1 Cost minimization

Consider a firm which uses only two inputs in production, capital (K) and labor (L). The firm has production function $q(K, L) = L^{\beta}K^{\alpha}$. Let w be the wage rate for L and r be the rental rate of capital. Suppose the firm wishes to produce 1080 units of the good. Let $\alpha = \frac{2}{3}$ and $\beta = \frac{1}{3}$, and let $w = \$4$ and $r = \$27$.

$1620^{(1/3)} * 480^{(2/3)} = 720.0$

1. Find the marginal product of capital.

   Answer:

   The marginal product of capital is simply the partial derivative of the production function with respect to K, or:

   $$\frac{\partial q}{\partial K} = \alpha K^{\alpha-1}L^{\beta}$$

2. Find the marginal product of labor.

   Answer:

   The marginal product of labor is simply the partial derivative of the production function with respect to L, or:

   $$\frac{\partial q}{\partial L} = \beta K^{\alpha}L^{\beta-1}$$

3. Find the marginal rate of technical substitution.

   Answer:

   The marginal rate of technical substitution is simply the negative of the ratio of the MP_L to the MP_K, or:

   $$\begin{aligned} MRTS &= -\frac{\beta K^{\alpha}L^{\beta-1}}{\alpha K^{\alpha-1}L^{\beta}} \\ MRTS &= -\frac{\beta K}{\alpha L} \end{aligned}$$

4. Find the cost-minimizing bundle of capital and labor for this firm, as well as the associated total cost amount.

   Answer:

   The short version is to set the MRTS equal to the price ratio, so that we have:

   $$\begin{aligned} -\frac{\beta K}{\alpha L} &= -\frac{w}{r} \\ K &= \frac{\alpha L w}{\beta r} \end{aligned}$$

Now substitute this into the production function to find:

$$\begin{aligned}
q &= L^{\beta}\left(\frac{\alpha L w}{\beta r}\right)^{\alpha} \\
q &= L^{\beta+\alpha}\left(\frac{\alpha w}{\beta r}\right)^{\alpha} \\
q\left(\frac{\beta r}{\alpha w}\right)^{\alpha} &= L^{\beta+\alpha} \\
\left(q\left(\frac{\beta r}{\alpha w}\right)^{\alpha}\right)^{\frac{1}{\beta+\alpha}} &= L
\end{aligned}$$

Now that we know L we can find K by plugging back in for L:

$$K = \frac{\alpha w}{\beta r}\left(q\left(\frac{\beta r}{\alpha w}\right)^{\alpha}\right)^{\frac{1}{\beta+\alpha}}$$

Now we can just plug in the values for q, β, α, w, and r to find:

$$\begin{aligned}
L &= \left(1080 * \left(\frac{\frac{1}{3} * 27}{\frac{2}{3} * 4}\right)^{\frac{2}{3}}\right)^{\frac{1}{\frac{1}{3}+\frac{2}{3}}} \\
L &= \left(1080 * \left(\frac{\frac{27}{3}}{\frac{8}{3}}\right)^{2/3}\right) \\
L &= 1080 * \left(\frac{27}{8}\right)^{2/3} \\
L &= 1080 * \frac{3}{2} \\
L &= 2430
\end{aligned}$$

Now that we have L it is easy to find K:

$$\begin{aligned}
K &= \frac{\alpha L w}{\beta r} \\
K &= \frac{\frac{2}{3} * 2430 * 4}{\frac{1}{3} * 27} \\
K &= \frac{\frac{19440}{3}}{\frac{27}{3}} \\
K &= \frac{19440}{27} \\
K &= 720
\end{aligned}$$

We can check to make sure that the firm should not use only K or only L – if the firm uses only L so that $K = 0$ then it produces nothing. The same is true if the firm uses only K so that $L = 0$. The total cost of this cost-minimizing bundle is:

$$TC = 720 * 27 + 2430 * 4 = 29,160$$

Suppose that the price of capital increases, so that now $r = \$64$. The firm still wishes to produce 1080 units.

5. Find the cost-minimizing bundle of capital and labor for this firm given the increase in r, as well as the associated total cost amount.

