

Correlation Uncertainty, Heterogeneous Beliefs and Asset Prices

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Abstract

We construct an equilibrium model in the presence of correlation uncertainty and heterogeneous ambiguity-averse investors. In this model the level of correlation uncertainty and asset characteristic jointly affect the asset prices and trading activities. The price of low *weighted volatility* asset declines and naive agent holds less position when the level of correlation uncertainty goes up; concurrently, the price of high *weighted volatility* asset increases and the naive agent holds larger position. The sophisticated agent provides liquidity in the market and always holds a well-diversified optimal portfolio with higher portfolio risk and better performance. Moreover, regardless low or high weighted volatility, higher trading volumes are associated with a larger dispersion of correlation uncertainty across agents. This model explains a number of well documented empirical puzzles on correlation, including *asymmetric correlation*, *under-diversification* and *limited participation*, *comovement* and *flight to quality* in the financial market.

A principle purpose of research in finance is to study the correlated structure among all financial assets. Correlated structure is pervasive in financial market and plays a central role in finance since Markowitz (1952)'s seminal work on the portfolio choice, arbitrage pricing theory and derivative pricing (Duffie et al, 2009), among many others. However, to estimate the correlated structure is difficult from both statistical and econometric perspective (Chan, Karceski and Lakonishok, 1999; Ledoit, Santa-Clara and Wolf, 2003). The challenge to estimate the correlated structure stems from several reasons such as lack of enough market data source, limitation in estimation methodology, correlation process being unstable or virtually complicated (Bursaschi, Porchia and Trojani, 2010; Engle, 2002), not to mention the increasingly interconnected market pattern, making the whole correlation analysis much more demanding than the estimation of the marginal distribution.¹ Besides its fundamental role in asset pricing, numerous studies have empirically documented some significant phenomenons arising from correlation structure, including asymmetric correlation (Ang and Chen, 2002; Longin and Solnik, 2001), under-diversification and limited participation (Vissing-Jorgensen, 2002; Calvet, Campbell and Sodini, 2009; Dimmock, Kouwenberg, Mitchell and Peijnenburg, 2016), comovement and contagion (Barberis, Shleifer and Wurgler, 2005; Vedlhamp, 2006; Kyle and Xiong, 2001), flight to quality and flight to safety (Caballero and Krishnamurthy, 2008; Vayanos, 2004).

The aim of our study is to explain the aforementioned empirical puzzles using a uniform theoretical model and offer novel predictions for optimal portfolio choice and asset pricing. We construct an equilibrium model with heterogeneous *correlation uncertainty among agents* by employing a multiple-priors setting in Gilboa and Schmeidler (1989).² Specifically, agents' concern on the correlated structure is interpreted by ambiguity-aversion on the correlated structure, or virtually the correlation coefficient estimation. Each agent is risk averse and

¹Alternatively, the correlated structure or the covariance matrix can be estimated from the option market. See Buss and Vilkov (2012); Kitiwattananachai and Pearson (2015). However, the complexity of the correlation process still preserves in spite of this implied estimation methodology.

²There is a growing body of research in asset pricing that applies the multiple-priors framework. See, for instance, Easley and O'Hara (2009, 2010); Garlappi, Uppal and Wang (2007); Cao, Wang and Zhang (2005); Uppal and Wang (2003). For other approaches to address the ambiguity and its implication to asset pricing, see Bossaerts, Ghirardato, Guarnaschelli and Zame (2010); Routledge and Zin (2014) and Illeditsch (2011).

ambiguity averse. Agents are heterogeneous in terms of ambiguity aversion, reflecting their various levels of sophistication to deal with statistical data and estimation methodology.³

To concentrate on the role of correlation ambiguity, we assume that agents have perfect knowledge of the marginal distributions for all assets; that is, they merely have concerns about the correlation structure. In this manner, our discussion on the correlation uncertainty significantly departs from previous literature in model uncertainty where either expected return or the variance estimation is a concern. For instance, Cao, Wang and Zhang (2005), Garlappi, Uppal and Wang (2007), Boyle, Garlappi, Uppal and Wang (2012), Easley and O’Hara (2009) investigate the expected return parameter uncertainty. Easley and O’Hara (2010), Epstein and Ji (2014) discuss volatility parameter uncertainty. In an information ambiguity setting, Illeditsch (2011) addresses the conditional distribution ambiguity of the signals. In all those previous studies, the correlation structure is always given as exogenous. Instead, allowing ambiguity on the correlation structure while the marginal distribution is known can cast a situation in which ambiguity-averse agent view the overall market as highly ambiguous rather than individual stocks (Boyle, Garlappi, Uppal and Wang, 2012; Uppal and Wang, 2003). In this regard the correlation uncertainty can be also explicated to some extent as systemic risk uncertainty since the ambiguity on the overall market attributes to the macroeconomic uncertainty (See Dicks and Fulghieri, 2015).⁴

In this paper, the unique equilibrium in the presence of correlation uncertainty is characterized in which the optimal correlated structure and the optimal portfolio for each agent are simultaneously determined. The characterization of the equilibrium demonstrates that the heterogeneous beliefs in correlation uncertainty, among others, plays a critical role in characterizing the equilibrium. Specifically, when the dispersion of correlation uncertainty among agents is small, each agent chooses the *highest* available correlation in a full participation equilibrium. On the other hand, when the uncertainty dispersion is large, while the sophisticated agent still chooses her highest possible correlation coefficient, the naive agent’s choice of correlation coefficient is no longer relevant in the equilibrium, even though his optimal portfolio is unique. We show that this *portfolio inertia* feature can be obtained in the

³There are both laboratory evidence and non-laboratory empirical evidence of ambiguity aversion heterogeneity. See Bossaerts, Ghirardato, Guarnaschelli and Zame, 2010; Dimmock, Kouwenberg, Mitchell and Peijnenburg, 2016.

⁴Examining the “best” joint distribution with fixed marginal distributions has also been well studied in classic statistical literature (Strassen, 1965; White 1976).

context of both portfolio choice and equilibrium under model uncertainty, and it turns out that a limited participation equilibrium emerges because of the portfolio inertia property.

In addition to generate the limited participation endogenously through the channel of correlation uncertainty, we show that the sophisticated agent always holds a more diversified (well-diversified) portfolio whereas the naive agent holds an under-diversified portfolio. We confirm that the naive agent's portfolio is less risky because his higher correlation uncertainty yields a higher implicit risk aversion.⁵ We also show that the sophisticated agent achieves a better performance on the optimal portfolio than the naive agent.

Our portfolio analysis heavily hinges upon a *dispersion measure*, inspired by the portfolio selection literature (Hennessy and Lapan, 2003; Ibragimov, Jaffee and Walden, 2011), as we propose to quantify the dispersion among economic variables. Previous studies on under-diversification and limited participation such as Cao, Wang and Zhang (2005), Wang and Uppal (2003), Easley and O'Hara (2009) focus on negligible positions on assets of which the marginal distribution is ambiguous. In contrast, we examine the optimal portfolio itself and compare it with a well-diversified market portfolio by using this dispersion measure in a precise manner. Our portfolio approach to the limited participation is consistent with the methodology in Calvet et al. (2009), Dimmock et al. (2016), Polkovnichenko (2005) and Hirshleifer, Huang and Teoh (2016). We further show that the optimal portfolio is less diversified with higher perceived level of correlation uncertainty across the agents, which is empirically documented in Dimmock et al. (2016).

This equilibrium model has *several* key implications in understanding the stylized facts about correlation structure. *First*, correlation asymmetry occurs endogenously. Since an agent's correlation ambiguity is often larger in the weak market than in the strong market, a higher correlation in a downside market movement is obtained in equilibrium. Moreover, a higher correlation is associated with a higher volatility of individual asset. This correlation asymmetry phenomenon (Ang and Chen, 2002; Longin and Solnik, 2001) is shown to be persistent despite the heterogeneity in correlation estimation.

Second, we find that the dispersion of Sharpe ratios *decreases* endogenously as correlation uncertainty increases regardless of the dispersion in correlation uncertainty. Previous

⁵See Garlappi, Uppal and Wang (2007); Gollier (2011); Wang and Uppal (2003) for the discussion that ambiguity leads to risk aversion implicitly.

studies document that assets moves closely together in a downside market and moves apart in an upside market (Pindyck and Rotemberg, 1993; Forbes and Rigobon, 2002; Barberis, Shleifer and Wurgler, 2005). Our model reveals that all risky assets are enforced to comove more with a higher level of correlation uncertainty, as a result of similar investment opportunities provided in the market. Therefore, our model sheds light to further understand asset comovement from an investment perspective.

Third, the level of the correlation uncertainty affects asset prices differently based on asset characteristics. We use individual asset's *weighted volatility* over the total weighted volatility of all assets, which we call as *eta* in this paper, to measure its relative risk contribution to the entire market while the asset size factor represents the weight. In Appendix C we demonstrate that the *eta* characterizes the sensitivity of individual asset return with respect to the market portfolio return. For a low *eta* asset, the higher the correlation uncertainty, the lower the price. Meanwhile, the price of a high *eta* asset increases as the level of correlation uncertainty increases. This cross-sectional asset price pattern with respect to agents' ambiguity aversion illustrates a significant heterogeneous paradigm. In a homogeneous equilibrium, the endogenous correlation coefficient is large when the perceived level of a representative agent's correlation uncertainty is large; therefore, one asset's price decline reduces the prices of all assets (contagion). Remarkably, a richer price pattern is obtained for assets with varying characteristics under heterogeneity of correlation uncertainty.

Fourth, the model divulges agents' trading activities when the level of correlation uncertainty varies. Each agent holds "long" position on *high eta* assets, but the naive agent might "sell" some *low eta* assets. When the dispersion among agents' correlation uncertainty is large or when the ambiguity on the entire market is large, the sophisticated agent's long position on the low *eta* asset *increases* whereas the naive agent's corresponding position decreases. For very low *eta* assets, the holding of the naive agent could even be negative (short positions). In contrast, he holds long positions for high *eta* assets. Since a high level of correlation uncertainty or a large dispersion among agents' ambiguity often comes with a stressed economy, the high *eta* assets can be used to hedge against the "catastrophic economic shock" or "correlation uncertainty". Therefore, we provide an alternative explanation of flight-to-safety or flight-to-quality episodes because those high *eta* assets enjoy a high demand from the naive agent and its price moves up sharply in a very weak economic situation. Studies

like Caballero and Krishnamurthy(2007), Vayanos (2004) and Guerrieri and Shimer (2014) develop a model to generate the flight-to-quality with liquidity risk or adverse selection. Our model generates flight-to-quality endogenously from a correlation uncertainty perspective.

Finally, the model implies that high trading volume is always associated with high level of correlation uncertainty or a large dispersion in correlation ambiguity among agents, for both low eta as well as high eta assets. In a very weak economic situation, the naive agent is easy to get panic and overreact to the market, thus purchases a significant amount of high eta assets and sells substantially on low eta assets. When we interpret the heterogeneity as disagreement, our result justifies theoretically a recent empirical findings in Carlin, Longstaff and Matoba (2014).

As one application, our theoretical results offer a concrete description about the financial market during the 2007-2009 crisis. Following Bloom (2009), Baele, Bekaert, Inghelbrecht and Wei (2013), we use VIX index to measure the ambiguity on the overall financial market. Figure 1 displays VIX as well as S&P 500 index from 2006 to 2016. As shown clearly, VIX is extremely high within the whole year of 2008, representing a very high level of uncertainty in the market. We observe the VIX index and S&P index move in opposite directions consistently for the entire period from 2006-2016. On the other hand, we compare VIX with gold price during the same time period 2006-2016, in Figure 2, and we see the gold price is to a large extent positively correlated to the VIX price. If we define the whole stock market as one risky asset and other type of assets including fixed income securities, foreign exchange and commodity market together as another risky asset, we can further argue that the stock market is a low eta asset class while the financial market excluding the stock market is a high eta asset class. We present detailed analysis in Section 4.⁶ Following this interpretation, the naive agent's market demand on high eta assets (including bond, hard commodity, in particular gold and silver market) sharply increases when the perceived level of uncertainty moves up, and the higher the uncertainty level the larger the positions. Meanwhile, the naive agent reduce his positions on the low eta assets, i.e. stocks. This analysis explains investors' flight to safety activities in 2007-2009. Furthermore, the trading volume on both the stock

⁶Briefly speaking, the volume of the fixed income market alone is about triple size as the stock market globally. Since the volatility of stock market is about three times of the fixed-income market, their weighted volatility is about the same. Adding a comparable size of commodity market and a much larger volume of the foreign exchange market into the weighted volatility of non-stock financial market, the eta of the stock market is quite small compared to the eta of the non-stock market.

market and these uncertainty-hedging asset classes increases significantly if the dispersion of correlation uncertainty is large, as shown from our model as well as the market data.

The model also highlights important role of sophisticated agents (institutional investors) on the financial market. With increasing number of institutional investors, naive agents are more likely to hold a limited participation portfolio and sophisticated agents therefore dominate the trading activities and asset prices. Moreover, each agent's optimal portfolio performance is improved given a large number of sophisticated agents. Hence, introducing more sophisticated agents into the market, and enhancing investors' finance education/training can be beneficial to every investor as well as the whole financial market.⁷ More importantly, we show that the premium of low eta asset decreases and the premium of high eta asset increases because of large number of institutional investor; thus, our model also helps to explain the disappearance of the small firm premium due to the increased institutional investor demands (Gompers and Metrick, 2001).

The rest of the paper is organized as follows. Section 1 presents a model setting of correlation uncertainty. In Section 2, we study the portfolio choice problem under correlation uncertainty and demonstrate the persistent portfolio inertia feature. We also characterize the equilibrium in a homogeneous environment as a baseline model. In Section 3 we present the unique market equilibrium under heterogeneous correlation uncertainty. We explain in Section 4 the joint effect of correlation uncertainty and asset characteristic on asset price and risk premium. Section 5 presents the optimal portfolio analysis. We establish that the sophisticated agent always has a well-diversified, higher risk and better performance optimal portfolio than the naive agent. The trading activities and trading volumes are examined carefully in Section 6, in which the effect of the heterogeneity in correlation uncertainty is discussed extensively. We further investigate the effect of the percentage of sophisticated agent as well as the dispersion of asset characteristics (eta) on the equilibrium. Section 7 extends the model to a block equicorrelation structure and Section 8 concludes. Proofs and technical arguments are collected in Appendices.

⁷Easley and O'Hara (2009) and Easley, O'Hara and Yang (2015) present similar insights from alternative perspectives.

1 A Model of Correlation Uncertainty

There are N risky assets and one risk-free asset which serves as a numeraire in a two-period economy (date $t = 0$ and $t = 1$). The payoffs or dividends of these N risky assets are $\tilde{a}_1, \tilde{a}_2, \dots, \tilde{a}_N$, respectively at time $t = 1$. The risk-free asset is in zero net supply, and the per capita endowment of risky asset i is $\bar{x}_i, i = 1, \dots, N$. Each risky asset can be viewed as an investment asset, an investment fund, an asset class, or a market portfolio in an international market.

To focus entirely on the correlated risk and its effect on asset pricing, we investigate the correlation structure instead of the joint distribution of $(\tilde{a}_1, \dots, \tilde{a}_N)$.⁸ In other words, given known marginal distribution of each \tilde{a}_i , we study the asset pricing implication within a plausible set of joint distribution. Since the correlation coefficient between the payoffs is identical to the correlation coefficient between asset returns, our assumption is equivalent to the ambiguity purely on asset returns' correlation structure and the agent has no ambiguous on the marginal distribution on the asset return; but this marginal distribution is characterized endogenously in our setting. In this paper we confine ourselves on a *nonnegative correlated* financial market.⁹

In our specification of the correlation matrix, we employ Engle and Kelly (2012)'s dynamic equicorrelation (DECO) model; that is, any two distinct risky assets have a same correlation coefficient ρ , i.e., $corr(\tilde{a}_i, \tilde{a}_j) = \rho$ for each $i \neq j$. This assumption on the correlation structure will be relaxed in a block equicorrelation model in Section 7. Engle and Kelly (2012) show that the (block) DECO estimation of U.S. stock return data can display a better fit for the data than a general dynamic conditional correlation (DCC in Engle, 2002) model. For simplicity, we assume that $(\tilde{a}_1, \dots, \tilde{a}_N)$ has a multivariate Gaussian distribution as in Cao, Wang and Zhang (2005), Easley and O'Hara (2009, 2012). That is, each agent is confident in the estimation of expected mean \bar{a}_i and variance σ_i^2 for each risky

⁸By a copula theory (see McNeil, Frey and Embrechts, 2015), the joint distribution of $\tilde{a} = (\tilde{a}_1, \dots, \tilde{a}_N)$ is characterized by the marginal distribution of each \tilde{a}_i , and a copula function which determines the correlation structure. In other words, the correlation structure can be independent of the marginal distributions. We virtually assume a parametric single-factor Gaussian copula correlation structure in this paper, but our setting itself is general enough to encounter an arbitrary copula correlation structure.

⁹Technically speaking, our results hold when all correlation coefficients are strictly larger than $-\frac{1}{N-1}$. A positive correlated structure is driven by common shocks, or some factors in a financial market. Both the diversification benefits and synchronization are more critical in a positive correlated economy than in a negative correlated environment.

asset $i = 1, \dots, N$; however, they are seriously concerned about the estimation of ρ which represents the ambiguity aversion on the correlated structure.

There is a group of agents in this economy. Each agent has the CARA-type risk preference to maximize the worst-case diversification benefit

$$\min_{\rho} \mathbb{E}^{\rho} [u(W)], u(W) = -e^{-\gamma W}, \quad (1)$$

where ρ runs through a plausible correlation coefficient region, $\mathbb{E}^{\rho}[\cdot]$ represents the expectation operator under corresponding correlation coefficient ρ , and γ is the agent's absolute risk aversion parameter and we assume that each agent has the same absolute risk aversion. In this regard our setting departs from Ehling and Larsen (2013) which studies the correlation structure through the channel of heterogeneity in risk aversions.

Every agent encounters some correlation uncertainty, and we assume that the correlation coefficient is a plausible region instead of one particular number. The agents are heterogeneous in their estimations of the correlation matrix. There are two types of agent, *sophisticated* (“she”) and *naive* (“he”). For the sophisticated agent, her available correlation coefficient region is $[\underline{\rho}_1, \bar{\rho}_1]$; while for the naive agent, his correlation coefficient region is $[\underline{\rho}_2, \bar{\rho}_2]$.¹⁰ We assume $[\underline{\rho}_1, \bar{\rho}_1] \subseteq [\underline{\rho}_2, \bar{\rho}_2]$, which reflects the fact that the estimation on correlation coefficient for the sophisticated agent is more accurate than the naive agent. The percentage of sophisticated agents in the market is ν and naive agents' percentage is $1 - \nu$.

