium price measure.

v. Does the given price system permit arbitrage?
Solution. Since an equilibrium price measure exists, the given price system does not permit arbitrage.

vi. Find the equilibrium prices of the following securities.

(a) A call option on security 1 with an exercise price of 550.
Solution. The payoffs of this option are given by $X = [50, 0, 0]$. Using the state price vector Ψ found above, the price of X is $P(X) = X \cdot \Psi = 10$.

(b) A put option on security 2 with an exercise price of 1150.
Solution. The payoffs of this option are given by $X = [50, 0, 0]$. Thus the price of X is $P(X) = X \cdot \Psi = 10$.

(c) A security whose payoff in each state is the maximum payoff of securities 1 through 3 in that state less the average payoff of securities 1 through 3 in that state.
Solution. The payoffs of this security are given by $X = [433\,1/3, 533\,1/3, 570]$. Thus the price of X is $P(X) = X \cdot \Psi = 254$.

3. There are $S = 2$ states and $N = 3$ securities with payoffs

$$D = \begin{pmatrix} 20 & 44 & 12 \\ 48 & 36 & 12 \end{pmatrix}$$

and prices $p = [24, 40, 12]'$.

i. Is this market complete? What is the set $\mathcal{M}$ of attainable contingent claims?
Solution. Since

$$\begin{bmatrix} 20 & 44 & 12 \\ 48 & 36 & 12 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & \frac{2}{29} \\ 0 & 1 & \frac{7}{29} \end{bmatrix},$$

$rank(D) = 2 = S$, so the market is complete.

ii. Is the consumption process

$$c(0) = 10,\ c(T, \omega_1) = 6,\ c(T, \omega_2) = 5$$

attainable? If so, find the initial endowment and trading strategy that attain it.
Solution. The trading strategy $\theta = [1/348, 47/348, 0]$ attains it, but there are other trading strategies that also work.

iii. Is the consumption process

$$c(0) = 0,\ c(T, \omega_1) = 9,\ c(T, \omega_2) = 1$$

attainable? If so, find the initial endowment and trading strategy that attain it.
Solution. The trading strategy $\theta = [-35/174, 103/348, 0]$ attains it, but there are other trading strategies that also work.

iv. Do equilibrium price measures exist? If so, find all of them.
Solution. Since

$$\left[\begin{array}{cc|c} 20 & 48 & 24 \\ 44 & 36 & 40 \\ 12 & 12 & 12 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]$$

there does not exist a state price vector.

v. Does the given price system permit arbitrage? If so, find an arbitrage trading strategy.
Solution. Since a state price vector does not exist, the price system permits arbitrage. A good place to look for arbitrage strategies is in the null space of D, that is, the set of all θ such that $D\theta = 0$. Now

$$\left[\begin{array}{ccc|c} 20 & 44 & 12 & 0 \\ 48 & 36 & 12 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & \frac{2}{29} & 0 \\ 0 & 1 & \frac{7}{29} & 0 \end{array}\right]$$

so θ is in the null space of D if $\theta_2 = -7/29\theta_3$ and $\theta_1 = -2/29\theta_3$. Taking $\theta_3 = -29$, we then have $\theta_1 = 2$ and $\theta_2 = 7$. This trading strategy costs $(2)(24)+(7)(40)+(-29)(12) = -20$, but pays off 0 in each state.

vi. Find the equilibrium prices of the following securities:

(a) A call option on security 1 with an exercise price of 25.
Solution. Since there does not exist an EPM, the security cannot be priced.

(b) A call option on security 2 with an exercise price of 40.
Solution. Since there does not exist an EPM, the security cannot be priced.