   Answer:

   This is pretty straightforward. The results from part 4 still hold except for when the numbers are plugged in. So we have:

$$\begin{aligned} L &= \left(q\left(\frac{\beta r}{\alpha w}\right)^{\alpha}\right)^{\frac{1}{\beta+\alpha}} \\ K &= \frac{\alpha L w}{\beta r} \end{aligned}$$

   Skipping some steps we have:

$$\begin{aligned} L &= 1080 * \left(\frac{64}{8}\right)^{2/3} \\ L &= 4320 \\ K &= \frac{\frac{2}{3} * 4320 * 4}{\frac{1}{3} * 64} \\ K &= \frac{\frac{34560}{3}}{\frac{64}{3}} \\ K &= \frac{34560}{64} \\ K &= 540 \end{aligned}$$

   So the cost-minimizing bundle when r increases to \$64 is $K = 540$ and $L = 4320$. The total cost of this cost-minimizing bundle is:

$$TC = 64 * 540 + 4 * 4320 = 51,840$$

2 Production

Consider the following production functions, where q is the quantity produced of the good, K is the quantity of capital used, and L is the quantity of labor used:

Production function 1

$$q(K, L) = K^{\alpha} L^{\beta}$$

Production function 2

$$q\left(K,L\right)=K^{\alpha}+L^{\beta}$$

1. For production function 1, for what values of α and β will this production function exhibit (a) increasing, (b) constant, and (c) decreasing returns to scale?

   Answer:

   We can just check to see my multiplying each input by δ. So:

$$\begin{aligned}
q\left(\delta K,\delta L\right) &= \left(\delta K\right)^{\alpha}\left(\delta L\right)^{\beta}\\
q\left(\delta K,\delta L\right) &= \delta^{\alpha+\beta}K^{\alpha}L^{\beta}\\
q\left(\delta K,\delta L\right) &= \delta^{\alpha+\beta}q\left(K,L\right)
\end{aligned}$$

   If $\alpha+\beta<1$, then the production function has decreasing returns to scale. If $\alpha+\beta=1$, then the production function has constant returns to scale. If $\alpha+\beta>1$, then the production function has increasing returns to scale. This is just the example from class.

2. For production function 2, for what values of α and β will this production function exhibit (a) increasing, (b) constant, and (c) decreasing returns to scale?

   Answer:

   Again, just multiply the inputs by δ:

$$\begin{aligned}
q\left(\delta K,\delta L\right) &= \left(\delta K\right)^{\alpha}+\left(\delta L\right)^{\beta}\\
q\left(\delta K,\delta L\right) &= \delta^{\alpha}K^{\alpha}+\delta^{\beta}L^{\beta}
\end{aligned}$$

   It's not as easy now. If $\alpha=\beta$, then one could factor out the δ^{α} and have $\delta^{\alpha}\left(K^{\alpha}+L^{\beta}\right)$. In the case where the two exponents are equal, if they are equal to some number less than 1 then there is decreasing returns to scale. If $\alpha=\beta=1$, then there is constant returns to scale. If $\alpha=\beta>1$, then there is increasing returns to scale.

3 Monopoly

Suppose that a monopolist faces the following inverse demand curve, $P\left(Q\right)=65-5Q$. The monopolist's total cost function is: $TC=2.4Q^{3}-19Q^{2}+66.5Q+40$.

1. Find the monopolist's marginal revenue function.

   Answer:

The monopolist's marginal revenue function is simply the derivative of total revenue with respect to quantity, or:

$$\begin{aligned} TR &= (65 - 5Q)\, Q \\ \frac{\partial TR}{\partial Q} &= 65 - 10Q \end{aligned}$$

2. Find the monopolist's marginal cost function.

   Answer:

   The monopolist's marginal cost function is simply the derivative of total cost with respect to quantity, or:

$$\begin{aligned} TC &= 2.4Q^3 - 19Q^2 + 66.5Q + 40 \\ \frac{\partial TR}{\partial Q} &= 7.2Q^2 - 38Q + 66.5 \end{aligned}$$

3. Find the monopolist's profit-maximizng price and quantity in this market, as well as the monopolist's profit at this price and quantity.

   Answer:

   One could differentiate the profit function and set it equal to zero and solve for Q, but since we already have MR and MC:

$$\begin{aligned} MR &= MC \\ 65 - 10Q &= 7.2Q^2 - 38Q + 66.5 \\ 0 &= 7.2Q^2 - 28Q + 1.5 \end{aligned}$$

   Solving this quadratic gives an optimal quantity of about 3. 8.

4 Cournot with Different MC

Assume the following: there are two firms competing in a Cournot (quantity) game. The firms face the following inverse demand function: $P(Q) = a - bQ = 15000 - 50Q$. Firm 1 has a cost structure such that $TC_1 = c_1 * q_1$, so that Firm 1's marginal cost is $MC_1 = c_1$. Firm 2 has a cost structure such that $TC_2 = c_2 * q_2$, so that Firm 2's marginal cost is $MC_2 = c_2$. Let $c_1 = 50$ and $c_2 = 100$, so that we have $c_1 < c_2$.