The plausible correlation coefficient region $[\underline{\rho}, \bar{\rho}]$ represents the correlation *magnitude* as well as *uncertainty* and its presence can be justified as follows. An agent has in mind a benchmark or reference correlation coefficient that represents his best estimate on the correlation, and the benchmark/reference correlation is implied by $\frac{\underline{\rho} + \bar{\rho}}{2}$; on the other hand, the level of correlation uncertainty indicates how far away plausible correlation coefficients move upon and below the benchmark and this uncertainty is measured by $\frac{\bar{\rho} - \underline{\rho}}{2}$. In most situations, econometricians are able to find the benchmark correlation coefficient through the calibration to a stochastic matrix process, and treat it as a market reference with some estimation errors (Buraschi, Porchia and Trojani, 2010; Chan, Karceski and Lakonishok,

¹⁰It is well known that the plausible linear correlation coefficient between any two variables X and Y is an interval, $[\rho_{min}, \rho_{max}]$. See McNeil, Frey and Embrechts (2015).

1999; Engle, 2002). By abuse of notations, we use ρ^{avg} to denote the benchmark and ϵ the level of uncertainty.

As a special case to notice, both types of agents can agree on the same benchmark correlation coefficient, but have different ambiguity degrees with respect to the correlation coefficient estimation. If so, the available correlation coefficient for the sophisticated agent is $[\rho^{avg} - \epsilon_1, \rho^{avg} + \epsilon_1]$, and the naive agent's plausible correlation coefficient is $[\rho^{avg} - \epsilon_2, \rho^{avg} + \epsilon_2]$ and $\epsilon_1 < \epsilon_2$. An extreme situation is $\epsilon_1 = 0$, where the sophisticated agent becomes a Savage investor with perfect knowledge about the correlation structure.¹¹ We illustrate our results with these special cases later.

2 Equilibrium in a Homogeneous Environment

We first solve the portfolio choice problem under correlation uncertainty, then present the equilibrium in a homogeneous environment.

2.1 Portfolio Choices

Let x_i be the number of shares on the risky asset i , $i = 1, \dots, N$, and W_0 is the initial wealth of the agent, then the final wealth at time 1 is

$$W = W_0 + \sum_{i=1}^N x_i(\tilde{a}_i - P_i),$$

where P_i is the price of the risky asset i at time $t = 0$. Assuming the plausible range of the asset correlation coefficient is $[\underline{\rho}, \bar{\rho}]$ and there is no trading constraint, the optimal portfolio choice problem for the agent is

$$\max_{x \in \mathbb{R}^N} \min_{\rho \in [\underline{\rho}, \bar{\rho}]} \mathbb{E}^\rho [-e^{-\gamma W}]. \quad (2)$$

¹¹Alternatively, by adopting Cao et al. (2005), we can assume that there is a continuum of agents, say $[\rho^{avg} - \epsilon, \rho^{avg} + \epsilon]$, each type of investor's correlation uncertainty is captured by the parameter ϵ while ϵ is uniformly distributed among agents on $[\bar{\epsilon} - \delta, \bar{\epsilon} + \delta]$ with density $1/(2\delta)$. The main insights of this setting are fairly similar to ours whereas the impact of sophisticated agents in our current setting has a clearer expression. Our setting is in a manner reminiscent of Easley and O'Hara (2009, 2012) on the heterogeneity of agents' ambiguity aversion.

Under the CARA preference and the multivariate Gaussian distribution assumption of the asset returns, the above problem is reduced to be $u(A)$ and

$$A \equiv \max_x \min_{\rho \in [\underline{\rho}, \bar{\rho}]} CE(x, \rho) \quad (3)$$

where $CE(x, \rho) = (\bar{a} - p) \cdot x - \frac{\gamma}{2} x^T \sigma^T R(\rho) \sigma x$ is the mean-variance utility of the agent when the demand vector on the risk assets is $x = (x_1, \dots, x_N)^T$, σ is a diagonal $N \times N$ matrix with diagonal vector $(\sigma_1, \dots, \sigma_N)$ and $R(\rho)$ is a correlation matrix with a common correlation coefficient ρ . We use \cdot^T to denote the transpose operator of a matrix. The certainty-equivalent of the uncertainty-averse agent is

$$CE(x) = \min_{\rho \in [\underline{\rho}, \bar{\rho}]} CE(x, \rho) \quad (4)$$

and we have

$$CE(x) = \begin{cases} CE(\bar{\rho}, x), & \text{if } \sum_{i \neq j} \sigma_i x_i \sigma_j x_j > 0, \\ CE(\underline{\rho}, x), & \text{if } \sum_{i \neq j} \sigma_i x_i \sigma_j x_j < 0, \\ \sum_{i=1}^N ((\bar{a}_i - p_i)x_i - \frac{\gamma}{2} \sigma_i^2 x_i^2), & \text{if } \sum_{i \neq j} \sigma_i x_i \sigma_j x_j = 0. \end{cases} \quad (5)$$

The insight of Equation (5) is straightforward. When the vector of shares x satisfies that $\sum_{i \neq j} \sigma_i x_i \sigma_j x_j > 0$, the agent chooses the highest possible correlation coefficient to compute the certainty-equivalent in the worst case scenario under correlation uncertainty. On the other hand, if the holding positions on the risky assets are rather opposite such that $\sum_{i \neq j} \sigma_i x_i \sigma_j x_j < 0$, the correlation coefficient to compute the certainty-equivalent in the worst case scenario must be the smallest possible correlation coefficient. Finally, if *limited participation* occurs in the sense that $\sum_{i \neq j} \sigma_i x_i \sigma_j x_j = 0$,¹² then the choice of the correlation coefficient is irrelevant to compute the certainty-uncertainty as $CE(\rho, x) = \sum_{i=1}^N ((\bar{a}_i - p_i)x_i - \frac{\gamma}{2} \sigma_i^2 x_i^2)$ for each $\rho \in [\underline{\rho}, \bar{\rho}]$.

¹²When $x_1 x_2 = 0$ for $N = 2$, either $x_1 = 0$ or $x_2 = 0$. Similarly, when $N = 2$ and $x_1 x_2 < 0$, it is a pair trading or a market-neutral strategy; and if $x_1 x_2 > 0$, the portfolio yields a synchronization strategy. For an arbitrary N and each $x_i \geq 0$, the equation $\sum_{i \neq j} x_i x_j = 0$ ensures that at most one component of x is non-zero, equivalently, only one risky asset is invested or less than that. This is anti-diversification in the sense of Goldman (1979).

To elaborate the certainty-equivalent and solve the portfolio choice problem, we introduce a dispersion measure, $\Omega(w)$, of a vector $w = (w_1, \dots, w_N)$ with $\sum_{i=1}^N w_i \neq 0$ by

$$\Omega(w) \equiv \sqrt{\frac{1}{N-1} \left(N \frac{\sum_{i=1}^N w_i^2}{(\sum_{i=1}^N w_i)^2} - 1 \right)}. \quad (6)$$

If further $\sum_{i=1}^N w_i = 1$, then $\Omega(w)^2 = \frac{1}{N-1} \left(N \sum_{i=1}^N w_i^2 - 1 \right)$ is up to a linear transformation the Herfindahl index $\sum_{i=1}^N w_i^2$. In Appendix B we present a formal justification of $\Omega(\cdot)$ being a dispersion measure.¹³ This dispersion measure plays a crucial role in our equilibrium analysis for the endogenous correlation structure and its asset pricing implications are further discussed in Section 2 - Section 6.

By using the dispersion measure, we now reformulate the certainty-equivalent utility of the uncertainty-averse agent as

$$CE(x) = \begin{cases} CE(\bar{\rho}, x), & \text{if } \Omega(\sigma x) < 1, \\ CE(\underline{\rho}, x), & \text{if } \Omega(\sigma x) > 1, \\ \sum_{i=1}^N ((\bar{a}_i - p_i)x_i - \frac{\gamma}{2}\sigma_i^2 x_i^2), & \text{if } \Omega(\sigma x) = 1. \end{cases} \quad (7)$$

Therefore, the optimal portfolio choice problem for the uncertainty-averse agent becomes

$$A = \max_x \left\{ \max_{\Omega(\sigma x) < 1} CE(\bar{\rho}, x), \max_{\Omega(\sigma x) > 1} CE(\underline{\rho}, x), \max_{\Omega(\sigma x) = 1} CE(\rho, x) \right\}. \quad (8)$$

The solution to the optimal portfolio choice problem (2) is given by the next result.

Proposition 1 *Let $\Omega(s)$ be the dispersion of the Sharpe ratios vector $s = (s_1, \dots, s_N)^T$ of all risky assets and assume that $S = \sum_{i=1}^N s_i \neq 0$, $s_i = (\bar{a}_i - p_i)/\sigma_i$. For each $\rho \in [\underline{\rho}, \bar{\rho}]$, $x_\rho \equiv \frac{1}{\gamma}\sigma^{-1}R(\rho)^{-1}s$ is the optimal portfolio in the absence of uncertainty when the correlation coefficient is ρ . $\tau(\cdot)$ is a linear fractional transformation: $\tau(t) \equiv \frac{1-t}{1+(N-1)t}$ for any real number $t \neq -\frac{1}{N-1}$.*

¹³The dispersion measure has been applied in the portfolio selection context. See Ibragimov, Jaffee and Walden (2011); Hennessy and Lapan (2003).

1. If $\underline{\rho} > \tau(\Omega(s))$, then the agent chooses the optimal correlation coefficient $\rho^* = \underline{\rho}$, and the optimal demand is $x^* = x_{\underline{\rho}}$ in Problem (2).
2. If $\bar{\rho} < \tau(\Omega(s))$, then the agent chooses the optimal correlation coefficient $\rho^* = \bar{\rho}$, and the optimal demand is $x^* = x_{\bar{\rho}}$ in Problem (2).
3. If $\underline{\rho} \leq \tau(\Omega(s)) \leq \bar{\rho}$, then the agent is allowed to choose any correlation coefficient as the optimal, that is $\rho^* \in [\underline{\rho}, \bar{\rho}]$, and the optimal demand is $x_{\tau(\Omega(s))}$ in Problem (2).

When all available correlation coefficients are large enough, $\underline{\rho} > \tau(\Omega(s))$, the worst case correlation coefficient is the lowest plausible one. We first explain briefly the technical argument while its intuition will be explained shortly. The dispersion of Sharpe ratios $\Omega(\sigma x_{\underline{\rho}})$ is strictly larger than 1 when $\underline{\rho} > \tau(\Omega(s))$. Therefore, $(\underline{\rho}, x_{\underline{\rho}})$ solves the optimal portfolio choice problem $CE(\underline{\rho}, x)$ under the demand constraint $\Omega(\sigma x) > 1$. By analyzing the dual problem of Problem (2), $\underline{\rho}$ is the best possible correlation coefficient for the uncertainty-averse agent in this situation; thus, $(\underline{\rho}, x_{\underline{\rho}})$ is the unique solution of the problem (2). Similarly, if all available correlation coefficients are small, $\bar{\rho} < \tau(\Omega(s))$, then the agent chooses the highest possible correlation for diversification.

Proposition 1 is particularly interesting when the agent's level of correlation uncertainty is large in the sense that $\underline{\rho} \leq \tau(\Omega(s)) \leq \bar{\rho}$. First, the agent holds a limited participation portfolio to resolve the correlation uncertainty concern since $\sum \sigma_i x_i^* \sigma_j x_j^* = 0$, the dispersion of $x_{\tau(\Omega(s))}$ is one. Specially for $N = 2$, $x_1^* x_2^* = 0$ ensures either $x_1^* = 0$ or $x_2^* = 0$, which is a classic limited participation portfolio. Second, the optimal demand x^* is unique and any other demand vector leads to a smaller maxmin expected utility in Problem (2). Third, while the agent's optimal demand is uniquely determined, the choice of optimal correlation coefficient is irrelevant, that is, the *portfolio inertia* occurs. It is well documented that a high ambiguity might result in portfolio inertia since Dow-Werlang (1992), Epstein-Schneider (2008) and Illeditsch (2011). However, this feature does not emerge naturally in the Gilboa-Schmeidler maxmin expected utility setting where either mean or volatility is unknown and the worst case scenario corresponds to the extreme parameters, for example, Garlappi, Uppal, Wang (2007), Easley and O'Hara (2009) and Epstein and Ji (2014).

According to Proposition 1, the dispersion of Sharpe ratios $\Omega(s)$ is fundamental in characterizing the optimal portfolio. Both the maxmin expected utility for a CRRA-type agent and the Sharpe ratio of the agent's optimal portfolio is positively related to $\Omega(s)$. By its definition, $\tau(\Omega(s))$ measures the "similarities" among investment opportunities offered by assets and it is determined endogenously. As will be shown later, the optimal trading strategy under correlation uncertainty largely depends on the level of $\Omega(s)$. Moreover, $\Omega(s)$ plays a pivotal role in understanding many correlated phenomena.

We take an example of $N = 2$ to explain the intuition behind the very unusual portfolio inertia feature under correlation uncertainty. In an economy with two risky assets,

$$\Omega(s) = \left| \frac{s_1 - s_2}{s_1 + s_2} \right|, \quad \tau(\Omega(s)) = \begin{cases} \min \left\{ \frac{s_1}{s_2}, \frac{s_2}{s_1} \right\}, & \text{if } s_1 s_2 > 0, \\ \max \left\{ \frac{s_1}{s_2}, \frac{s_2}{s_1} \right\}, & \text{if } s_1 s_2 < 0, \\ 0, & \text{if } s_1 s_2 = 0. \end{cases}$$

Clearly, $\tau(\Omega(s))$ measures the *similarity* of two investment opportunities offered by each risky asset. In an extreme situation where one asset (say, the first risky asset) has a very small Sharpe ratio, $\tau(\Omega(s))$ is close to zero. Since the expected return of holding the first risky asset is almost the same as holding the risk-free rate, the only reason to hold it in an optimal portfolio is for the diversification purpose. In order to take advantage of the diversification benefit, the optimal strategy in a mean-variance analysis for these two positively correlated assets must be one long and one short, the smallest possible correlation is thus chosen. Now assume another extreme case where $s_1 = s_2$, hence $\tau(\Omega(s)) = 1$. These two risky assets have the same Sharpe ratios, so the unknown correlation coefficient becomes a major concern for diversification. Therefore, to hedge the correlation uncertainty in the worst-case scenario, the agent chooses naturally the highest possible correlation coefficient. In a general case with arbitrary $s_1 > 0, s_2 > 0$, the diversification benefit with a specific correlation coefficient ρ is

$$u \left(\frac{1}{2\gamma} \left(\frac{s_1^2 + s_2^2 - 2\rho s_1 s_2}{1 - \rho^2} \right) \right).$$

If the agent's correlation uncertainty is large such that his plausible correlation coefficient range contains $\tau(\Omega(s))$, the agent can choose any correlation coefficient while the optimal

demand is decided given that $\tau(\Omega(s))$ is treated as the “true” correlation coefficient. The intuition of Proposition 1 for $N \geq 3$ is similar.

2.2 A Homogeneous Equilibrium

We characterize the equilibrium in a baseline model (with one representative agent) as follow.

Proposition 2 *Assume the plausible correlation coefficient is $[\underline{\rho}, \bar{\rho}]$ in a homogeneous environment. There exists a unique uncertainty equilibrium in which the representative agent’s endogenous correlation coefficient is the highest plausible correlation coefficient $\bar{\rho}$.*

The price of the risky asset i is given by

$$p_i = \bar{a}_i - \gamma\sigma_i(1 - \bar{\rho})\sigma_i\bar{x}_i - \gamma\sigma_i\bar{\rho} \left(\sum_{n=1}^N \sigma_n\bar{x}_n \right). \quad (9)$$

Proposition 2 follows from Proposition 1 inherently. Since optimal portfolio must be the market portfolio $\sum_{i=1}^N \bar{x}_i\tilde{a}_i$ in equilibrium, $\Omega(\sigma x^*) = \Omega(\sigma\bar{x}) < 1$. Then by Proposition 1, the representative agent chooses the highest possible correlation coefficient.

There are several important asset pricing implications of Proposition 2. To highlight the effect of correlation uncertainty, we write $\underline{\rho} = \rho^{avg} - \epsilon$, $\bar{\rho} = \rho^{avg} + \epsilon$. Therefore, the risk premium $\bar{a}_i - p_i$ can be written as a sum of two components:

$$\begin{aligned} \bar{a}_i - p_i = & \overbrace{\gamma(1 - \rho^{avg})\sigma_i^2\bar{x}_i + \gamma\sigma_i\rho^{avg} \sum_{n=1}^N \sigma_n\bar{x}_n} \\ & + \epsilon\gamma \left(\sigma_i \sum_{j \neq i} \sigma_j\bar{x}_j \right) \end{aligned} \quad (10)$$

where the first component represents the risk premium at the absence of correlation uncertainty, and the second one is the correlation-uncertainty premium.

First of all, Proposition 2 is useful to explain the equity premium puzzle due to a positive uncertainty premium. As an illustrative example, we report the percentage of the correlation-uncertainty premium to the uncertainty-free component, $\gamma(1 - \rho^{avg})\sigma_i\bar{x}_i + \gamma\rho^{avg} \sum_{n=1}^N \sigma_n\bar{x}_n$,

in this baseline model in Table 1. Given parameters $\sigma_1 = 9\%$, $\bar{x}_1 = 1$, $\sigma_2 = 10\%$, $\bar{x}_2 = 5$, $\sigma_3 = 12\%$, $\bar{x}_3 = 10.5$ and $\gamma = 1$, we have $\Omega(\sigma\bar{x}) = 0.5558$. Assume that $\rho^{avg} = 0.4$. Without correlation uncertainty, $(s_1, s_2, s_3) = (0.79, 1.49, 2.70)$. Letting the level of uncertainty ϵ move between 0 to 0.2, we observe that the correlation-uncertainty premium increases in a reasonable amount. For instance, when $\epsilon = 0.08$, the percentage of the correlation-uncertainty premium adds about 17 percent, 16 percent and 12 percent to each asset respectively. With a high level of uncertainty, $\epsilon = 0.2$, the correlation-uncertainty premium is quite significant, adding 40 percent to the reference correlation coefficient.

By comparing the effect of correlation ambiguity with the mean and volatility ambiguity, the correlation uncertainty offers a richer channel to affect the excess equity premium. For simplicity, we consider two independent risky assets and one risk-free asset with zero return. To be consistent, the joint distribution of asset returns is assumed to be a bivariate Gaussian distribution and the representative agent has a CARA-type preference. In economy A, the representative agent has no ambiguity on the variance of each asset, but the expected return $a_i \in [\bar{a}_i - \epsilon_i, \bar{a}_i + \epsilon_i]$ for each risky asset $i = 1, 2$. In economy B, the representative agent has no ambiguity concern on the expected return of each asset, but the plausible volatility $\sigma_i \in [\bar{\sigma}_i - \epsilon_i, \bar{\sigma}_i + \epsilon_i]$ for each asset i . It can be shown that the Sharpe ratios in economy A and economy B are¹⁴

$$s_i^A = s_i + \frac{\epsilon_i}{\sigma_i}; s_i^B = s_i \frac{\bar{\sigma}_i + \epsilon_i}{\bar{\sigma}_i} \quad (11)$$

where s_i is the Sharpe ratio in the absence of ambiguity (with expected mean \bar{a}_i and volatility $\bar{\sigma}_i$ for each risky asset). In Equation (11), the uncertainty premium of each risky asset depends only on the ambiguity of the marginal distribution estimation. By contrast, the correlation-uncertainty premium of each risky asset i depends on the entire market structure, in particular, $\sigma_j, \bar{x}_j, j \neq i$.

