FINN 6203 Homework 3

1. There are $S = 3$ states and $N = 3$ securities with payoffs

$$D = \begin{pmatrix} 60 & 110 & 30 \\ 50 & 120 & 30 \\ 53 & 127 & 30 \end{pmatrix}$$

and prices $p = [28, 59, 15]'$. The actual probability measure is given by $\mathbb{P} = [1/2, 1/10, 2/5]$.

i. Find the discount factor m that can price all securities in this market.
Solution. Since

$$\left[\begin{array}{ccc|c} 60 & 50 & 53 & 28 \\ 110 & 120 & 127 & 59 \\ 30 & 30 & 30 & 15 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{6}{25} \\ 0 & 1 & 0 & \frac{3}{50} \\ 0 & 0 & 1 & \frac{1}{5} \end{array}\right]$$

the unique equilibrium price measure is $Q = [12/25, 6/50, 2/5]$ and $R_f = 2$. So

$$m = \frac{Q}{\mathbb{P}}\left(\frac{1}{1+r}\right) = [12/25, 3/5, 1/2].$$

ii. Let R_1, R_2 and R_3 be the columns of the matrix of returns. Find $E(R_1)$, $\sigma(R_1)$, $E(R_2)$, $\sigma(R_2)$, $E(R_f)$, $E(m)$, $E(m^2)$, $\sigma(m)$, $\text{cov}(R_1, R_2)$, $\text{cov}(R_1, m)$, $\text{cov}(R_2, m)$.
Solution.
$E(R_1) = 281/140 = 2.00714$
$\sigma(R_1) = \sqrt{379}/140 = 0.13906$
$E(R_2) = 589/295 = 1.99661$
$\sigma(R_2) = \sqrt{1619}/295 = 0.13640$
$E(R_f) = 2$
$E(m) = 1/2$
$\sigma(m) = \sqrt{3}/50 = 0.03464$
$E(m^2) = 157/625$
$\text{cov}(R_1, R_2) = E(R_1R_2) - E(R_1)E(R_2) = -699/41300 = -0.01692$
$\text{cov}(R_1, m) = E(mR_1) - E(m)E(R_1) = 1 - E(m)E(R_1) = -1/280 = -0.00357$
$\text{cov}(R_2, m) = 1/590 = 0.00169$

iii. Find the mean-variance frontier and graph it. On your graph, identify R_f, R_1, R_2, $m/E(m^2)$.
Solution.

$$E(R) = R_f \pm \frac{\sigma(m)}{E(m)}\sigma = 2 \pm \frac{\sqrt{3}}{25}\sigma.$$

iv. Find the minimum-variance portfolio consisting of the risky securities R_1 and R_2.
Solution. Define $\sigma^2(w) = w^2\sigma^2(R_1) + (1-w)^2\sigma^2(R_2) + 2w(1-w)\text{cov}(R_1, R_2)$. If

$$\left.\frac{\partial \sigma^2}{\partial w}\right|_{w^*} = 0,$$

then

$$w^* = \frac{\sigma^2(R_2) - \text{cov}(R_1, R_2)}{\sigma^2(R_1) + \sigma^2(R_2) - 2\text{cov}(R_1, R_2)}.$$

So $w^* = 0.49490$, that is, 49.49% in security 1 and 50.51% in security 2.