1. Find the best-response functions for Firm 1 and Firm 2.

   Answer:

   Firm 1 solves the following maximization problem:

$$\max_{q_1 \geq 0} \ \pi_1 = (a - b(q_1 + q_2))\, q_1 - c_1 q_1$$

with first order condition:

$$\frac{\partial \pi_1}{\partial q_1} = a - 2bq_1 - bq_2 - c_1 \leq 0, \text{ with equality if } q_1 > 0.$$

Setting this inequality to an equality we get:

$$q_1 = \frac{a - bq_2 - c_1}{2b}.$$

The best response function for Firm 1 is then:

$$b_1(q_2) = Max\left[0, \frac{a - bq_2 - c_1}{2b}\right],$$

where Firm 1 will produce 0 if $q_2 \geq \frac{a-c_1}{b}$. Firm 2 solves:

$$\max_{q_2 \geq 0} \ \pi_2 = (a - b(q_1 + q_2))\,q_2 - c_2 q_2$$

with first order condition:

$$\frac{\partial \pi_2}{\partial q_2} = a - 2bq_2 - bq_1 - c_2, \text{ with equality if } q_1 > 0.$$

Setting this inequality to an equality we get:

$$q_2 = \frac{a - bq_1 - c_2}{2b}.$$

The best response function for Firm 2 is then:

$$b_2(q_1) = Max\left[0, \frac{a - bq_1 - c_2}{2b}\right],$$

where Firm 2 will produce 0 if $q_1 \geq \frac{a-c_2}{b}$. Note that if $a < c_2$ then Firm 2 will produce 0, as will Firm 1 if $a < c_1$.

2. Find the Nash equilibrium for this game.

   Answer:

   Let (q_1^*, q_2^*) be the NE quantities. To find the NE note that if $a \leq c_1 < c_2$ then $(q_1^*, q_2^*) = (0, 0)$. If $c_1 < a \leq c_2$ then $(q_1^*, q_2^*) = \left(\frac{a-c_1}{2b}, 0\right)$. If $a > c_2 > c_1$ then we have a more interesting problem. We can solve the following system of equations:

$$\begin{aligned} q_1 &= \frac{a - bq_2 - c_1}{2b} \\ q_2 &= \frac{a - bq_1 - c_2}{2b}. \end{aligned}$$

Solving this system we find:

$$
\begin{aligned}
q_1^* &= \frac{a + c_2 - 2c_1}{3b} \\
q_2^* &= \frac{a + c_1 - 2c_2}{3b}
\end{aligned}
$$

Plugging in the numbers for a, b, c_1, and c_2 we have:

$$
\begin{aligned}
q_1^* &= \frac{15000 + 100 - 2 * 50}{3 * 50} \\
q_1^* &= \frac{15000}{150} = 100 \\
q_2^* &= \frac{15000 + 50 - 2 * 100}{3 * 50} \\
q_2^* &= \frac{14850}{150} = 99
\end{aligned}
$$

3. Find the market price and resulting firm profits at the Nash equilibrium.

   Answer:

   The price is given by:

$$
\begin{aligned}
P(Q) &= a - bQ \\
P(Q) &= 15000 - 50Q \\
P(Q) &= 15000 - 50q_1 - 50q_2 \\
P(Q) &= 15000 - 50 * 99 - 50 * 100 \\
P(Q) &= 5050
\end{aligned}
$$

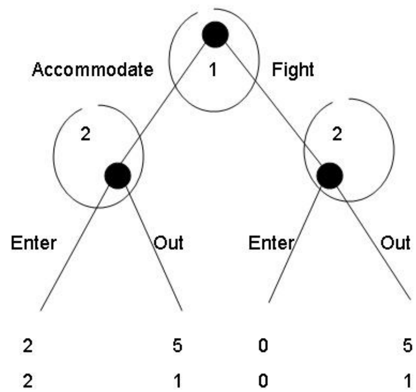
   Firm profits are simply:

$$
\begin{aligned}
\Pi_1 &= P * q_1 - c_1 * q_1 \\
\Pi_1 &= (5050 - 50) * 100 \\
\Pi_1 &= 500,000 \\
\Pi_2 &= P * q_2 - c_2 * q_2 \\
\Pi_2 &= (5050 - 100) * 99 \\
\Pi_2 &= 490050
\end{aligned}
$$