Second, the Sharpe ratio is given by $s_i = \gamma(1 - \bar{\rho})\sigma_i\bar{x}_i + \gamma\bar{\rho}L$, $L \equiv \sum_{n=1}^N \sigma_n\bar{x}_n$, and

$$\frac{\partial p_i}{\partial \epsilon} = -\gamma\sigma_i \sum_{j \neq i} \sigma_j \bar{x}_j < 0; \frac{\partial s_i}{\partial \epsilon} = \gamma \sum_{j \neq i} \sigma_j \bar{x}_j > 0, \quad (12)$$

¹⁴These types of portfolio choice problem have been studied in Garlappi, Uppal and Wang (2007) and Easley and O'Hara (2009). We can apply the same method to prove Proposition 1 in two economies and derive the unique equilibrium in a homogeneous environment. The details are available upon request from the authors.

It asserts that there is a price decline in each risky asset with the increase of correlation uncertainty; moreover, the variance (risk) of the market portfolio, $\sum \bar{x}_i \hat{a}_i$, increases because of larger endogenous correlation coefficient. In this connection Proposition 2 offers an intuitive illustration about the 2007-2009 financial crisis in which a representative agent has high ambiguity on the correlation structure, so the agent chooses the highest possible correlation coefficient in equilibrium thus asset prices drop. Furthermore, a high level of correlation uncertainty leads to significant increase in total market risk, excess covariance risk, or comovement in the equity market.¹⁵ The 2007-2009 financial crisis will be explained more robustly in a heterogeneous equilibrium in Section 4 - Section 6.

Third, Proposition 2 explains the asymmetric correlation phenomenon. Notice that the level of uncertainty displays a counter-cyclical feature (see Krishnan, Petkova and Ritchken, 2009; Caballero and Simsek, 2013) so as to the market correlation $\rho^{avg} + \epsilon$. Therefore, Proposition 2 is consistent with the correlation asymmetric phenomenon as documented in Ang and Chen (2002), where the market endogenous correlation coefficient is often larger in a weak market than in a strong economy. Moreover, the stylized fact (Longin and Solnik, 2001) that the volatility of risky asset is higher in a weak market can be also explained by Proposition 2. To see it, let $\hat{\sigma}_i$ be the volatility of the asset i 's return, σ_i is a product of the volatility and its price p_i . Therefore, with a higher level of correlation uncertainty, the price p_i declines, as a consequence, $\hat{\sigma}_i$ moves up.

Fourth, the dispersion of Sharpe ratio in the baseline model is

$$\Omega(s) = \Omega(\sigma\bar{x})\tau(\bar{\rho}) \tag{13}$$

and it thus depends negatively on the level of correlation uncertainty. Its intuition is simple. When the correlation uncertainty becomes higher, all risks assets move more closely together and offer similar investment opportunities (Sharpe ratios), so the dispersion of all Sharpe ratio is reduced.

Finally, with increasing level of correlation uncertainty, the market portfolio's Sharpe ratio increases. It means that the representative agent's market portfolio has better perfor-

¹⁵The excess covariance risk is closely linked to the comovement and it can not be explained entirely by economic fundamentals. See Barberis, Shleifer and Wurgler (2005), Pindyck and Rotemberg (1993) and Vedlhamp (2006). Our result provides an alternative explanation of the excess covariance due to correlation uncertainty.

mance if he is more ambiguous about the correlated structure.¹⁶ The intuition is as follows. While the volatility of the market portfolio is high under a large degree of correlation uncertainty, the risk premium in Equation (10) increases more significantly due to its substantial decrease in asset price. Therefore, the Sharpe ratio of the market portfolio is high and the overall market becomes more attractive from the ambiguity-averse investor's perspective.

3 A General Equilibrium

We characterize the equilibrium under heterogeneous correlation uncertainty in this section. To characterize the equilibrium, it is vital to determine the optimal correlation coefficient for each agent. A representative agent can choose either the highest, the lowest one, or even any possible correlation coefficient in a portfolio choice setting, but the market clearing condition enforces the agent to choose the highest possible correlation coefficient under the diversification concern. In a heterogeneous environment, however, the different choices among agents and the dispersion of correlation uncertainty lead to strikingly new features in equilibrium. The asset pricing implications of this general equilibrium are presented in subsequent sections.

For each real number $x, y \neq 1, -\frac{1}{N-1}$, put

$$m(x, y) \equiv \frac{\nu}{1-x} + \frac{1-\nu}{1-y}, n(x, y) \equiv \frac{\nu x}{(1-x)(1+(N-1)x)} + \frac{(1-\nu)y}{(1-y)(1+(N-1)y)}.$$

Proposition 3 *The market equilibrium is presented in two separable cases.*

1. (Full Participation Equilibrium) *If $\bar{\rho}_1$ is large enough such that*

$$\bar{\rho}_1 \geq \frac{1}{N-1} \left\{ \frac{\nu}{1-\nu} \frac{\Omega(\sigma\bar{x})}{1-\Omega(\sigma\bar{x})} N - 1 \right\}, \quad (14)$$

¹⁶It can be shown that the square of the market portfolio's Sharpe ratio is

$$G(\bar{\rho}) = \frac{(\gamma L)^2}{N} \times \{1 + (N-1)\Omega(\sigma\bar{x})^2 + (N-1)(1-\Omega(\sigma\bar{x})^2)\bar{\rho}\}.$$

or if $\bar{\rho}_2$ is strictly smaller than $K(\bar{\rho}_1)$, there exists a unique equilibrium in which each agent chooses the corresponding highest correlation coefficient, respectively. The function $K(\cdot)$ is given by Equation (A-24) in Appendix A.

2. (Limited Participation Equilibrium) If Equation (14) fails and $\bar{\rho}_2$ is larger than $K(\bar{\rho}_1)$, there exists a unique equilibrium in which the sophisticated agent chooses her highest possible correlation coefficient $\bar{\rho}_1$ while the choice of the naive agent is irrelevant. However, the naive agent's optimal demand $x^{(n)}$, is uniquely determined by the endogenous Sharpe ratios in the equilibrium (see Equation (16) below).
3. In either equilibrium, the market price for asset i is, for each $i = 1, \dots, N$,

$$p_i = \bar{a}_i - \frac{\gamma\sigma_i}{m(\bar{\rho}_1, \rho_2)} \left(\sigma_i \bar{x}_i + \frac{n(\bar{\rho}_1, \rho_2)}{m(\bar{\rho}_1, \rho_2) - Nn(\bar{\rho}_1, \rho_2)} \sum_{j=1}^N \sigma_j \bar{x}_j \right). \quad (15)$$

where $\rho_2 = \bar{\rho}_2$ in the full participation equilibrium and $\rho_2 = K(\bar{\rho}_1)$ in the limited participation equilibrium. Furthermore, each risky asset is priced at discount in equilibrium. That is, $p_i < \bar{a}_i$ and $s_i > 0$ for each $i = 1, \dots, N$.

The full participation equilibrium prevails when all agents participate in the market. If agents are relatively homogeneous, i.e. both $\bar{\rho}_1$ and $\bar{\rho}_2$ are large under condition (14) or both $\bar{\rho}_1$ and $\bar{\rho}_2$ are small in the sense that $\bar{\rho}_2 < K(\bar{\rho}_1)$, both agents participate in the market by choosing the corresponding highest possible correlation coefficient under the diversification concern. This full participation condition (14) indicates that whether agents participate in the market relies on both the heterogeneity of agents and the dispersion of correlation uncertainty.

In particular, there are two cases for which Equation (14) holds. The first situation relates to a high reference correlation coefficient, say $\rho^{avg} \geq \frac{1}{N-1} \left\{ \frac{\nu}{1-\nu} \frac{\Omega(\sigma\bar{x})}{1-\Omega(\sigma\bar{x})} N - 1 \right\}$, which often applies to certain asset class in which each pair of assets displays a high correlation by nature. In the second situation, the benchmark correlation coefficient is small, but each agent has high correlation uncertainty, which is often true in a weak market. Both agents in a full participation equilibrium choose the highest possible correlation coefficient in equilibrium in

order to hedge the worst-case correlation uncertainty, regardless of the uncertainty dispersion between two agents.

In contrast to the full participation equilibrium, if there is a large amount of heterogeneity of correlation estimation among agents, a limited participation equilibrium is generated in Proposition 3, (2). For instance, when there are many sophisticated agents and a high risk dispersion such that $\nu + \Omega(\sigma\bar{x})$ is greater than 1, the condition (14) fails. However, if the naive agent is very ambiguous on the correlated structure such that $\bar{\rho}_2 > K(\bar{\rho}_1)$, any choice of the correlation coefficient in his plausible range is feasible but irrelevant in the limited equilibrium. This indicates that when there are many sophisticated agents in the market, sophisticated agents dominate the trading in the market, and naive agents with big uncertainty concern choose limited participation as their optimal strategies.

Figure 4 illustrates the threshold, $K(\bar{\rho}_1)$, of the naive agent's level of uncertainty when a limited participation equilibrium emerges. The function $K(\bar{\rho}_1)$ is drawn with respect to $\bar{\rho}_1$ for different values of ν . As shown, $K(\bar{\rho}_1)$ denoted by the dark curve line is strictly above the straight line of $\bar{\rho}_1$ and the upward pattern indicates its positive reliance on the ambiguity of the sophisticated agent. The limited participation equilibrium occurs only in the yellow region whereas a full participation equilibrium happens elsewhere. We also observe that as the correlation uncertainty of the sophisticated agent increases, the endogenous correlation of naive agent rises consequently.

It is worth mentioning that the naive agent *does* participate for $N \geq 3$ in the limited participation equilibrium and his optimal demand, $x^{(n)}$, is uniquely determined by the endogenous Sharpe ratios in the equilibrium such that

$$x_i^{(n)} = \frac{1}{\gamma\sigma_i} \frac{1 + (N-1)\Omega}{N\Omega} \left(s_i - \frac{1-\Omega}{N} S \right) \quad (16)$$

where $\Omega = \tau(K(\bar{\rho}_1))$ and each s_i is determined by Proposition 3, (3). By a limited participation equilibrium we mean that the naive agent's choice of the correlation coefficient is irrelevant to the equilibrium, which is equivalent to anti-diversification only when $N = 2$. Proposition 3, (2) is consistent with an endogenous limited participation equilibrium as shown in Cao, Wang and Zhang (2005) when investors are heterogeneous in terms of ex-

pected mean uncertainty with two risky assets; but for $N \geq 3$, Proposition 3, (2) offers remarkable new insights regarding the limited participation or under-diversification issue.

In a recent study of household portfolio choice, Calvet, Campbell and Sodini (2009) present evidence suggesting that the proportion of equity held in individual stocks on top of well-diversified portfolio (mutual fund) is a reasonable proxy for portfolio under-diversification. Hirshleifer et al (2016) also show that investors have non-negligible holdings of assets they know little about, and investors hold a common risk-adjusted market portfolio regardless of their information set, instead of anti-diversification. Therefore, our concept of a limited participation equilibrium is consistent with the methodology in Calvet et al. (2009), Polkovnichenko (2005) and Hirshleifer et al. (2016) to study the optimal portfolio and compare it with a well-diversified portfolio. In this regard Equation (16) resembles to the risk-adjusted market portfolio of Hirshleifer et al. (2016) since the naive agent’s optimal holdings in Equation (16) is clearly not anti-diversified. In contrast to Hirshleifer et al. (2016), we show that the naive agent’s optimal portfolio is also fundamental in characterizing equilibrium. A detailed study of each agent’s optimal portfolio is presented in the portfolio analysis section.

4 The Risk Premium and Sharpe Ratio

In this section we study the endogenous risk premium and the Sharpe ratios in equilibrium. We present several testable properties about the risk premium and asset prices under correlation uncertainty, in particular, how the level of correlation uncertainty impacts the asset prices with respect to different asset characteristics that is explained next.

Let $\hat{\sigma}_i$ be the (return) volatility of asset i . Then $\sigma_i = \hat{\sigma}_i p_i$ and $\sigma_i \bar{x}_i = \hat{\sigma}_i (p_i \bar{x}_i)$. Notice that $p_i \bar{x}_i$ is the market capitalization of asset i , and $w_i \equiv \frac{p_i \bar{x}_i}{\sum_{i=1}^N p_i \bar{x}_i}$ represents the “size factor” of asset i .¹⁷ Therefore, $\sigma_i \bar{x}_i$ is proportional to $\hat{\sigma}_i w_i$, a *product of the volatility and the size factor*. $\hat{\sigma}_i w_i$ can be understood in two essentially equivalent ways. On one hand, $\hat{\sigma}_i w_i$ is a *weighted volatility* where the size factor contributes to the weight; on the other hand, $\hat{\sigma}_i w_i$ is a risk-adjusted size factor, either one captures the asset’s size factor and risk factor together.

¹⁷As documented in Moskowitz (2003), the firm size factor is the most powerful among others to predict future covariation.

In contrast to simple size factor and volatility, $\hat{\sigma}_i w_i$ is large only when both the size factor and the volatility (risk factor) is high or at least one factor is extremely large; and $\hat{\sigma}_i w_i$ is small if both the size factor and the volatility are small or at least one factor is very small. More importantly, $\hat{\sigma}_i w_i$ also tells us how the weighted volatility contributes to the total market risk whereas the correlation coefficient is disentangled from the covariance, by the following equation:

$$\hat{\sigma}_m = \sum_{i=1}^N \hat{\sigma}_i w_i \cdot \text{corr}(\tilde{R}_i, \tilde{R}_m) \quad (17)$$

where $\hat{\sigma}_m$ is the market return volatility. Furthermore, we introduce $\eta_i \equiv \frac{\hat{\sigma}_i w_i}{\sum_{n=1}^N \hat{\sigma}_n w_n}$, and Equation (17) ensures that η_i is a good proxy to represent the individual asset's risk contribution to the market.¹⁸

We use *eta* to identify asset characteristics and discuss the joint effect of asset characteristics and the level of correlation different ambiguity to Sharpe ratio in the next proposition.

- Proposition 4**
1. *The Sharpe ratio $s_i \geq s_j$, if and only if $\eta_i \geq \eta_j$. Moreover, s_i is larger than the average Sharpe ratio, $\frac{S}{N}$, if and only if $\eta_i \geq \frac{1}{N}$.*
 2. *The relative Sharpe ratio $\frac{s_i}{S}$ always decreases with respect to the level of correlation uncertainty and increases with more sophisticated agents when $\eta_i > \frac{1}{N}$; and it displays opposite monotonic feature when $\eta_i < \frac{1}{N}$;*
 3. *$\Omega(s)$ depends negatively on the level of correlation uncertainty, but it increases with respect to the number of sophisticated agents.*
 4. *For asset i with $\eta_i < \frac{1}{N}$, the higher the level of uncertainty the larger its Sharpe ratio; the effect of correlation uncertainty on the Sharpe ratio of asset i is negative if η_i is large enough.*

¹⁸In an equicorrelation model, we can uniquely determine the correlation coefficient $\text{corr}(\tilde{R}_i, \tilde{R}_m)$, given η_i of each individual asset. We can also characterize η_i with $\text{corr}(\tilde{R}_i, \tilde{R}_m)$ vice versa. The denominator, $\sum_{i=1}^N \hat{\sigma}_i w_i$, in η_i is also proportional to $\hat{\sigma}_m$. See Appendix C, Equation (C-6). The detailed proofs are provided in Appendix C.

Proposition 4 displays the symmetric property between the Sharpe ratio and the eta among all assets. Proposition 4, (1) states that the a larger Sharpe ratio always corresponds to a higher eta among all risky assets. By the same reason, a risky asset's Sharpe ratio is above the average Sharpe ratio only when its eta is above the average level $\frac{1}{N}$.

Other components of Proposition 4 concern with the comparative analysis of the Sharpe ratio s_i , the dispersion of Sharpe ratios $\Omega(s)$ and the relative Sharpe ratio $\frac{s_i}{S}$ when the level of correlation uncertainty changes or the percentage of sophisticated agents varies. We focus on the effect of correlation uncertainty in this section and postpone the discussion about the effect of ν in Section 6. Proposition 4,(2) follows from the following decomposition of relative Sharpe ratio $\frac{s_i}{S}$:

$$\frac{s_i}{S} - \frac{1}{N} = \frac{m - Nn}{m} \left(\eta_i - \frac{1}{N} \right), \quad (18)$$

which reveals how the relative Sharpe ratio departs from $\frac{1}{N}$ is proportional to the distance between its eta and $\frac{1}{N}$. Therefore, the sensitivity of the relative Sharpe ratio with respect to the level of correlation uncertainty relies on how large its eta to be, i.e, the sign of $\eta_i - \frac{1}{N}$. Precisely, this sensitivity is negative for a high eta asset, and positive for assets with low eta. With increasing correlation uncertainty, $\frac{s_i}{S}$ decreases if $\eta_i > \frac{1}{N}$ and increases if $\eta_i < \frac{1}{N}$.

Proposition 4,(2) is important to examine the effect of correlation uncertainty to different assets respectively. For a low eta asset, its relative Sharpe ratio increases with increasing perceived level of correlation uncertainty; however, the relative Sharpe ratio decreases otherwise for a high eta asset. The relative Sharpe ratios $\frac{s_i}{S}$ of “high eta” and “low eta” asset are displayed in Figure 5 in a full participation equilibrium. As demonstrated, for a high eta asset (in the upper plot of Figure 5), $\frac{s_i}{S}$ decreases with respect to the level of correlation uncertainty. It means that a high level of uncertainty results in a high eta asset less attractive. On the other hand, the low eta asset becomes relatively more attractive given an increase in correlation uncertainty. In the end, all assets intend to comove under high level of uncertainty. Figure 6 displays the comparative analysis about $\Omega(s)$. Table 2 also reports $\frac{s_i}{S}$ and $\Omega(s)$ in a limited participation equilibrium with the same property.

