FINN 6203 Homework 4

3. There are $K=5$ states, $N=4$ securities and $T=2$ periods.

   i. Is the price system an equilibrium price system?
   Solution. Check that all

$$D'(t,\omega)Q[t,S_{t-1}(\omega)]=(1+r)p(t-1,\omega)$$

   have positive solutions. The first system is

$$\begin{bmatrix} 1.5 & 3.25 & 5 \\ 2.5 & 2.75 & 6 \\ 3.5 & 4.75 & 10 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} Q(1,f_{11}) \\ Q(1,f_{12}) \\ Q(1,f_{13}) \end{bmatrix} = (1+0) \begin{bmatrix} 3.6 \\ 4 \\ 6.6 \\ 1 \end{bmatrix}$$

   which has solution

$$\begin{bmatrix} Q(1,f_{11}) \\ Q(1,f_{12}) \\ Q(1,f_{13}) \end{bmatrix} = \begin{bmatrix} \frac{1}{5} \\ \frac{2}{5} \\ \frac{2}{5} \end{bmatrix}.$$

   Similarly, the three remaining systems have solutions

$$\begin{bmatrix} Q(2,\omega_1) \\ Q(2,\omega_2) \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}, \begin{bmatrix} Q(2,\omega_3) \\ Q(2,\omega_4) \end{bmatrix} = \begin{bmatrix} \frac{3}{4} \\ \frac{1}{4} \end{bmatrix}, Q(2,\omega_5)=1.$$

   So

$$Q = \begin{bmatrix} \frac{1}{5}\times\frac{1}{2} \\ \frac{1}{5}\times\frac{1}{2} \\ \frac{2}{5}\times\frac{3}{4} \\ \frac{2}{5}\times\frac{1}{4} \\ \frac{2}{5}\times 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{10} \\ \frac{1}{10} \\ \frac{3}{10} \\ \frac{1}{10} \\ \frac{2}{5} \end{bmatrix}$$

   ii. Is this a complete price system?
   Solution. Check that the rank of the price matrix at each node equals the value of the splitting function at that node.

$$Rank\begin{pmatrix} 1.5 & 2.5 & 3.5 & 1 \\ 3.25 & 2.75 & 4.75 & 1 \\ 5 & 6 & 10 & 1 \end{pmatrix} = 3, Rank\begin{pmatrix} 2 & 4 & 5 & 1 \\ 1 & 1 & 2 & 1 \end{pmatrix} = 2$$

$$Rank\begin{pmatrix} 3 & 2 & 4 & 1 \\ 4 & 5 & 7 & 1 \end{pmatrix} = 2, Rank\begin{pmatrix} 5 & 6 & 10 & 1 \end{pmatrix} = 1$$

   So, yes, it is complete.

   iii. (a) $p=0.65$
   (b) $p=0.60$

(c) $p = 6.60$

4. There are $K = 6$ states, $N = 3$ securities and $T = 2$ periods.

i. Is the price system an equilibrium price system?
Solution. Check that all

$$D'(t,\omega)Q[t,S_{t-1}(\omega)] = (1+r)p(t-1,\omega)$$

have positive solutions. The first system is

$$\begin{bmatrix} 1 & 1 & 1 \\ 0.8 & 1.5 & 0.73 \\ 0.2 & 1.25 & 0.095 \end{bmatrix} \begin{bmatrix} Q(1,f_{11}) \\ Q(1,f_{12}) \\ Q(1,f_{13}) \end{bmatrix} = (1+0) \begin{bmatrix} 1 \\ 1.129 \\ 0.6935 \end{bmatrix}$$

which has solution

$$\begin{bmatrix} Q(1,f_{11}) \\ Q(1,f_{12}) \\ Q(1,f_{13}) \end{bmatrix} = \begin{bmatrix} 0.53 - 1.1x \\ 0.47 + 0.1x \\ x \end{bmatrix}.$$

Similarly, the three remaining systems have solutions

$$\begin{bmatrix} Q(2,\omega_1) \\ Q(2,\omega_2) \end{bmatrix} = \begin{bmatrix} 0.2 \\ 0.8 \end{bmatrix}, \begin{bmatrix} Q(2,\omega_3) \\ Q(2,\omega_4) \end{bmatrix} = \begin{bmatrix} 0.25 \\ 0.75 \end{bmatrix}, \begin{bmatrix} Q(2,\omega_5) \\ Q(2,\omega_6) \end{bmatrix} = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}.$$

So

$$Q = \begin{bmatrix} 0.2(0.53 - 1.1x) \\ 0.8(0.53 - 1.1x) \\ 0.25(0.47 + 0.1x) \\ 0.75(0.47 + 0.1x) \\ 0.5x \\ 0.5x \end{bmatrix}$$

ii. What is the dimension of $M(p)$?
Solution. Form the H matrix and find that $Rank(H) = 7$. Then $Dim(M(p)) = Rank(H) = 7$.

iii. What is the initial equilibrium price of a call option on security 2 with exercise price of $a = 0.7$?
Solution. The payoff of the call is

$$X = \begin{bmatrix} 0 \\ 0.3 \\ 0 \\ 1.3 \\ 0.12 \\ 0 \end{bmatrix}.$$

With $x = 0.1$ in Q, the price of the option is 0.5748.