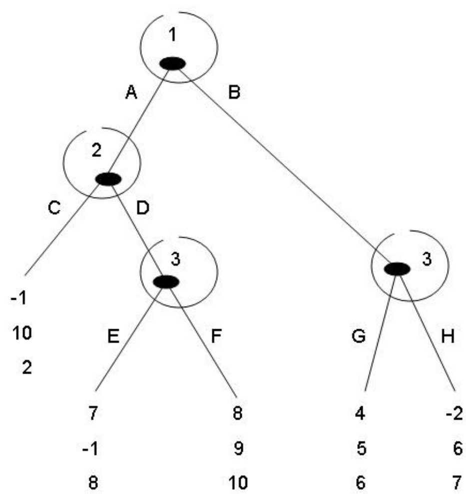
   $P * q_1$

5 Game Trees

Here are 2 extensive form games. Answer the questions below.



Game tree 1



Game tree 2

1. In Game tree 1, find the subgame perfect NE (SPNE). Also find a pure strategy Nash Equilibrium that is not subgame perfect.

   Answer:

   In the first game the SPNE is that Player 1 chooses Fight, and Player 2 chooses Enter if 1 chooses Accommodate and Out if 1 chooses Fight.

For instructional purposes I provide the normal form version of this first game below with the best responses circled. Note that Player 2 has FOUR strategies:

		Player 2: E if A, E if F	E if A, O if F	O if A, E if F	O if A, O if F
Player 1	Fight (F)	0, 0	$\boxed{5}, \boxed{1}$	0, 0	$\boxed{5}, \boxed{1}$
	Accommodate (A)	$\boxed{2}, \boxed{2}$	$2, \boxed{2}$	$\boxed{5}, 1$	$\boxed{5}, 1$

Note that there are 3 PSNE to this game: One where Player 1 Accommodates and Player 2 "Always Enters", a second where Player 1 Fights and Player 2 Enters if 1 Accommodates and chooses Out if 1 Fights, and a third where Player 1 Fights and Player 2 always chooses Out. Note that the first equilibrium relies on a non-credible threat by Player 2 (Enter if Fight), the 2^{nd} equilibrium is the SPNE, and the third is simply one where Player 2 is not choosing optimally off-the-equilibrium path.

2. In Game tree 2, find the subgame perfect NE (SPNE). Explain why the set of strategies A, D, F, G (which would yield the efficient payoff) will NOT constitute a NE (so not only is that set of strategies not SPNE, it is not even NE).

   Answer:

   The SPNE to the second game is that Player 1 chooses A, Player 2 chooses C, and Player 3 chooses F if 1 chooses A and 2 chooses D and H if 1 chooses B. In shorthand, the SPNE is A,C,F,H, which leads to payoffs of -1 for Player 1, 10 for Player 2, and 2 for Player 3.

   The set of strategies A,D,F,G yield the efficient payoff (in the sense that payoff is maximized for the group as a whole) but would NOT consitute a NE because Player 2 would never choose D because Player 2's highest payoff comes from choosing C and Player 2 can ensure himself 10 if he chooses C (he can ensure that he receives 10 because the game ends if he chooses C). We can write up the normal form for this game as:

Player 1: A

		Player 2: C	D
Player 3	E,G	$\boxed{2}, \boxed{10}, -1$	$8, -1, \boxed{7}$
	E,H	$\boxed{2}, \boxed{10}, \boxed{-1}$	$8, -1, \boxed{7}$
	F,G	$\boxed{2}, \boxed{10}, -1$	$\boxed{10}, 9, \boxed{8}$
	F,H	$\boxed{2}, \boxed{10}, \boxed{-1}$	$\boxed{10}, 9, \boxed{8}$

Player 1: B

		Player 2: C	D
Player 3	E,G	$6, \boxed{5}, \boxed{4}$	$6, \boxed{5}, 4$
	E,H	$\boxed{7}, \boxed{6}, -2$	$\boxed{7}, \boxed{6}, -2$
	F,G	$6, \boxed{5}, \boxed{4}$	$6, \boxed{5}, 4$
	F,H	$\boxed{7}, \boxed{6}, -2$	$\boxed{7}, \boxed{6}, -2$

Note that Player 3's (row player) payoffs are listed first, Player 2's (column player) payoffs are listed second, and Player 1's (one listed at the bottom) payoffs are listed third. The normal form shows that there are only 2 PSNE – one is the SPNE (A,C,F,H) and the other yields the same payoffs (A,C,E,H) but has Player 3 changing his strategy at an information set that is not reached.