As all relative Sharpe ratios in the market, $\frac{s_1}{S}, \dots, \frac{s_N}{S}$, move closer to each other with higher correlation uncertainty, the dispersion of Sharpe ratios decreases as presented in Proposition 4, (3). Indeed, the dispersion of Sharpe ratios and the dispersion of $\eta(s)$ has a similar relation as in Equation (18):

$$\Omega(s) = \frac{m - Nn}{m} \Omega(\eta). \quad (19)$$

To understand further Proposition 4 (1)-(3), we introduce $\kappa \equiv \frac{\Omega(\eta)}{\Omega(s)}$, the dispersion measure of η and the dispersion measure of Sharpe ratio, to illustrate the fundamental relation between the η and the Sharpe ratio by Equation (18) and Equation (19). We argue that κ measures the effect of correlation uncertainty to comovement/contagion of all risky assets in the market for the following reasons. On one hand, fix the η of each asset, the higher κ the smaller dispersion of Sharpe ratio is; put differently, the higher likelihood of assets moving together. On the other hand, fixing the dispersion of Sharpe ratio, the smaller κ the smaller dispersion of η is; hence, each asset contributes similar market risk (weighted volatility) in the economy. Remarkably, κ depends purely on the endogenous correlation of each agent and his/her proportion in the market, regardless the marginal distribution of each asset.¹⁹

Last but not the least, Proposition 4, (4) demonstrates the effect of the correlation uncertainty to the risk premium. We decompose the Sharpe ratio into two components:

$$s_i = \frac{S}{N} + \overbrace{\frac{\gamma L}{m} \left(\eta_i - \frac{1}{N} \right)} \quad (20)$$

where the first component is the *average Sharpe ratio*, and the second one represents how much it differs from the average Sharpe ratio. We call the second component as a *specific Sharpe ratio*. The specific Sharpe ratio of asset i is proportional to the difference between the relative weighted volatility and the average level, $\eta_i - \frac{1}{N}$. Equation (20) is an alternative version of Equation (18).

¹⁹For instance, in a homogeneous equilibrium, $\kappa = \frac{1+(N-1)\rho}{1-\rho}$ increases with the correlation coefficient ρ . It displays similar property of the contagion measure, $\rho^* = \frac{\rho}{\sqrt{\rho^2+(1-\rho^2)/(1+\delta)}}$, suggested in Forbes and Rigobon (2002). A complete discussion of the contagion in the presence of correlation uncertainty is beyond the scope of this paper as we intend to focus on its asset pricing implications of the correlation uncertainty.

According to Proposition 4, (4), the level of the correlation uncertainty affects the Sharpe ratio very differently on assets with high and low eta. For a low eta asset (say, its eta is smaller than the average), the level of correlation uncertainty always *increases* the Sharpe ratio through two distinct channels: by increasing the average Sharpe ratio and by increasing the specific Sharpe ratio.²⁰ Thus, the Sharpe ratio increases and the price drops in an increasing correlation uncertainty environment.

However, for asset whose eta is larger than the average, the effect of correlation uncertainty is not straightforward, due to the opposing effects of the average Sharpe ratio and the specific Sharpe ratio. As the uncertainty increases, the average Sharpe ratio always increases, but the specific Sharpe ratio decreases. When the eta is large enough, the negative effect of the specific Sharpe ratio dominates the positive effect of the average Sharpe ratio, thus reaching an overall negative effect on the risk premium and the Sharpe ratio; as a consequence, the price increases.

Precisely, $\frac{\partial s_i}{\partial \bar{\rho}} < 0$ when

$$-\frac{\partial}{\partial \bar{\rho}} \left(\frac{1}{m} \right) \left(\eta_i - \frac{1}{N} \right) > \frac{\partial}{\partial \bar{\rho}} \left(\frac{1}{(m - Nn) * N} \right).$$

We numerically draw the Sharpe ratios of all three assets in a full participation equilibrium in Figure 7. The market parameters are $N = 3$, $\bar{a}_1 = 50$, $\sigma_1 = 9\%$, $\bar{x}_1 = 1$; $\bar{a}_2 = 60$, $\sigma_2 = 10\%$, $\bar{x}_2 = 5$; $\bar{a}_3 = 15$, $\sigma_3 = 12\%$, $\bar{x}_3 = 10.5$. By calculation, $\Omega(\sigma\bar{x}) = 0.56$, $\eta_1 = 0.048$, $\eta_2 = 0.027$ and $\eta_3 = 0.68$, hence both asset 1 and asset 2 are low eta assets and asset 3 has a high eta. Figure 7 shows how the Sharpe ratio changes when the perceived level of correlation uncertainty moves in a full participation equilibrium. Given the price movement under correlation uncertainty, high eta assets can be used to hedge “economic catastrophe shock” or ambiguity on the correlated structure, for instance, in a very weak market with high level of uncertainty. Proposition 4, (4) has an important implication to explain the asset price pattern in the 2007-2009 financial crisis time period. In a homogeneous environment with a representative agent, asset price declines with increase of correlation

²⁰By the expression of $m(x, y)$, $n(x, y)$ and the fact that $S = \frac{\gamma L}{m(x, y) - Nn(x, y)}$ and $\bar{\rho}_1 < \rho_2^*$ in Proposition 3, it is easy to see that both the average Sharpe ratio and $m(\bar{\rho}_1, \rho_2)$ depend positively on the level of uncertainty.

uncertainty. As an example, the entire stock market price drops in the 2007-2009 financial crisis time period (Caballero and Simsek, 2013) due to a high level of uncertainty.

To conduct the same analysis in a heterogeneous environment, we now introduce two types of asset classes. We treat the entire stock market as one asset class and those non-stock financial market (including fixed income, commodity and foreign exchange market) as another asset class. Although the volatility of the stock market is larger than the volatility in non-stock market, the volume of the non-stock market is much larger. It turns out that the stock market can be seen as a low eta asset and non-stock financial market as a high eta asset. To illustrate, we follow McKinsey Global Institute research (www.mckinsey.com/mgi) and report, in Figure 3, the global stock market and fixed income markets (including public debt market, financial Bonds, corporate bonds, securitized loan and unsecuritized loans outstanding) between 2005 to 2014. The total volume of the fixed income market is more than triple as the stock market's. Given the volatility of stock market is about three times of the fixed income market from historical data, the weighted volatility of each market should be about the same. If we further consider the comparable size of commodity market and a much larger size of foreign exchange market to be included into the weighted volatility of the non-stock market, the eta of the financial market without equity should be quite large compared to the eta of the stock market. Hence, the stock market is a low eta asset class while the financial market excluding the stock market, as another asset class, is a high eta asset class.

In the 2007-2009 financial crisis time period, the investor is very ambiguous about the entire financial market. Consistent with Proposition 4, the price of low eta asset, stock market, drops significantly, and at the same time, we observe a price increase of high eta assets, in particular, the government bond and hard commodity (gold, silver) market. Other features such as trading positions and trading volumes under correlation uncertainty will be discussed in Section 5-6 to provide more theoretical insights about the financial market phenomenon during the 2007-2009 financial crisis.

5 Optimal Portfolios

In this section we discuss and compare in details the optimal portfolio, $x^{(s)}$ and $x^{(n)}$, where $x^{(j)}$ is the optimal demand vector for agent $j \in \{s, n\}$. Our comparison between these two optimal portfolios is summarized by the next proposition.

- Proposition 5** 1. (Under-diversification and well-diversification) *Compared with the market portfolio $\sum_{i=1}^N \bar{x}_i \tilde{a}_i$, the naive agent always has an under-diversified portfolio while the sophisticated agent has a well-diversified portfolio. Moreover, increasing perceived level of correlation uncertainty induces less diversified optimal portfolio for each agent.*
2. (Portfolio Risk) *The sophisticated agent holds a riskier portfolio than the naive agent. Specifically, the variance of $\sum_i \tilde{a}_i x_i^{(s)}$ is strictly larger than the variance of $\sum_i \tilde{a}_i x_i^{(n)}$.*
3. (Portfolio Position) *The sophisticated agent holds long position on all risky assets; the naive agent holds long positions on high eta assets but short positions on the low eta assets.*
4. (Portfolio Performance) *The sophisticated agent has a better portfolio performance in the sense that her optimal portfolio's Sharpe ratio is strictly larger than the naive agent's. Moreover, the sophisticated agent has a higher maxmin expected utility than the naive agent.*
5. (Comovement) *From both agents' perspective, the optimal portfolio displays a comovement feature. Specifically, the covariance of these two optimal portfolios are greater than $(\frac{S}{N})^2$, which is always increasing with higher level of correlation uncertainty.*

For each agent $j \in \{s, n\}$, we write $\sigma_i x_i^{(j)} = \hat{\sigma}_i p_i x_i^{(j)}$, which is proportional to $\hat{\sigma}_i \omega_i^{(j)}$ and $\omega_i^{(j)}$ represents agent's wealth weight invested in asset i . We use the dispersion $\Omega(\hat{\sigma}_i \omega_i)$ of the risk-adjusted weights for each agent to measure the quantity of under-diversification of the optimal portfolio and to compare it with the market portfolio (Hirshleifer, Huang and Teoh, 2016). Proposition 5, (1) sheds some lights on the under-diversification or limited participation puzzle by virtue of the dispersion of risk-adjusted weights. Indeed, we prove

in a precise manner that the sophisticated agent has a better diversified portfolio than the market portfolio and the naive agent's optimal portfolio is less diversified. Hence, the sophisticated agent always chooses a well-diversified optimal portfolio and the naive agent's optimal portfolio is always under-diversified.

Extent theoretical studies posit that under-diversification occurs from different perspectives such as model misspecification (Uppal and Wang, 2003; Easley and O'Hara, 2009), heterogeneous beliefs (Milton and Vorkink, 2008; Hirshleifer, Huang and Teoh, 2016) and costly information (Van Nieuwerburgh and Veldkamp, 2010). For instance, Easley and O'Hara (2009) demonstrate limited participation happens at the presence of the marginal distribution ambiguity while assets are assumed to be independent. Uppal and Wang (2003) consider the ambiguity of both the joint distribution and the marginal distributions from a portfolio choice setting. Uppal and Wang (2003) show that numerically, when the overall ambiguity on the joint distribution is high, a small difference in ambiguity on the marginal return distribution will result in an under-diversified portfolio. By contrast, our result shows that under-diversification can be generated endogenously from the dispersion of correlation uncertainty, even without ambiguity on any marginal distribution. Furthermore, we demonstrate a well-diversified portfolio is associated with a better estimation on the correlated structure.

In a recent empirical study on the portfolio choice puzzles, Dimmock, Kouwenberg, Mitchell and Peijnenburg (2016) examine ambiguity-averse investors who view the overall market as a highly ambiguous under-diversified portfolio, using the under-diversification measure proposed in Calvet et al. (2009). They find that a one standard deviation increase in ambiguity aversion leads to a 38.9 percentage point increase in the fraction of equity allocated to individual stocks for those who view the overall market as highly ambiguous. In contrast, sophisticated agents with good stock market knowledge allocate little to individual stock. Proposition 5 (1) also justifies the evidence theoretically across agents in equilibrium.

We draw numerically in Figure 8 the dispersion of the optimal portfolios for both agents in a full participation equilibrium. Clearly, the dispersion of the sophisticated agent's optimal portfolio, as drawn in the upper plot, is smaller than the corresponding dispersion of the naive agent in the lower plot.

To demonstrate the result forcibly, we also consider a limited participation equilibrium and compute the optimal portfolio's dispersion for each agent, in Table 3. As shown, the naive agent's dispersion is fairly close to one, which reflects to his extremely under-diversified portfolio. By contrast, the dispersion of the sophisticated agent's optimal portfolio is between 0.625 and 0.629, indicating a more diversified holding.

Proposition 5, (2) concerns with the portfolio risk, which states that the sophisticated agent is willing to take a riskier portfolio than the naive agent due to the dispersion of correlation uncertainty among agents. The intuition is simple. Since ambiguity-aversion leads to risk-aversion, the naive agent behaves more risk-averse; hence, he has a smaller risk in his optimal portfolio. Proposition 5, (2), demonstrates that ambiguity-aversion leads to risk-aversion in the correlated structure,²¹ and a higher level of correlation uncertainty yields a higher level of risk-aversion; thus the corresponding optimal portfolio is less risky.

The comparison of portfolio risks can also be derived by Proposition 5, (3) which gives the position on each risky asset. By Proposition 5, (3), all positions on the risky asset in the sophisticated agent's optimal portfolio at time 1 must be "long" positions whereas the naive agent only hold long positions on high eta assets. In other words, the naive agent could have short positions on very low eta assets. As a consequence, the portfolio risk of the sophisticated agent is higher than the naive agent. This result about portfolio positions can be also illustrated by the 2007-2009 financial crisis time period. When the level of correlation uncertainty is high, all agents have long positions on very high eta assets (such as fixed income market or commodity), and thus, these asset prices move up because of market demand. However, since the naive agent is panic and overreacts to the market uncertainty, he reduces his positions on the stock market, or could even shorts the stock market; but the sophisticated agent still holds long positions in the stock market. Figure 9 displays the portfolio risk for the sophisticated agent as well as the naive agent, when $\bar{\rho}_1$ and $\bar{\rho}_2$ moves in a full participation equilibrium.

Proposition 5, (4) further compares the performance between these two optimal portfolios. As expected, the sophisticated agent has a higher Sharpe ratio portfolio than the naive agent. By the same token, the sophisticated agent also has a higher maxmin expected utility.

²¹It is well known that uncertainty aversion yields risk aversion in the ambiguity literature. See Easley and O'Hara (2009, 2010); Cao, Wang and Zhang (2005); Gollier (2011); Garlappi, Uppal and Wang (2007).

At last, Proposition 5, (5) investigates the comovement between the agents' optimal portfolios from their own perspectives and we observe a robust comovement pattern among agents.

6 Trading Volumes

After studying the optimal portfolio, we examine next the trading positions on individual assets. We further study the trading volume on each asset and investigate how the level of correlation uncertainty affects the trading position and trading volume. At last, we examine two important parameters: the number of sophisticated agents (institutional investors) and eta (risk) dispersion and their influence on the equilibrium.

For simplicity we assume that the sophisticated agent has a perfect knowledge about the market, a Savage investor who knows the true correlation coefficient ρ , and the naive agent's plausible range of correlation coefficient is $[\rho - \epsilon, \rho + \epsilon]$. To study how the level of correlation uncertainty affects the risk-sharing among agents, we assume that each agent holds a market portfolio initially (without the correlation uncertainty), so as to analyze the ambiguity effects on the trading volume precisely.

Proposition 6 1. *For low eta asset i with $\eta < \frac{1}{N}$, $x_i^{(s)}$ always increases and $x_i^{(n)}$ decreases with respect to ϵ . The effect of correlation uncertainty on $x_i^{(s)}$ and $x_i^{(n)}$ is completely opposite for very high eta asset.*

2. (Trading Pattern) *Put*

$$J(\epsilon, \nu) = \frac{1}{1 + (N - 1)\rho + (N - 1)\nu\epsilon}.$$

The sophisticated agent always sells high eta assets satisfying $\eta > J(\epsilon, \nu)$ and purchases low eta assets with $\eta < J(\epsilon, \nu)$; The naive agent always purchases high eta assets with $\eta > J(\epsilon, \nu)$ and sells low eta asset with $\eta < J(\epsilon, \nu)$.

3. (Trading Volume) *The higher the correlation uncertainty, the larger the trading volume, $|x_i^n - \bar{x}_i|$ and $|x_i^s - \bar{x}_i|$ for the sophisticated agent and the naive agent respectively.*

By Proposition 5,(3), the sophisticated agent always holds long position on each risky asset, but her position relies on the naive agent’s ambiguity about the correlated structure as described in Proposition 6, (1). Specially, when the naive agent’s perceived level of uncertainty increases, the sophisticated agent holds larger position on low eta assets and smaller positions on very high eta assets. On the contrary, the naive agent holds smaller positions on low eta assets or even short positions for very low eta assets (Proposition 5), and purchases more shares of very high eta assets. This property of the portfolio position under correlation uncertainty is displayed in Figure 10.

Proposition 6 (1) has a nice implication to asset pricing. We again interpret the stock market as a low eta asset. Proposition 6 (1) states that the sophisticated agent holds more on the stock market sine she has a perfect knowledge about the overall market and the price decline of the stock market virtually follows from the naive agent’s over selling on the stock market. Moreover, the more ambiguous the naive agent about the whole market, the less positions he holds on the stock market; in turn, the sophisticated agent holds more equity positions. Similarly, for the high eta asset (for instance, government bonds and gold) which is used to hedge against the “economic catastrophe risk”, the naive agent holds more and more positions.

We examine next the trading volume under assumption that each agent’s initial position is a well-diversified market portfolio. In other words, each agent has a well risk-sharing position initially at the absence of correlation uncertainty. As shown in Proposition 6, (2), the naive agent sells the low eta asset ($\eta < J(\nu, \epsilon)$) and purchases high eta asset ($\eta > J(\nu, \epsilon)$). Correspondingly, the sophisticated agent buys low eta asset and sells high eta assets. More importantly, the trading volume for each agent is increasing on almost all assets regardless low eta or high eta. Put differently, the more ambiguous the naive agent on the correlated structure, the more trading or overreaction in the market. Take the 2007-2009 financial crisis time period as an illustrative example, when the naive agent has a very high perceived level of ambiguity aversion on the entire financial market, there exists dramatic trading activities and extreme price declines for the stock market and substantial price increase pattern for government bonds, golds and other economical risk-hedging assets (high eta assets).

By Proposition 6, a flight to safety or flight to quality episode is generated, that is, a simultaneous price decline in one asset class with fire sale and an increase of price and volume

in another asset class. Our model offers an intuitive explanation about the flight-to-quality and flight-to-safety phenomenon in 2007-2009 financial crisis time period. Baele, Bekaert, Inghelbrecht and Wei (2013) identify empirically that the flight to safety episode coincides with significant increases in the VIX. Caballero and Krishnamurthy (2007), Vayanos (2004) and Guerrieri and Shimer (2014) characterize the flight-to-quality in other contexts of model uncertainty, liquidity risk or adverse selection. We provide a complementary to these previous studies to demonstrate that correlation uncertainty could generate flight-to-quality endogenously.

Proposition 6 (3) has a remarkable interpretation when we see ϵ as one type form of disagreement between the agents. Under this interpretation, Proposition 6 shows that a larger trading volume is associated with a larger disagreement between the sophisticated agent and the naive agent, and higher volatility is also associated with increasing disagreement. Our finding is consistent with the empirical evidence documented in Carlin, Longstaff and Matoba (2014).²²In the paper, they argue that the high volatility itself does not lead to higher trading volume, rather, it is only when disagreement arises in the market that higher uncertainty is associated with more trading. Proposition 6 presents a theoretical explanation of their important empirical findings through the correlation uncertainty mechanism.

The major insights on the trading volume in Proposition 6, (3) are explained briefly. Under the assumption that each agent has initial endowment \bar{x}_i , we obtain that $x_i^{(s)} < \bar{x}_i$ for high eta asset with $\eta_i > J(\epsilon, \nu)$ (“sells” the asset). The trading volume is thus $\bar{x}_i - x_i^{(s)}$ and the marginal increasing amount becomes

$$\frac{\partial}{\partial \epsilon} \left(\bar{x}_i - x_i^{(s)} \right) = -\frac{\partial}{\partial \epsilon} \left(x_i^{(s)} \right).$$

As shown in Proposition 6, (1), for such a high eta asset, a higher level of the naive agent’s correlation uncertainty the smaller $x_i^{(s)}$. Hence, $\frac{\partial}{\partial \epsilon} \left(\bar{x}_i - x_i^{(s)} \right) > 0$. On the other hand, for low eta asset and, in particular, $\eta_i < \frac{1}{N}$, the sophisticated agent wants to buy asset since

²²According to the construction of disagreement index, the disagreement largely depends on the dispersion of agents’ forecast which is also often used to measure the ambiguous level.

$x_i^{(s)} > \bar{x}_i$ and the trading volume satisfy (by Proposition 6, (1) again)

$$\frac{\partial}{\partial \epsilon} (x_i^s - \bar{x}_i) = \frac{\partial}{\partial \epsilon} (x_i^{(s)}) > 0.$$

Overall, the sophisticated agent has a larger trading volume on almost all assets if the naive agent's ambiguous about the correlated structure becomes larger. The market clearing condition in equilibrium ensures that the naive agent has higher trading volume with his increasing level of correlation uncertainty.

6.1 Implications of ν and $\Omega(\eta)$

We explain briefly several effects of the sophisticated agent proportion on the asset pricing. First, the number of sophisticated agent decides different equilibrium cases. We use $K(\rho, \nu, \Omega(\eta))$ to highlight its impact of ν and $\Omega(\eta)$, and assume that the naive agent has a reasonable ambiguity on the market in the sense that

$$\epsilon + \rho > \lim_{\nu \rightarrow 1} K(\rho, \nu, \Omega(\eta)).$$

For a small number of sophisticated agents, $\epsilon + \rho \leq K(\rho, \nu, \Omega(\eta))$, a full participation equilibrium is generated. However, if more and more sophisticated agents participate in the market such that $\epsilon + \rho > K(\rho, \nu, \Omega(\eta))$, a limited participation equilibrium prevails. In an extreme case, $\nu \rightarrow 1$, the limited participation equilibrium becomes the homogeneous equilibrium as in Proposition 2.

Second, given a larger number of sophisticated agent, $\Omega(s)$ is increased (by Proposition 4) so as for the expected utilities of each agent and their corresponding optimal portfolios perform better (according to Proposition 5). Because of this positive effect to market equilibrium, it provides strong incentives to have more sophisticated agent in the market from a regulatory perspective (See Easley and O'Hara and Yang (2015)'s similar argument to reduce the ambiguity of agents).

Third, a growing number of sophisticated agents has completely different effects to assets with low eta and high eta. Consider a low eta asset with $\eta_i < \frac{1}{N}$, it is easy to see

$$\frac{\partial}{\partial \nu} (s_i) = \frac{\partial}{\partial \nu} \left(\frac{S}{N} \right) + \frac{\partial}{\partial \nu} \left(\frac{\gamma L}{m} \right) \left(\eta_i - \frac{1}{N} \right) < 0. \quad (21)$$

Therefore, the risk premium of low eta asset drops with the increasing number of sophisticated agents (institutional investors). Similarly, the risk premium of high eta asset increases with the increasing number of institutional investors. In this regard Equation (21) is closely related to the empirical findings in Gompers and Metrick (2001), which empirically document that the small-company stock premium drops due to increasing demand of institutional investors. By the same reason, the institutional investors' demand on high eta firm would increase the premium. Equation (21) asserts that the small eta firm premium decreases and the high eta firm premium increases with increasing institutional investors.

To finish the discussion of the equilibrium analysis, we examine the effect of the risk dispersion, $\Omega(\eta)$, on equilibrium. The eta distribution $\Omega(\eta)$ performs another vital role in the heterogeneous equilibrium for two reasons.

On one hand, when each asset contributes similar weighted volatility risk (eta) to the market, the condition (14) in Proposition 3 is satisfied; thus, a full participation equilibrium is generated. In one extreme case, if each asset contributes the same risk, $\Omega(\eta) = 0$, then $\Omega(s) = 0$.

On the other hand, when assets have various contributions to the market risk, or alternatively, they offer a skewed eta distribution and $\Omega(\eta)$ is close to one, a limited participation equilibrium is obtained by Proposition 3. The intuition is very simple. We observe that

$$\lim_{\Omega(\eta) \rightarrow 1} K(\rho, \nu, \Omega(\eta)) = \rho. \quad (22)$$

Therefore, any naive agent must hold a limited portfolio for a skewed enough eta distribution. This analysis demonstrates another channel for the limited participation phenomena, that is, a limited participation can be obtained due to a large dispersion among asset characteristics. For instance, since U. S. equity market dominates other equity markets in an international finance setting in terms of eta, U.S. households concentrate their investments mainly on the U.S equity market instead of the international stock market.

7 Extension to block equicorrelation structure

So far, we assume that any two risky assets have the same, or very close, correlation coefficient. In this section, we extend it to Engle and Kelly (2013)'s block equicorrelation structure.

The block equicorrelation structure is justified as follows. We decompose these risky assets into K sectors based on their firm characteristics, business lines and so on. Therefore, the risky assets in each sector have relatively close correlation coefficient. Specifically, we group assets into several classes, A_1, A_2, \dots, A_k , such that the pairwise correlation coefficients among assets in each class A_i is very close to each other and the common correlation coefficient is denoted by ρ_i . We assume that $\rho_1 \gg \rho_2 \gg \dots \gg \rho_k$.²³ In this way, ρ_1 represents the largest possible correlation coefficient among all assets and any pair of assets in class A_1 has a correlation coefficient (or very close to) ρ_1 . The correlation coefficient ρ_2 denotes the second largest possible correlation coefficient, to some contexts, among all assets in the market. For this reason we assume that ρ_1 dominates ρ_2 , written as $\rho_1 \gg \rho_2$.

Under the above hierarchical correlation structure, we are able to extend the model setting in Section 2 to each asset class A_1, A_2, \dots, A_k separately, since the pairwise correlation coefficient in each asset class A_i is fairly close to each other. In fact, this result can be readily applied to any asset class in which the pairwise correlation coefficient is close to each other, when the correlation uncertainty within this asset class is a concern. For instance, because of the too big to fail issue and the substantial systemic risk, it is essential to understand the correlation structure among these big financial institutions as a group and how it changes with respect to macro-economic shocks.²⁴ Therefore, our results provide ideal environment to investigate the correlation uncertainty among those big financial institutions in this setting.

To demonstrate our extended setting, we consider two different types of correlation matrices among asset classes in the hierarchical structure. In the first one, we assume that each asset class is independent to each other, or has extremely small correlation coefficients

²³We use $a \gg b$ to represent that the number a is *way larger* than b . This decomposition of the correlation structure is similar to the principal component analysis in which the eigenvalues of the covariance matrix has a decreasing order in terms of " \gg ".

²⁴For instance, for big financial institutions such as Bank of America, Citi, AIG, Wells Fargo, JP Morgan, our computation shows that any pair of two big financial institutions in this group stays between 70% to 80% from 2001 to 2014, including both pre-crisis and post-crisis time periods.

at the least. For simplicity we use two asset classes to illustrate, the correlation matrix can be written as

$$R_1 = \begin{bmatrix} R(\rho_1) & 0 \\ 0 & R(\rho_2) \end{bmatrix}.$$

with $\rho_1 \gg \rho_2$ and each $R(\rho_i)$ is a correlation matrix with element ρ_i off the diagonal and the component being one along the diagonal.

For the second type of correlation matrix, we consider two asset classes while the first asset class contains asset $i \in I$ and the second asset class contains all other assets $j \in J$. Given the hierarchical structure, we assume that for each pair $i_1 \neq i_2 \in I$, $\text{corr}(\tilde{a}_{i_1}, \tilde{a}_{i_2}) = \rho_1$, and for each pair $j_1 \neq j_2 \in J$, $\text{corr}(\tilde{a}_{j_1}, \tilde{a}_{j_2}) = \rho_2$. We assume that $\rho_1 \gg \rho_2$. Since the correlation between asset class I and asset class J is not substantial, we assume that $\text{corr}(\tilde{a}_i, \tilde{a}_j) = \rho_1\rho_2$ for $i \in I, j \in J$. Precisely, the correlation matrix is written as

$$R_2 = \begin{bmatrix} R(\rho_1) & \rho_1\rho_2 B \\ \rho_1\rho_2 B' & R(\rho_2) \end{bmatrix}.$$

with $\rho_1 \gg \rho_2$ and $R(\rho_1)$ is a $n_1 \times n_1$ matrix, $R(\rho_2)$ is a $n_2 \times n_2$ matrix and B is a $n_1 \times n_2$ matrix with all entries being one. Since we are interested in the positive correlated environment, we assume that $\rho_i > 0$ for each $i = 1, \dots, K$ in each correlated structure.²⁵

Proposition 7 *In a homogeneous environment the representative agent has a correlation uncertainty $[\bar{\rho}_k - \epsilon_k, \bar{\rho}_k + \epsilon_k]$ on the correlation coefficient ρ_k in the asset class A_k for $k = 1, 2$. We assume that the correlation uncertainty is also consistent with the correlated structure in the sense that $\bar{\rho}_1 - \epsilon_1 > \bar{\rho}_2 + \epsilon_2$ for the first hierarchical structure, and $\bar{\rho}_1 - \epsilon_1 \gg \bar{\rho}_2 + \epsilon_2$. The agent chooses the highest possible correlation coefficients in equilibrium in the above two correlation structures with correlation uncertainty.*

According to Proposition 7, the representative agent chooses the highest plausible correlation coefficients in a relatively straightforward hierarchical correlated structure. Similar

²⁵Both R_1 and R_2 are two special examples of a block equicorrelation correlated structure with $\rho_{12} \in \left(-\sqrt{\frac{\rho_1(n_1-1)+\rho_2(n_2-1)+1}{n_1n_2}}, \sqrt{\frac{\rho_1(n_1-1)+\rho_2(n_2-1)+1}{n_1n_2}} \right)$ where $\text{corr}(\tilde{a}_i, \tilde{a}_j) = \rho_{12}$ for each $i \in I, j \in J$. See Engle and Kelly (2013).

to our previous discussion in Section 1, $\bar{\rho}_k + \epsilon_k$ relies on both the benchmark and the level of correlation uncertainty for each asset class A_k . In an extremely weak market situation where the level of correlation uncertainty is high, the endogenous correlation coefficients between assets will be high, and the endogenous risk premiums will increase and prices will drop enormously. Therefore, our previous arguments can be applied to the general correlation structure.

The endogenous correlation pattern in hierarchical correlated structure is richer than the one exhibited in Section 6, attributable to the heterogeneous uncertainty levels among agents. The endogenous market correlation is time-varying, depending on the benchmark of the correlation coefficients in each asset class and the level of uncertainty. Besides, the endogenous correlation inside each asset class or between asset classes increases with the number of sophisticated agents.

8 Conclusion

To investigate the on-going complicated correlation structure among asset classes and the nature of well-documented stylized facts on correlated structure, this paper develops an equilibrium model at the presence of correlation uncertainty where two types of agents have heterogeneous beliefs in their correlation estimation. We find that those correlation-related phenomena can be inherently connected through the disagreement among agents on the correlation structure, the heterogeneity of agents and asset risks, when the marginal distribution of each risky asset is perfectly estimated.

Specifically, (1)when the disagreement on correlation estimation is large; or (2)when more sophisticated agents emerge in the market; or (3)when the dispersion of asset risks is high, the choice of correlation coefficient for the naive agent is irrelevant even though his optimal portfolio is uniquely determined in equilibrium. Otherwise, each agent chooses the corresponding highest plausible correlation coefficient. Our portfolio analysis demonstrates that the sophisticated agent always holds a diversified portfolio versus the naive agent who is under-diversified. The optimal portfolio becomes less diversified when the perceived level of the correlation uncertainty increases. This equilibrium model is helpful to explain several empirical puzzles on correlation including asymmetric correlation, comovement, under-

diversification and flight to quality. Overall, this paper offers a theoretical description of the financial market in the 2007-2009 financial crisis time period and contributes further to the literature on the asset pricing implications of Knightian uncertainty.

Appendix A: Proof

The following Sherman-Morrison formula in linear algebra is useful in the subsequent derivations.

Lemma 1 *Suppose A is an invertible $s \times s$ matrix and u, v are $s \times 1$ vectors. Suppose further that $1 + v^T A^{-1} u \neq 0$. Then the matrix $A + uv^T$ is invertible and*

$$(A + uv^T)^{-1} = A^{-1} - \frac{A^{-1}uv^T A^{-1}}{1 + v^T A^{-1}u}. \quad (\text{A-1})$$

The next lemma computes the dispersion of optimal demand in terms of the dispersion of the Sharpe ratios.

Lemma 2 *Let $\rho \neq 1, \rho \neq -\frac{1}{N-1}$. Then*

$$\Omega(\sigma x_\rho) = \Omega(s) \frac{1 + (N-1)\rho}{1 - \rho}; \Omega(s) = \Omega(\sigma x_\rho) \frac{1 - \rho}{1 + (N-1)\rho} \quad (\text{A-2})$$

Proof. Note that $\sigma x_\rho = \frac{1}{\gamma} R(\rho)^{-1} s$. By Lemma 1,

$$R(\rho)^{-1} = \frac{1}{1 - \rho} I_N - \frac{1}{1 + (N-1)\rho} \frac{\rho}{1 - \rho} ee^T. \quad (\text{A-3})$$

Then $\Omega(\sigma x_\rho) = \Omega(t)$, where $t_i = s_i - \frac{\rho}{1 + (N-1)\rho} S$. We obtain $\sum_{i=1}^N t_i = \frac{1 - \rho}{1 + (N-1)\rho} S$, and

$$\begin{aligned} \sum_{i=1}^N t_i^2 &= \sum_{i=1}^N \left(s_i^2 - 2s_i S \frac{\rho}{1 + (N-1)\rho} + \left(\frac{\rho}{1 + (N-1)\rho} \right)^2 S^2 \right) \\ &= \sum_{i=1}^N s_i^2 - \frac{(N-2)\rho^2 + 2\rho}{(1 + (N-1)\rho)^2} S^2. \end{aligned}$$

By straightforward computation, we obtain

$$\Omega(t) = \Omega(s) \frac{1 + (N-1)\rho}{1 - \rho}. \quad (\text{A-4})$$

□

Proof of Proposition 1.

By Sion's theorem (1958),

$$\begin{aligned} A &= \max_{x \in \mathbb{R}^N} \min_{\rho \in [\underline{\rho}, \bar{\rho}]} \left\{ (\bar{a}_i - p_i)x_i - \frac{\gamma}{2} \sum_{i,j=1}^N x_i x_j \sigma_i \sigma_j R_{ij} \right\} \\ &= B \equiv \min_{\rho \in [\underline{\rho}, \bar{\rho}]} \max_{x \in \mathbb{R}^N} \left\{ (\bar{a}_i - p_i)x_i - \frac{\gamma}{2} \sum_{i,j=1}^N x_i x_j \sigma_i \sigma_j R_{ij} \right\} \end{aligned}$$

and both A and B are equal to $\min_{\rho \in [\underline{\rho}, \bar{\rho}]} \frac{1}{2\gamma} G(\rho)$, where

$$G(\rho) \equiv s^T R^{-1} s = \frac{N \sum_{n=1}^N s_n^2 - (\sum_{n=1}^N s_n)^2}{N(1-\rho)} + \frac{(\sum_{n=1}^N s_n)^2}{N(1+(N-1)\rho)}. \quad (\text{A-5})$$

$G(\rho)$ can be rewritten in term of the dispersion measure as follows.

$$G(\rho) = \frac{S^2}{N} \left(\frac{(N-1)\Omega(s)^2}{1-\rho} + \frac{1}{1+(N-1)\rho} \right).$$

Lemma 3 *Notations as above. The solution of $\min_{\rho \in [\underline{\rho}, \bar{\rho}]} G(\rho)$ is given by, when $\Omega(s) \neq \frac{1}{N-1}$,*

$$\rho^* = \begin{cases} \underline{\rho}, & \text{if } \underline{\rho} > \tau(\Omega(s)), \\ \bar{\rho}, & \text{if } \bar{\rho} < \tau(\Omega(s)), \\ \tau(\Omega(s)), & \text{if } \tau(\Omega(s)) \in [\underline{\rho}, \bar{\rho}]. \end{cases} \quad (\text{A-6})$$

If $\Omega(s) = \frac{1}{N-1}$, then ρ^* is given as above in which $\tau(\Omega(s))$ is replaced by $\frac{N-2}{2(N-1)}$.

Proof of Lemma 3. For simplicity we assume that $\Omega(s) \neq \frac{1}{N-1}$ and let $\hat{\tau}(\Omega(s)) \equiv \frac{1+\Omega(s)}{1-(N-1)\Omega(s)}$. Then,

$$G'(\rho) = \frac{(N-1)S^2}{N(1-\rho)^2[1+(N-1)\rho]^2} [\Omega(s)^2(N-1)^2 - 1][\rho - \tau(\Omega(s))][\rho - \hat{\tau}(\Omega(s))]. \quad (\text{A-7})$$

Thus, $\rho^* = \operatorname{argmin}_{\rho \in [\underline{\rho}, \bar{\rho}]} G(\rho)$ is determined in the following three cases, respectively.

- If $\Omega(s) \geq \frac{2}{N-2}$, then $|\tau(\Omega(s))| \leq 1$, $|\hat{\tau}(\Omega(s))| \leq 1$ and $\hat{\tau}(\Omega(s)) < \tau(\Omega(s))$. Moreover $\hat{\tau}(\Omega(s)) \leq 0 \leq \underline{\rho}$.
- If $\frac{2}{N-2} > \Omega(s) > \frac{1}{N-1}$, then $\hat{\tau}(\Omega(s)) < -1$, and $0 < \tau(\Omega(s)) < 1$.
- If $\Omega(s) < \frac{1}{N-1}$, then $|\tau(\Omega(s))| \leq 1$, $\hat{\tau}(\Omega(s)) > 1$; but $\Omega(s)^2(N-1) - 1 < 0$ so $G'(\rho) = (-) \times (\rho - \tau(\Omega(s))) \times (\rho - \hat{\tau}(\Omega(s)))$.

Then the proof of Lemma 3 is finished. \square

Proof of Proposition 1, Continue.

By the above argument, we have shown that

$$A = B = \frac{1}{2\gamma} G(\rho^*) = CE(\rho^*, x_{\rho^*}). \quad (\text{A-8})$$

(1). If $\underline{\rho} > \tau(\Omega(s))$, then by Lemma 2, $\Omega(\sigma x_{\underline{\rho}}) > 1$ and $\max_{\Omega(\sigma x) > 1} CE(\underline{\rho}, x) = CE(\underline{\rho}, x_{\underline{\rho}}) = A$. By equation (A-8) and the property of $G(\rho)$ stated in Lemma 3, we know that $\max_{\Omega(\sigma x) < 1} CE(\bar{\rho}, x) \leq \max_x CE(\bar{\rho}, x) < CE(\underline{\rho}, x_{\underline{\rho}})$. Similarly, $\max_{\Omega(\sigma x) = 1} CE(\rho, x) < CE(\underline{\rho}, x_{\underline{\rho}})$ for each $\rho \in [\underline{\rho}, \bar{\rho}]$. Then the unique solution of the problem (2) is $\rho^* = \underline{\rho}$, and $x^* = x_{\underline{\rho}}$.

(2). If $\bar{\rho} < \tau(\Omega)$, by Lemma 2, $\max_{\Omega(\sigma x) < 1} CE(\bar{\rho}, x) = CE(\bar{\rho}, x_{\bar{\rho}}) = A$. Moreover, by equation (A-8) and Lemma 3, we have $\max_{\Omega(\sigma x) > 1} CE(\underline{\rho}, x) \leq CE(\underline{\rho}, x_{\underline{\rho}}) < CE(\bar{\rho}, x_{\bar{\rho}})$ and $\max_{\Omega(\sigma x) = 1} CE(\rho, x) < CE(\bar{\rho}, x_{\bar{\rho}})$ for each $\rho \in [\underline{\rho}, \bar{\rho}]$. Therefore, $\rho^* = \bar{\rho}$, $x^* = x_{\bar{\rho}}$ is the unique solution of the portfolio choice problem (2).

(3). Assume that $\underline{\rho} \leq \tau(\Omega) \leq \bar{\rho}$. By equation (A-8), $A = B = CE(\tau(\Omega(s)), x_{\tau(\Omega(s))})$. Moreover, by Lemma 2, $\Omega(x_{\tau(\Omega(s))}) = 1$. By straightforward calculation, we have

$$A = B = CE(\tau(\Omega(s)), x_{\tau(\Omega(s))}) = \frac{(\sum_i s_i)^2}{2\gamma} \left(\frac{1 + (N-1)\Omega(s)}{N} \right)^2.$$

For any x with $\Omega(\sigma x) < 1$, by Lemma 3 and the last equation, we have

$$CE(\bar{\rho}, x) \leq CE(\bar{\rho}, x_{\bar{\rho}}) < A = CE(\tau(\Omega(s)), x_{\tau(\Omega(s))}).$$

By the same reason, for any x with $\Omega(\sigma x) > 1$, we see that $CE(\underline{\rho}, x) < CE(\tau(\Omega(s)), x_{\tau(\Omega(s))})$. Finally, for each x^* with $\Omega(\sigma x^*) = 1$ and $CE(\tau(\Omega(s)), x^*) = \max_{\Omega(\sigma x)=1} CE(\tau(\Omega(s)), x^*)$, we have $CE(\tau(\Omega(s)), x^*) = CE(\tau(\Omega(s)), x_{\tau(\Omega(s))})$, and because of the uniqueness x_ρ for maximizing $CE(\rho, x)$, $x^* = x_{\tau(\Omega(s))}$. Therefore, we have shown that the unique demand for the uncertainty-averse agent is $x_{\tau(\Omega(s))}$, but the agent is irrelevant to choose any correlation coefficient $\rho \in [\underline{\rho}, \bar{\rho}]$ since $CE(\rho, x_{\tau(\Omega(s))}) = A$ for each $\rho \in [\underline{\rho}, \bar{\rho}]$.

The proof of Proposition 1 is completed. \square

Proof of Proposition 2.

The optimal demand x^* is presented by Proposition 1. By the market clearing condition, $x^* = \bar{x}$ in equilibrium. Then $\Omega(\sigma x^*) = \Omega(\sigma \bar{x})$. Since $\Omega(\sigma \bar{x}) < 1$. Therefore, the optimal demand in equilibrium satisfies $\Omega(\sigma x^*) < 1$. Then, by Proposition 1 again, the optimal correlation coefficient is $\rho^* = \bar{\rho}$, the highest possible correlation coefficient. \square

In what follows we do not distinguish agent $j = 1, 2$ or $j = s, n$ for sophisticated agent and naive agent, respectively.

Proof of Proposition 3.

The unique optimal demand of type $j = 1, 2$ agent is $x_j = \frac{1}{\gamma} \sigma^{-1} R_j^{-1} s$ and R_j corresponds to an endogenous correlation coefficient $\rho_j \in [\underline{\rho}_j, \bar{\rho}_j]$. Note that x_j is unique regardless of the optimal correlation coefficient in Proposition 1, (3) and in this case let $\rho_j = \tau(\Omega(s))$. In equilibrium, we have $\nu x_1 + (1 - \nu)x_2 = \bar{x}$. Then

$$\frac{1}{\gamma}(\nu R_1^{-1} + (1 - \nu)R_2^{-1}) \cdot s = \sigma \bar{x}. \quad (\text{A-9})$$

The coefficient matrix of the last equation, $X = \nu R_1^{-1} + (1 - \nu)R_2^{-1}$, is written as

$$X \equiv \left(\frac{\nu}{1 - \rho_1} + \frac{1 - \nu}{1 - \rho_2} \right) I_N - \left(\frac{\nu \rho_1}{(1 - \rho_1)(1 + (N - 1)\rho_1)} + \frac{(1 - \nu)\rho_2}{(1 - \rho_2)(1 + (N - 1)\rho_2)} \right) ee^T \quad (\text{A-10})$$

Let $m \equiv \frac{\nu}{1 - \rho_1} + \frac{1 - \nu}{1 - \rho_2}$, $n \equiv \frac{\nu \rho_1}{(1 - \rho_1)(1 + (N - 1)\rho_1)} + \frac{(1 - \nu)\rho_2}{(1 - \rho_2)(1 + (N - 1)\rho_2)}$ and notice that $m - Nn = \frac{\nu}{1 + (N - 1)\rho_1} + \frac{1 - \nu}{1 + (N - 1)\rho_2} > 0$. Then by Lemma 1, X is invertible and its inverse matrix is (with

$$\kappa \equiv \frac{m - Nn}{m}$$

$$X^{-1} = \frac{1}{m} \left(I_N + \frac{n}{\kappa m} ee^T \right). \quad (\text{A-11})$$

Therefore, $s = \gamma X^{-1} \cdot (\sigma x)$. Precisely,

$$s_i = \frac{\gamma}{m} \left(\bar{y}_i + \frac{n}{\kappa m} L \right), L \equiv \sum_{i=1}^N \sigma_i \bar{x}_i. \quad (\text{A-12})$$

Hence

$$\sum s_i = \frac{\gamma}{m} \left(1 + \frac{Nn}{\kappa m} \right) L = \frac{\gamma L}{m - nN} \quad (\text{A-13})$$

and

$$\begin{aligned} \sum \left(\frac{s_i}{\gamma} \right)^2 &= \frac{C^2}{m^2} + \frac{L^2}{m^2} \left(\frac{Nn^2}{\kappa^2 m^2} + 2 \frac{n}{\kappa m} \right) \\ &= \frac{C^2}{m^2} + \frac{L^2}{m^2} \left(\frac{2n}{m - Nn} + \frac{Nn^2}{(m - Nn)^2} \right) \\ &= \frac{C^2}{m^2} + \frac{L^2}{m^2} \frac{2mn - Nn^2}{(m - Nn)^2} \end{aligned}$$

where $C^2 = \sum_{i=1}^N (\sigma_i \bar{x}_i)^2$. Then

$$\begin{aligned} \frac{\sum s_i^2}{(\sum s_i)^2} &= \frac{C^2}{L^2} \frac{(m - Nn)^2}{m^2} + \frac{2mn - Nn^2}{m^2} \\ &= \frac{C^2}{L^2} \kappa^2 + \frac{1}{N} (1 - \kappa^2). \end{aligned}$$

Therefore,

$$\Omega(s)^2 = \frac{1}{N-1} \left(N \frac{\sum s_i^2}{(\sum s_i)^2} - 1 \right) = \kappa^2 \Omega(\sigma \bar{x})^2, \quad (\text{A-14})$$

and then a fundamental relationship between the dispersion of Sharpe ratios and the dispersion of risks as follows

$$\Omega(s) = \kappa \Omega(\sigma \bar{x}). \quad (\text{A-15})$$

Assume first that $\Omega(\sigma\bar{x}) = 0$, then each $\sigma_i\bar{x}_i = c$ and Equation (A-15) ensures that $\Omega(s) = 0$ and $\tau(\Omega(s)) = 1$. Therefore, each agent chooses her highest correlation coefficient in equilibrium by Proposition 1. Moreover, all Sharpe ratios are the same and equal to

$$s_i = \frac{c}{m} \left(1 + \frac{nN}{m - nN} \right) = \frac{c}{m - nN}. \quad (\text{A-16})$$

We next assume that $\Omega(\sigma\bar{x}) \in (0, 1)$ and characterize the equilibrium in general. By using Proposition 1, there are five different cases regarding the equilibrium.

Case 1: $\tau(\Omega(s)) \leq \underline{\rho}_2$. Each agent chooses the smallest possible correlation coefficient, respectively.

Case 2: $\underline{\rho}_2 < \tau(\Omega(s)) < \underline{\rho}_1$. The sophisticated agent chooses the smallest possible correlation coefficient; the naive agent is irrelevant of the correlation coefficient and his optimal holding is $x_{\tau(\Omega(s))}$.

Case 3: $\tau(\Omega(s)) \in [\underline{\rho}_1, \bar{\rho}_1]$. Each agent is irrelevant of the correlation coefficient and the corresponding optimal holding is $x_{\tau(\Omega(s))}$.

Case 4: $\bar{\rho}_1 < \tau(\Omega(s)) \leq \bar{\rho}_2$. The sophisticated agent chooses the highest possible correlation coefficient; the naive agent is irrelevant of the correlation coefficient and his optimal holding is $x_{\tau(\Omega(s))}$.

Case 5: $\bar{\rho}_2 < \tau(\Omega(s))$, each agent chooses her highest possible correlation coefficient.

We prove next that Case 1 - Case 3 is impossible in equilibrium and characterize the equilibrium condition in both Case 4 and Case 5.

To proceed we state a simple lemma below and investigate each case respectively. Recall

$$\kappa = \kappa(\rho_1, \rho_2) \equiv \frac{\frac{\nu}{1+(N-1)\rho_1} + \frac{1-\nu}{1+(N-1)\rho_2}}{\frac{\nu}{1-\rho_1} + \frac{1-\nu}{1-\rho_2}}. \quad (\text{A-17})$$

Lemma 4 *Assume that $\kappa = \frac{\nu a + (1-\nu)b}{\nu c + (1-\nu)d}$ with $a, b, c, d > 0$ and $\nu \in (0, 1)$. Then*

$$\min \left\{ \frac{a}{c}, \frac{b}{d} \right\} \leq \kappa \leq \max \left\{ \frac{a}{c}, \frac{b}{d} \right\}. \quad (\text{A-18})$$

The inequalities are strictly if $\frac{a}{c} \neq \frac{b}{d}$.

Case 1. We show that this situation is impossible in equilibrium. Since $\tau(\Omega(s)) \leq \underline{\rho}_2 < \underline{\rho}_1$ ensures that $\Omega(s) \geq \tau(\underline{\rho}_2) > \tau(\underline{\rho}_1)$. Then, by Equation (A-15),

$$\kappa = \kappa(\underline{\rho}_1, \underline{\rho}_2) = \frac{\Omega(s)}{\Omega(\sigma\bar{x})} > \Omega(s) > \tau(\underline{\rho}_1), \tau(\underline{\rho}_2), \quad (\text{A-19})$$

which is impossible by Lemma 4.

Case 2. We prove that it is impossible in equilibrium in this situation that $\underline{\rho}_2 < \tau(\Omega(s)) \leq \underline{\rho}_1$.

In this case, $\tau(\underline{\rho}_2) > \Omega(s) \geq \tau(\underline{\rho}_1)$. But

$$\kappa = \kappa(\underline{\rho}_1, \tau(\Omega(s))) > \kappa\Omega(\sigma\bar{x}) = \Omega(s) \geq \frac{1 - \underline{\rho}_1}{1 + \underline{\rho}_1(N - 1)}, \kappa > \Omega(s) = \frac{1 - \tau(\Omega(s))}{1 + (N - 1)\tau(\Omega(s))}. \quad (\text{A-20})$$

Hence, it is impossible by using Lemma 4 again.

Case 3. We prove that it is impossible that $\tau(\Omega(s)) \in [\underline{\rho}_1, \bar{\rho}_1]$ in equilibrium.

Otherwise, the optimal holding of each agent is $x_{\tau(\Omega(s))}$ by Proposition 1. Then the market clearing condition yields that $x_{\tau(\Omega(s))} = \bar{x}$. However, by Lemma 2, $\Omega(\sigma x_{\tau(\Omega(s))}) = 1$ but $\Omega(\sigma\bar{x}) < 1$. Therefore, Case 3 is not possible in equilibrium.

Case 4. We characterize the equilibrium in which $\bar{\rho}_1 < \tau(\Omega(s)) \leq \bar{\rho}_2$. In equilibrium, $\rho_1 = \bar{\rho}_1$, the optimal holding for the naive agent is $x_{\tau(\Omega(s))}$, and

$$\tau(\bar{\rho}_2) \leq \Omega(s) < \tau(\bar{\rho}_1). \quad (\text{A-21})$$

By definition of κ , we have

$$\kappa = \kappa(\bar{\rho}_1, \tau(\Omega(s))) = \frac{\Omega(s)}{\Omega(\sigma\bar{x})} = \frac{\frac{\nu}{1+(N-1)\bar{\rho}_1} + \frac{1-\nu}{N}(1 + (N-1)\Omega(s))}{\frac{\nu}{1-\bar{\rho}_1} + \frac{1-\nu}{\Omega(s)N}(1 + (N-1)\Omega(s))}. \quad (\text{A-22})$$

By solving the last equation in $\Omega(s)$, we obtain

$$\Omega(s) = \frac{\frac{\nu}{1+(N-1)\bar{\rho}_1}\Omega(\sigma\bar{x}) - \frac{1-\nu}{N}(1 - \Omega(\sigma\bar{x}))}{\frac{\nu}{1-\bar{\rho}_1} + \frac{(1-\nu)(N-1)}{N}(1 - \Omega(\sigma\bar{x}))}. \quad (\text{A-23})$$

and

$$K(\bar{\rho}_1) \equiv \tau(\Omega(s)) = \frac{\frac{1}{1-\bar{\rho}_1} - \frac{\Omega(\sigma\bar{x})}{1+(N-1)\bar{\rho}_1} + \frac{(1-\nu)(1-\Omega(\sigma\bar{x}))}{\nu}}{\frac{1}{1-\bar{\rho}_1} + \frac{(N-1)\Omega(\sigma\bar{x})}{1+(N-1)\bar{\rho}_1}}. \quad (\text{A-24})$$

In particular, $\Omega(s) \geq 0$ ensures that

$$\bar{\rho}_1 \leq \frac{1}{N-1} \left\{ \frac{\nu}{1-\nu} \frac{\Omega(\sigma\bar{x})}{1-\Omega(\sigma\bar{x})} N - 1 \right\}. \quad (\text{A-25})$$

Moreover, the left side of Equation (A-21) is translated as $\bar{\rho}_2 \geq \tau(\Omega(s)) = K(\bar{\rho}_1)$, and the right side of Equation (A-21) is $K(\bar{\rho}_1) \geq \bar{\rho}_1$ which holds always. Then there exists a unique equilibrium in Case 4, a limited participation equilibrium, under conditions presented in Proposition 3.

Case 5. We characterize the equilibrium in which $\tau(\Omega(s)) > \bar{\rho}_2$.

In equilibrium, $\rho_1 = \bar{\rho}_1, \rho_2 = \bar{\rho}_2$, and $\Omega(s) < \tau(\bar{\rho}_2) < \tau(\bar{\rho}_1)$, which is equivalent to $\kappa(\bar{\rho}_1, \bar{\rho}_2)\Omega(s) < \frac{1-\bar{\rho}_2}{1+(N-1)\bar{\rho}_2}$. By straightforward computation, this condition equals to

$$\left\{ \frac{\nu}{1-\bar{\rho}_1} + \frac{\Omega(\sigma\bar{x})\nu(N-1)}{1+(N-1)\bar{\rho}_1} \right\} \bar{\rho}_2 < \frac{\nu}{1-\bar{\rho}_1} - \frac{\Omega(\sigma\bar{x})\nu}{1+(N-1)\bar{\rho}_1} + (1-\nu)(1-\Omega(\sigma\bar{x})) \quad (\text{A-26})$$

or equivalently, $\bar{\rho}_2 < K(\bar{\rho}_1)$.

To the end, we note that when $\bar{\rho}_1$ is large enough such that

$$\frac{\Omega(\sigma\bar{x})\nu N}{(1-\nu)(1-\Omega(\sigma\bar{x}))} \leq 1 + (N-1)\bar{\rho}_1, \quad (\text{A-27})$$

or equivalently,

$$\bar{\rho}_1 \geq \frac{1}{N-1} \left\{ \frac{\nu}{1-\nu} \frac{\Omega(\sigma\bar{x})}{1-\Omega(\sigma\bar{x})} N - 1 \right\},$$

then $\bar{\rho}_2 < K(\bar{\rho}_1)$ holds by natural since $\bar{\rho}_2 < 1$. Then we have characterized the equilibrium in Proposition 3.

Finally, by using this characterization, we see that each $s_i > 0$. Therefore, each risky asset is priced at discount in equilibrium. \square

Proof of Proposition 4.

The first part of (1) follows from the characterization of the Sharpe ratio in Proposition

3. The statement for the sophisticated agent follows from the expression of $\sigma_i x_i^{(s)}$:

$$\gamma \sigma_i x_i^{(s)} = \frac{1}{1 - \bar{\rho}_1} \left\{ s_i - \frac{\bar{\rho}_1}{1 + (N - 1)\bar{\rho}_1} S \right\}. \quad (\text{A-28})$$

Then, $\sigma_i x_i^{(s)} \geq \sigma_j x_j^{(s)}$ if and only if $s_i \geq s_j$. The proof for the naive agent is the same.

To proceed, by Proposition 3, we have

$$s_i = \frac{\gamma}{m} \left(\sigma_i \bar{x}_i + \frac{n}{m - Nn} L \right)$$

and then

$$S = \frac{\gamma}{m - Nn} L.$$

Therefore, we obtain Equation (18), and (1) is proved.

(2). By using the fact that $\frac{\kappa(\rho_1, \rho_2)}{\partial \rho_1} < 0$, $\frac{\kappa(\rho_1, \rho_2)}{\partial \rho_2} < 0$, and $\frac{\kappa(\rho_1, \rho_2)}{\partial \nu} > 0$, (2) follows from Equation (18).

(3). The statement regarding the average Sharpe ratio S/N follows from the fact that $\frac{\partial}{\partial \bar{\rho}}(m - Nn) < 0$, $\frac{\partial}{\partial \nu}(m - Nn) > 0$. On the other hand, according to Equation (A-15), $\Omega(s) = \kappa \Omega(\sigma \bar{x})$. In the limited participation equilibrium, we have

$$\Omega(s) = \frac{\frac{\Omega(\sigma \bar{x}) \nu N}{1 + (N - 1)\bar{\rho}_1} - (1 - \nu)(1 - \Omega(\sigma \bar{x}))}{\frac{\nu N}{1 - \bar{\rho}_1} + (1 - \nu)(1 - \Omega(\sigma \bar{x}))(N - 1)}. \quad (\text{A-29})$$

Clearly, $\frac{\partial \Omega(\sigma \bar{x})}{\partial \bar{\rho}_1} < 0$ and $\frac{\partial \Omega(\sigma \bar{x})}{\partial \nu} > 0$. The proof for the full participation equilibrium follows from the fact that $\frac{\kappa(\rho_1, \rho_2)}{\partial \rho_1} < 0$, $\frac{\kappa(\rho_1, \rho_2)}{\partial \rho_2} < 0$, and $\frac{\kappa(\rho_1, \rho_2)}{\partial \nu} > 0$.

(4). As explained in the text, it follows from the following decomposition of the Sharpe ratio

$$s_i = \frac{S}{N} + \frac{\gamma}{m} \left(\sigma_i \bar{x}_i - \frac{L}{N} \right). \quad (\text{A-30})$$

The proof of Proposition 4 is finished. \square

Proof of Proposition 5.

(1). We compare the agents' optimal portfolios with the market portfolio in term of the dispersion measure $\Omega(\cdot)$. First, Equation (A-15) states that $\Omega(s) = \kappa\Omega(\sigma\bar{x})$. In the full participation equilibrium, Lemma 2 implies that $\Omega(s) = \Omega(\sigma x^{(j)})\frac{1-\rho_j}{1+(N-1)\rho_j}$. Then, by Lemma 4 we have

$$\Omega(\sigma x^{(s)}) < \Omega(\sigma\bar{x}) < \Omega(\sigma x^{(n)}). \quad (\text{A-31})$$

The proof in the limited participation equilibrium is the same. Since $\bar{\rho}_1 \leq \tau(\Omega(s))$, Lemma 2 and Lemma 4 together imply that $\Omega(\sigma x^{(s)}) < \Omega(\sigma\bar{x})$. Moreover $\Omega(\sigma\bar{x}) < 1 = \Omega(\sigma x^{(n)})$.

To examine the effect of the level of the correlation uncertainty to the quantity of the under-diversified measure, we note that $\Omega(\sigma x^{(s)}) = \Omega(\sigma\bar{x})\kappa(\bar{\rho}_1, \rho_2)\frac{1+(N-1)\bar{\rho}_1}{1-\bar{\rho}_1}$. Since $\kappa(\bar{\rho}_1, \rho_2)\frac{1+(N-1)\bar{\rho}_1}{1-\bar{\rho}_1}$ is clearly increasing when $\bar{\rho}_1$ moves up, thus $\Omega(\sigma x^{(s)})$ increases so the optimal portfolio of the sophisticated agent becomes more under-diversified. Other remaining cases are proved similarly.

(2). Let $X^{(s)} = \sum_i \tilde{a}_i x_i^{(s)}$ be the optimal portfolio of the sophisticated agent and $X^{(n)}$ is the optimal portfolio of the naive agent. We have $Var(X^{(s)}) = \frac{1}{\gamma^2} s'R(\bar{\rho}_1)^{-1}s$ where the correlation coefficient is ρ_1 , and $Var(X^{(n)}) = s'R(\rho_2)^{-1}s$ with the correlation coefficient ρ_2 . By using the same notation as above, we have

$$Var(X^{(s)}) - Var(X^{(n)}) = \frac{1}{\gamma^2} \{G(\bar{\rho}_1) - G(\rho_2)\}. \quad (\text{A-32})$$

Applying Proposition 3, since the portfolio choice is under the worst-case correlation uncertainty, $G(\bar{\rho}_1) > G(\rho_2)$ holds. Therefore, $Var(X^{(s)}) > Var(X^{(n)})$.

(3). We have

$$\gamma\sigma_i x_i^{(s)} = \frac{1}{1-\bar{\rho}_1} \left(s_i - \frac{\bar{\rho}_1}{1+(N-1)\bar{\rho}_1} S \right), i = 1, \dots. \quad (\text{A-33})$$

Then $x_i^{(s)} > 0$ if and only if $\frac{s_i}{S} > \frac{\bar{\rho}_1}{1+(N-1)\bar{\rho}_1}$. By Equation (20), we have

$$\frac{s_i}{S} = \frac{m - Nn}{m} \left(\frac{\sigma_i \bar{x}_i}{L} \right) + \frac{n}{m}.$$

By Lemma 4, $\frac{n}{m} > \frac{\bar{\rho}_1}{1+(N-1)\bar{\rho}_1}$, then it follows $x_i^{(s)} > 0$. On the other hand, $x_i^{(n)} > 0$ if, and only if

$$\eta_i > \frac{\nu}{1 - \bar{\rho}_1} \frac{\rho_2 - \bar{\rho}_1}{1 + \nu(N-1)\rho_2 + (1-\nu)(N-1)\bar{\rho}_1}.$$

Therefore, there is a short position on asset i for the naive agent if, and only if asset i has a small relative weighted volatility,

$$\eta_i < \frac{\nu}{1 - \bar{\rho}_1} \frac{\rho_2 - \bar{\rho}_1}{1 + \nu(N-1)\rho_2 + (1-\nu)(N-1)\bar{\rho}_1}.$$

(4). $\mathbb{E}[X^{(s)}] = \sum_{i=1}^N x_i^{(s)}(\bar{a}_i - p_i) = (\sigma x^{(s)})'s = \frac{1}{\gamma} s'R(\bar{\rho}_1)s$. By (2), the variance of $X^{(s)}$ is $\frac{1}{\gamma^2} s'R(\bar{\rho}_1)^{-1}s$. Then the Sharpe ratio of the portfolio $X^{(s)}$ is

$$SR(X^{(s)}) = \sqrt{s'R(\bar{\rho}_1)^{-1}s}.$$

Therefore, $SR(X^{(s)}) > SR(X^{(n)})$ follows from $G(\bar{\rho}_1) > G(\rho_2)$. Moreover, by the proof of Proposition 5, the sophisticated agent's maxmin expected utility is $CH(\bar{\rho}_1, x_{\bar{\rho}_1}) = \frac{1}{2\gamma}G(\bar{\rho}_1)$. Again, the fact that $G(\bar{\rho}_1) > G(\rho_2)$ ensures that the sophisticated agent has a higher maxmin expected utility than the naive agent.

(5). For the sophisticated agent, the covariance between two portfolios $X^{(s)}$ and $X^{(n)}$ can be calculated as $(\sigma x^{(s)})'R(\bar{\rho}_1)(\sigma x^{(n)})'$, which is the same as $s'R(\rho_2)^{-1}s$. Similarly, the covariance of the naive agent's optimal portfolio is $s'R(\bar{\rho}_1)^{-1}s$. By the proof in Proposition 1, both covariances are bounded from below by $s'R(\tau(\Omega(s)))^{-1}s$. It suffices to show that $s'R(\tau(\Omega(s)))^{-1}s > 0$. By simple calculation, we have

$$s'R(\tau(\Omega(s)))^{-1}s = S^2 \left(\frac{1 + (N-1)\Omega(s)}{N} \right)^2 \quad (\text{A-34})$$

which is always greater than $\left(\frac{S}{N}\right)^2$. □

Proof of Proposition 6.

(1). By Proposition 4, $\frac{s_i}{S}$ is increasing with respect to ϵ when $\eta_i < \frac{1}{N}$. Since the average Sharpe ratio S is always increasing with respect to the level of correlation uncertainty, and

$$\gamma\sigma_i x_i^{(s)} = \frac{1}{1 - \bar{\rho}} S \left(\frac{s_i}{S} - \frac{\bar{\rho}}{1 + (N - 1)\bar{\rho}} \right),$$

thus $\frac{\partial}{\partial \epsilon} \left(x_i^{(s)} \right) > 0$. By using the market clearing equation, $\nu x_i^{(s)} + (1 - \nu)x_i^{(n)} = \bar{x}_i$, we have

$$\frac{\partial}{\partial \rho_1} \left(x_i^{(n)} \right) < 0.$$

If η_i is large, the level of correlation uncertainty to s_i is negative, by Proposition 4, (4), but S is positively relates to ϵ . Therefore, Equation (A-33) implies that $\frac{\partial}{\partial \epsilon} \left(x_i^{(s)} \right) < 0$ and

$$\frac{\partial}{\partial \epsilon} \left(x_i^{(n)} \right) > 0.$$

(2). By using the expression of $x_i^{(s)}$ and the expression of Equation (A-33) and the expression of s_i, S , it is easy to check that $\gamma\sigma_i x_i^{(s)} < \gamma\sigma_i \bar{x}_i$ is equivalent to $\eta_i > J(\epsilon, \nu)$. Because of the market clearing condition, $x_i^{(n)} < \bar{x}_i$ if and only if $\eta_i < J(\epsilon, \nu)$.

We next examine the trading volume. First, the naive agent buys the high eta asset with $\eta > J(\epsilon, \nu)$, so the trading volume is $|x_i^{(n)} - \bar{x}_i| = x_i^{(n)} - \bar{x}_i$. Then, by Proposition 6, (1),

$$\frac{\partial}{\partial \epsilon} \left(x_i^{(n)} - \bar{x}_i \right) = \frac{\partial}{\partial \epsilon} \left(x_i^{(n)} \right) > 0.$$

On the other hand, for the low eta asset, the naive trading volume is $\bar{x}_i - x_i^{(n)}$ since he needs to sell the initial position, thus, by Proposition 6, (1), we obtain

$$\frac{\partial}{\partial \epsilon} \left(\bar{x}_i - x_i^{(n)} \right) = -\frac{\partial}{\partial \epsilon} \left(x_i^{(n)} \right) > 0.$$

By the similar argument, we can show that, for the sophisticated agent,

$$\frac{\partial}{\partial \epsilon} \left| \bar{x}_i - x_i^{(s)} \right| > 0.$$

□

Proof of Proposition 7.

For the first correlated structure with K classes, the inverse matrix of the correlation matrix is

$$(R_1)^{-1} = \begin{bmatrix} R(\rho_1)^{-1} & \cdot & \cdots & 0 \\ \cdot & R(\rho_2)^{-1} & \cdots & 0 \\ \cdot & \cdots & \cdots & \cdot \\ 0 & \cdot & \cdots & R(\rho_K)^{-1} \end{bmatrix}.$$

Moreover, from the portfolio choice perspective, the optimal correlation coefficient is, for each $k = 1, \dots, K$, determined by Proposition 1.

For the second correlated structure, it is straightforward to derive (for $J = 1$)

$$(R_2)^{-1} = \begin{bmatrix} X & -\rho_2 X e \\ -\rho_2 e' X & 1 + \rho_2^2 e' X e \end{bmatrix}.$$

where X is the inverse matrix of the $M \times M$ matrix $R(\rho_1) - \rho_2^2 e e'$. By using the Sherman-Morrison formula, we have

$$X = \frac{1}{1 - \rho_1} \left(I_M - \frac{\rho_1 - \rho_2^2}{1 + (M - 1)\rho_1 - M\rho_2^2} e e' \right). \quad (\text{A-35})$$

Therefore, the diversification benefits, $G(\rho_1, \rho_2) \equiv (s_1, \dots, s_M, s_{M+1}) \cdot (R^{II})^{-1} \cdot (s_1, \dots, s_M, s_{M+1})'$, with the correlation coefficient ρ_1 and ρ_2 in the second correlated structure, can be written as

$$G(\rho_1, \rho_2) = s' X s + 2 \underline{\rho} t \sum_{i=1}^M s_i + t^2 \bar{\rho} \quad (\text{A-36})$$

where

$$s = (s_1, \dots, s_M)', t = s_{M+1}, \underline{\rho} = -\frac{\rho_2}{1 + (M - 1)\rho_1 - M\rho_2^2}, \bar{\rho} = 1 + \frac{M\rho_2^2}{1 + (M - 1)\rho_1}.$$

By simple algebra, we have

$$\frac{\partial G(\rho_1, \rho_2)}{\partial \rho_2} = \frac{2M\rho_2}{1 + (M - 1)\rho_1} t^2 - \frac{2(1 + (M - 1)\rho_1 + M\rho_2^2)}{(1 + (M - 1)\rho_1 - M\rho_2^2)^2} t \sum_{i=1}^M s_i. \quad (\text{A-37})$$

Therefore, for a region of ρ_2 with small values compared with ρ_1 , $G(\rho_1, \rho_2)$ is decreasing with respect to ρ_2 . For the discussion on ρ_1 , the proof is similar to the proof in Proposition 2. Therefore, the representative agent always chooses the highest possible correlations $\bar{\rho}_1 + \epsilon_1, \bar{\rho}_2 + \epsilon_2$ in equilibrium. \square

Appendix B: A Dispersion Measure

Given a time series of economic variables $X_1(t), \dots, X_N(t), t = 1, 2, \dots$, there are many approaches to estimate how these economic variables are correlated, interconnected and co-dependent. The literature is largely concerned with common factors in these economic variables. On the other hand, if these economic variables are fairly dispersed, it is not likely to have significant common factor. Therefore, a dispersion measure to some extent also measures the correlated information as we explain below.

A function $f : (X_1, \dots, X_N) \in \mathbb{R}^N \rightarrow [0, 1]$ is a dispersion measure if it satisfies the following three properties:

1. (Positively homogeneous property) Given any $\lambda > 0$, $f(\lambda X_1, \dots, \lambda X_N) = f(X_1, \dots, X_N)$;
2. (Symmetric property) Given any translation $\sigma : \{1, \dots, N\} \rightarrow \{1, \dots, N\}$, $f(X_1, \dots, X_N) = f(X_{\sigma(1)}, \dots, X_{\sigma(N)})$;
3. (Majorization property) Assuming (X_1, \dots, X_N) weakly dominates (Y_1, \dots, Y_N) , then $f(X_1, \dots, X_N) \geq f(Y_1, \dots, Y_N)$.

By a vector $a = (a_1, \dots, a_N)$ weakly dominates $b = (b_1, \dots, b_N)$ we mean that

$$\sum_{i=1}^k a_i^* \geq \sum_{i=1}^k b_i^*, k = 1, \dots, N \quad (\text{B-1})$$

where a_i^* is the element of a stored in decreasing order. When $f(Y) \leq f(X)$ for a dispersion measure we call Y is more dispersed than X under the measure f . One example is the portfolio weight in a financial market so the dispersion measure captures how one portfolio is more dispersed than another. Samuelson's famous theorem (Samuelson, 1967) states that equally-weighted portfolio is always the optimal one for a risk-averse agent when the risky assets have IID return. Boyle et al (2012) also find that equally-weight portfolio beats many optimal asset allocation under parameter uncertainty. As another example, the dispersion measure can be used to investigate the diversification of optimal portfolio in a general setting. See Ibragimov, Jaffee and Walden (2011) and Hennessy and Lapan (2003).

This paper concerns with one example of dispersion measure.

Lemma 5 Given a non-zero vector $X = (X_1, \dots, X_N)$,

$$f(X_1, \dots, X_N) = \sqrt{\frac{1}{N-1} \left(N \frac{\sum X_i^2}{(\sum_i X_i)^2} - 1 \right)}$$

is a dispersion measure.

Proof: Both the positive homogeneous property and the symmetric property are obviously satisfied. To prove the majorization property, we first assume that, because of the positively homogeneous property, $\sum X_i = \sum Y_i = 1$. Let $g(x_1, \dots, x_N) = N(\sum_i x_i^2) - (\sum_i x_i)^2$. Notice that $g(\cdot)$ is a Schur convex function in the sense that

$$(x_i - x_j) \left(\frac{\partial g}{\partial x_i} - \frac{\partial g}{\partial x_j} \right) \geq 0, \forall i, j = 1, \dots, N.$$

Then by the majorization theorem, see Marshall and Olkin (1979), $g(X) \geq g(Y)$. Then $f(X) \geq f(Y)$. □

Appendix C: The eta (η_i) and $\text{corr}(\tilde{R}_i, \tilde{R}_m)$

We explain the economic meaning of the asset characteristic parameter “eta” η_i . As shown in Proposition 8 below, η_i can be determined by $\text{corr}(\tilde{R}_i, \tilde{R}_m)$ for $i = 1, \dots, N$ or vice versa in an equicorrelation model, thus η_i characterizes the sensitivity of \tilde{R}_i with respect to the market portfolio return \tilde{R}_m . Our subsequent analysis holds under any distribution assumption of the asset return \tilde{R}_i . In a classic linear factor asset pricing model, it is well known that $\text{corr}(\tilde{R}_i, \tilde{R}_m)$ is in essence characterized by the *beta* parameter. Here we will show that our eta parameter is closely related to the *beta* in CAPM.

Proposition 8 *In an equicorrelation model with $\text{corr}(\tilde{R}_i, \tilde{R}_j) = \rho, \forall i \neq j$ and the market portfolio return $\tilde{R}_m = \sum_{i=1}^N w_i \tilde{R}_i$.*

1. *Given η_i , where $i = 1, \dots, N$, the correlation coefficient between individual asset return with the market portfolio return, $\text{corr}(\tilde{R}_i, \tilde{R}_m)$, is given by the following equation*

$$\text{corr}(\tilde{R}_i, \tilde{R}_m) = \frac{\rho + \eta_i(1 - \rho)}{\sqrt{\rho + \frac{(N-1)\Omega(\eta)^2 + 1}{N}(1 - \rho)}} \quad (\text{C-1})$$

where $\Omega(\eta)$ is the dispersion measure of η_1, \dots, η_N .

2. *Conversely, let $\alpha_i = \text{corr}(\tilde{R}_i, \tilde{R}_m), i = 1, \dots, N$ with $\sum_{i=1}^N \alpha_i \neq 0$, then*

$$\eta_i = \frac{\alpha_i}{\sum_{i=1}^N \alpha_i} \cdot \frac{1 + (N-1)\rho}{1 - \rho} - \frac{\rho}{1 - \rho} \quad (\text{C-2})$$

Proposition 8, (1) determines explicitly the correlation coefficient $\text{corr}(\tilde{R}_i, \tilde{R}_m)$ in terms of its eta and the dispersion of etas. In particular, if each individual asset has the same weighted volatility, they have the same eta and each $\eta_i = \frac{1}{N}$ due to the fact that $\sum_{i=1}^N \eta_i = 1$. Moreover, $\Omega(\eta) = 0$ if each eta is $1/N$. Then, for each asset $i = 1, \dots, N$, it has the same correlation coefficient with the market portfolio by

$$\text{corr}(\tilde{R}_i, \tilde{R}_m) = \sqrt{\rho + \frac{1 - \rho}{N}}.$$

By Proposition 8, (2), we see that η_i is up to a linear transformation of the relative correlation with the market portfolio $\frac{corr(\tilde{R}_i, \tilde{R}_m)}{\sum_{i=1}^N corr(\tilde{R}_i, \tilde{R}_m)}$. It is worth noting that the correlation coefficient $\alpha_i = corr(\tilde{R}_i, \tilde{R}_m)$ can not be arbitrarily given by Proposition 8. We can easily derive an equation of these correlation coefficients $corr(\tilde{R}_i, \tilde{R}_m)$ by using Equation (C-1), Equation (C-2) and the definition of $\Omega(\eta)$.

Proof: Since $corr(\tilde{R}_i, \tilde{R}_j) = \rho, \forall i \neq j$, we have

$$Cov(\tilde{R}_i, \tilde{R}_m) = Cov\left(\tilde{R}_i, \sum w_j \tilde{R}_j\right) = \left(\sum_j w_j \hat{\sigma}_j\right) \hat{\sigma}_i \rho + w_i \hat{\sigma}_i^2 (1 - \rho). \quad (C-3)$$

It follows that

$$corr(\tilde{R}_i, \tilde{R}_m) = \frac{\left(\sum_j w_j \hat{\sigma}_j\right) \rho + w_i \hat{\sigma}_i (1 - \rho)}{\hat{\sigma}_m}. \quad (C-4)$$

Moreover, Equation (C-3) yields the variance of the market portfolio,

$$\hat{\sigma}_m^2 = \left(\sum_j w_j \hat{\sigma}_j\right)^2 \rho + \sum_j w_j^2 \hat{\sigma}_j^2 (1 - \rho), \quad (C-5)$$

then by using the dispersion $\Omega(\eta) = \Omega(w\hat{\sigma})$, we obtain

$$\hat{\sigma}_m = \left(\sum_{j=1}^N w_j \hat{\sigma}_j\right) \sqrt{\rho + \frac{(N-1)\Omega(\eta)^2 + 1}{N}(1 - \rho)}. \quad (C-6)$$

By plugging Equation (C-6) into Equation (C-4), we derive Equation (C-1) as desired.

Conversely, let x denote the denominator in Equation (C-1), then by Equation (C-1) again with $\alpha_i = corr(\tilde{R}_i, \tilde{R}_m)$, we obtain

$$\rho + \eta_i(1 - \rho) = \alpha_i x. \quad (C-7)$$

By using $\sum_{i=1}^N \eta_i = 1$, it follows that

$$\sum_{i=1}^N \alpha_i x = N\rho + 1 - \rho, \quad (C-8)$$

yielding

$$x = \frac{1 + (N - 1)\rho}{\sum_{i=1}^N \alpha_i}. \quad (\text{C-9})$$

Equation (C-2) follows from Equation (C-7). □

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Figure 1: VIX vs S&P 500 index

This figure displays the VIX and S&P 500 index from Jan 2006 to Jan 2016. Resource: Chicago Board Option Exchange. As documented in Bloom (2009), VIX measure the macroeconomic uncertainty as well as the ambiguity on the entire financial market. The high level of VIX in 2007-2009 reflects to higher level of Knightian uncertainty. See Caballero and Krishnamurthy (2008). It is clear that the stock market largely moves in opposite direction with the VIX, in particular, in 2007-2009.

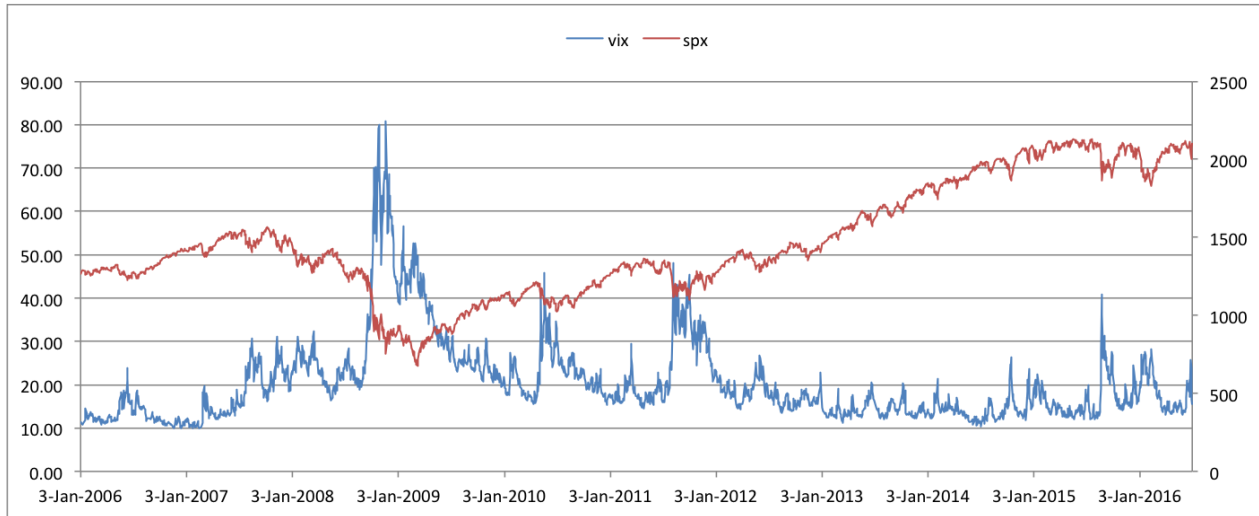


Figure 2: VIX vs Gold price

This figure compares VIX and the gold price from Jan 2006 to Jan 2016. Resource: Chicago Board Option Exchange. It can be seen that as “uncertainty hedging” asset, the gold price increases significantly when the VIX is on a high level in 2007-2009. It is also true for other uncertainty hedging assets such as government bonds.

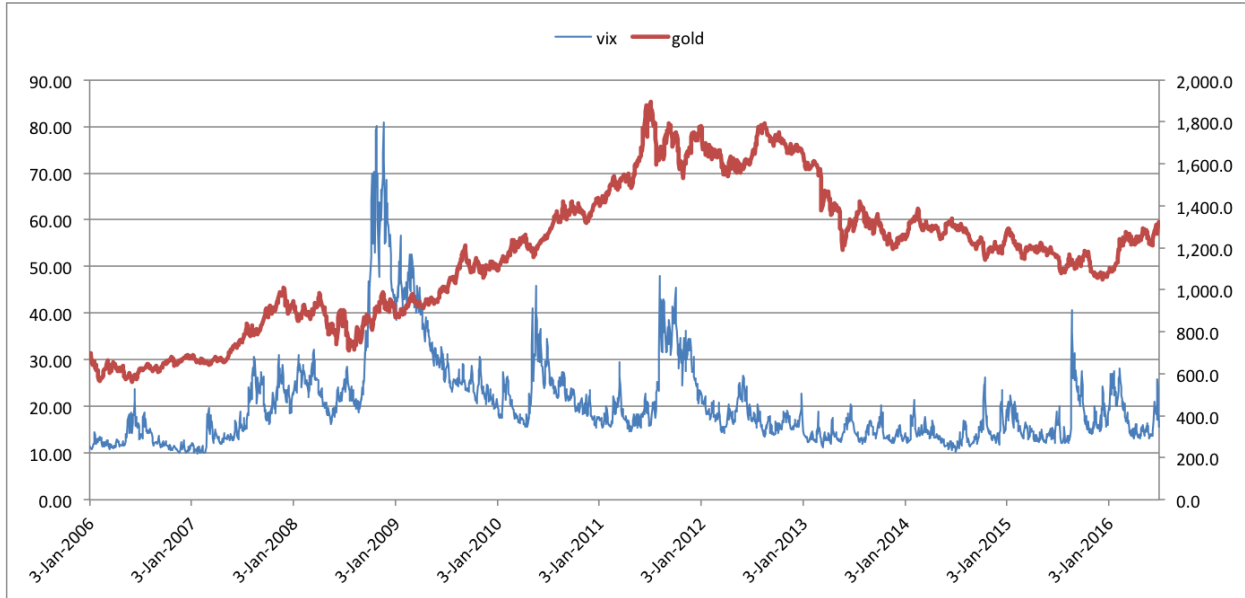


Figure 3: Comparison between equity market and debt/loan markets.

The objective of this graph to show that the stock market is a low eta asset class comparing with all other financial markets. Resource: McKinsey Global Institute research (www.mckinsey.com/mgi). In this graph, “others” denotes the debt/loan market, including “Public Debt market”, “Financial Bonds”, “Corporate Bonds”, “Securitized Loan Market” and “Unsecuritized Loans Outstanding Market”. We report the percentage of the equity market in the larger market consists of “equity” and “others” between 2005 to 2014. As can be seen, the equity market is around 20 percent to 30 percent. Even though the volatility of equity market is larger than the volatility of the debt/loan market, its weighted volatility is fairly comparable with the debt/loan market. Therefore, if we put commodity market, the foreign exchange market, and the debt/loan market together as one asset class, the stock market is a low eta asset class and all others can be interpreted as a high eta asset class.

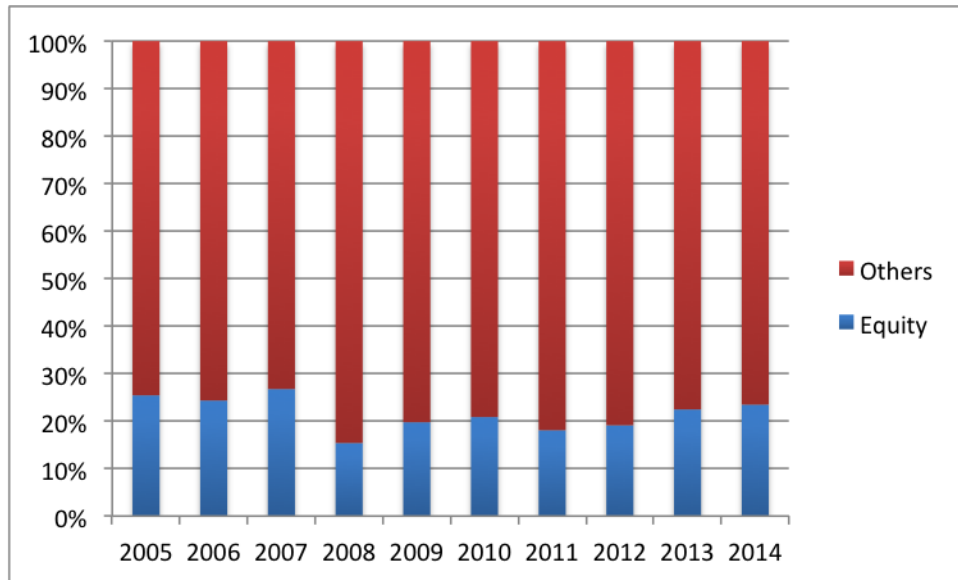


Figure 4: $K(\rho)$ with respect to ρ

This graph displays the property of the correlation boundary, $K(\rho)$, over the region $0 \leq \rho \leq \frac{1}{N-1} \left\{ \frac{\nu}{1-\nu} \frac{\Omega(\sigma\bar{x})}{1-\Omega(\sigma\bar{x})} N - 1 \right\}$. As shown, $K(\rho)$ is always strictly larger than ρ , and $K(\rho)$ increases when ρ increases. The parameters are $N = 3$, $\nu = 0.6$, $\bar{a}_1 = 50$, $\sigma_1 = 2\%$, $\bar{x}_1 = 0.5$; $\bar{a}_2 = 60$, $\sigma_2 = 8\%$, $\bar{x}_2 = 3$; $\bar{a}_3 = 15$, $\sigma_3 = 12\%$, $\bar{x}_3 = 10.5$. Notice that $\Omega(\sigma\bar{x}) = 0.7631$ and $\frac{1}{N-1} \left\{ \frac{\nu}{1-\nu} \frac{\Omega(\sigma\bar{x})}{1-\Omega(\sigma\bar{x})} N - 1 \right\} = 6.75$.

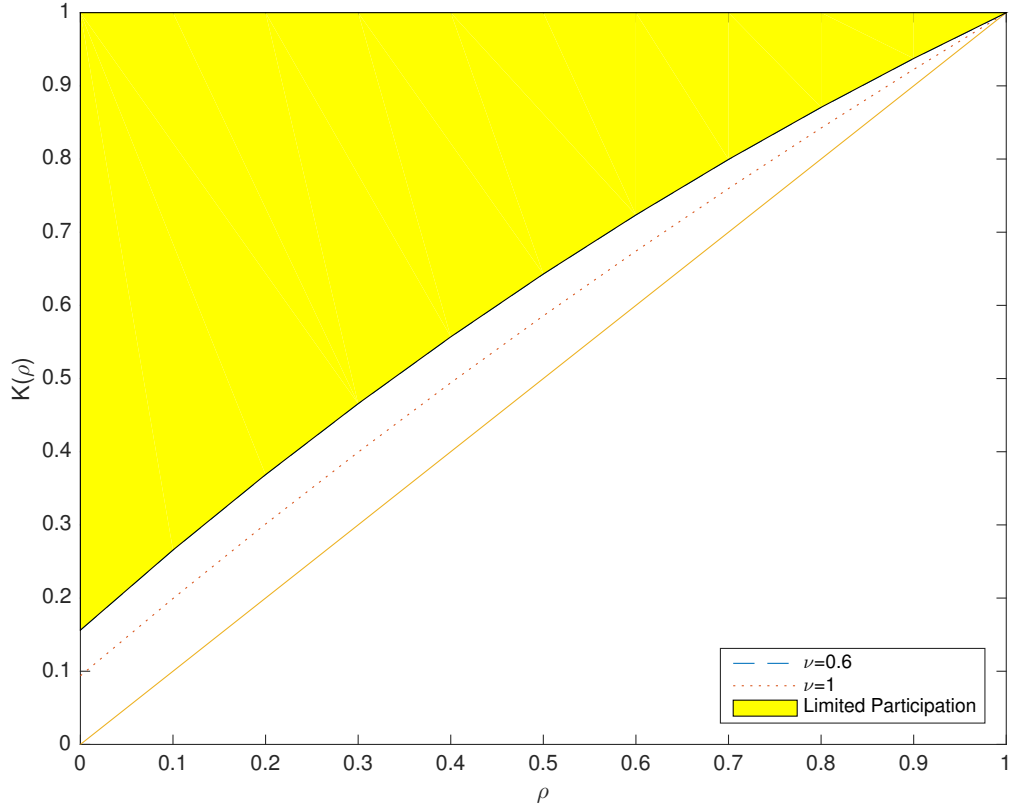


Figure 5: Relative Sharpe ratios

This three dimensional figure reports the relative Sharpe ratios, s_i/S , for high eta asset and low eta asset. The parameters in this figure are $N = 3$, $\nu = 0.3$, $\sigma_1 = 9\%$, $\bar{x}_1 = 1$; $\sigma_2 = 10\%$, $\bar{x}_2 = 5$; $\sigma_3 = 12\%$, $\bar{x}_3 = 10.5$. By computation $\Omega = 0.56$ and $\frac{1}{N-1} \left\{ \frac{\nu}{1-\nu} \frac{\Omega}{1-\Omega} N - 1 \right\} = 0.3$. Moreover $\eta_1 = 0.048$, $\eta_2 = 0.0938$ and $\eta_3 = 0.8582$. Thus, Asset 3 is a *high eta* asset while both Asset 1 and Asset 2 are *low eta* assets.

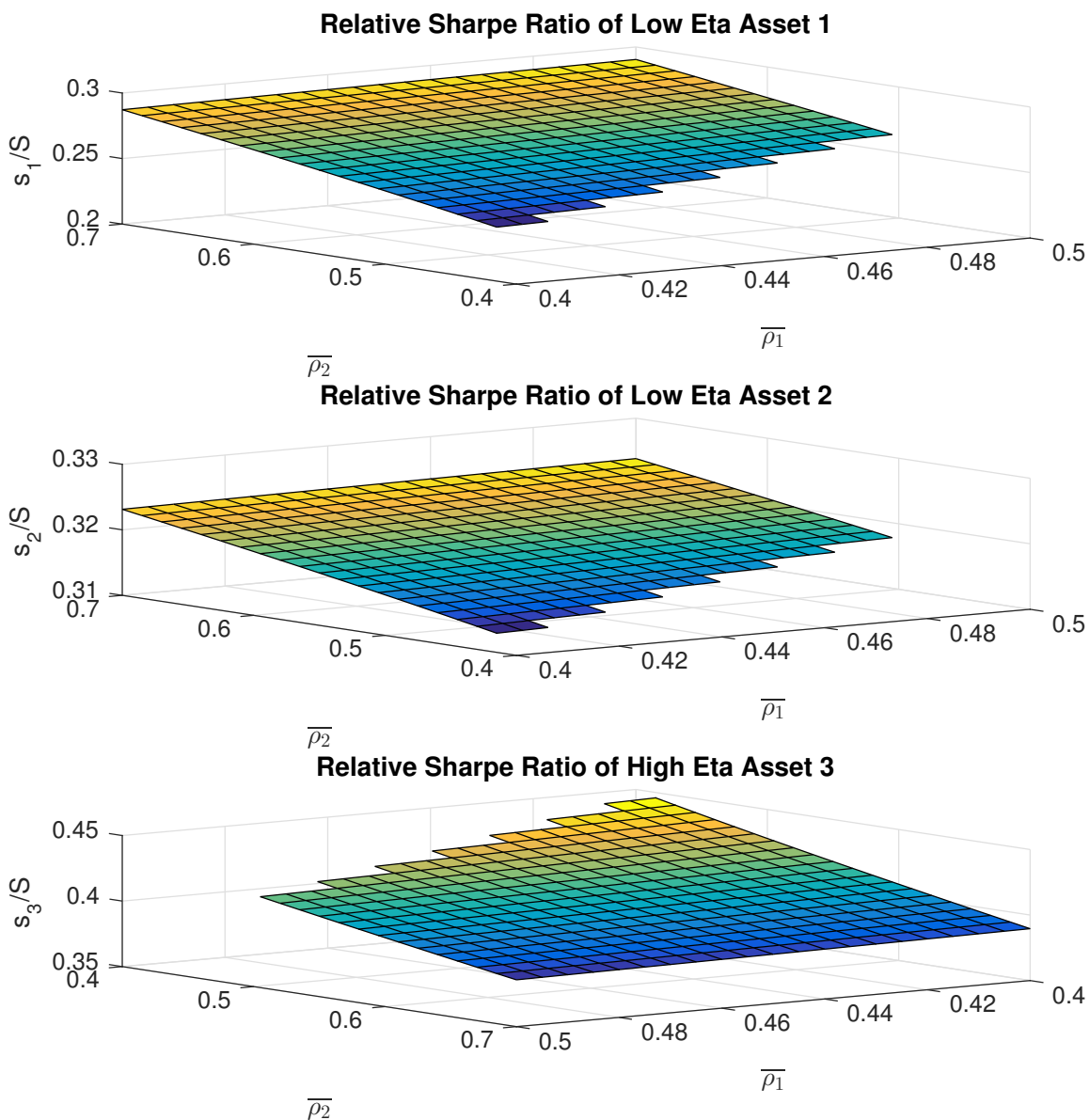


Figure 6: Dispersion of Sharpe Ratios

This three dimensional figure plots the dispersion, $\Omega(s)$, of Sharpe ratios in a full participation equilibrium. As shown the dispersion decreases with increasing of $\bar{\rho}_1$ and $\bar{\rho}_2$, of sophisticated and naive respectively. The same decreasing property of the dispersion in the homogeneous environment and in a limited participation equilibrium are reported in Table 2 and Table 3. The parameters in this figure are $N = 3$, $\nu = 0.3$, $\sigma_1 = 9\%$, $\bar{x}_1 = 1$; $\sigma_2 = 10\%$, $\bar{x}_2 = 5$; $\sigma_3 = 12\%$, $\bar{x}_3 = 10.5$. Notice that $\Omega(\sigma\bar{x}) = 0.56$ and $\frac{1}{N-1} \left\{ \frac{\nu}{1-\nu} \frac{\Omega(\sigma\bar{x})}{1-\Omega(\sigma\bar{x})} N - 1 \right\} = 0.3$.

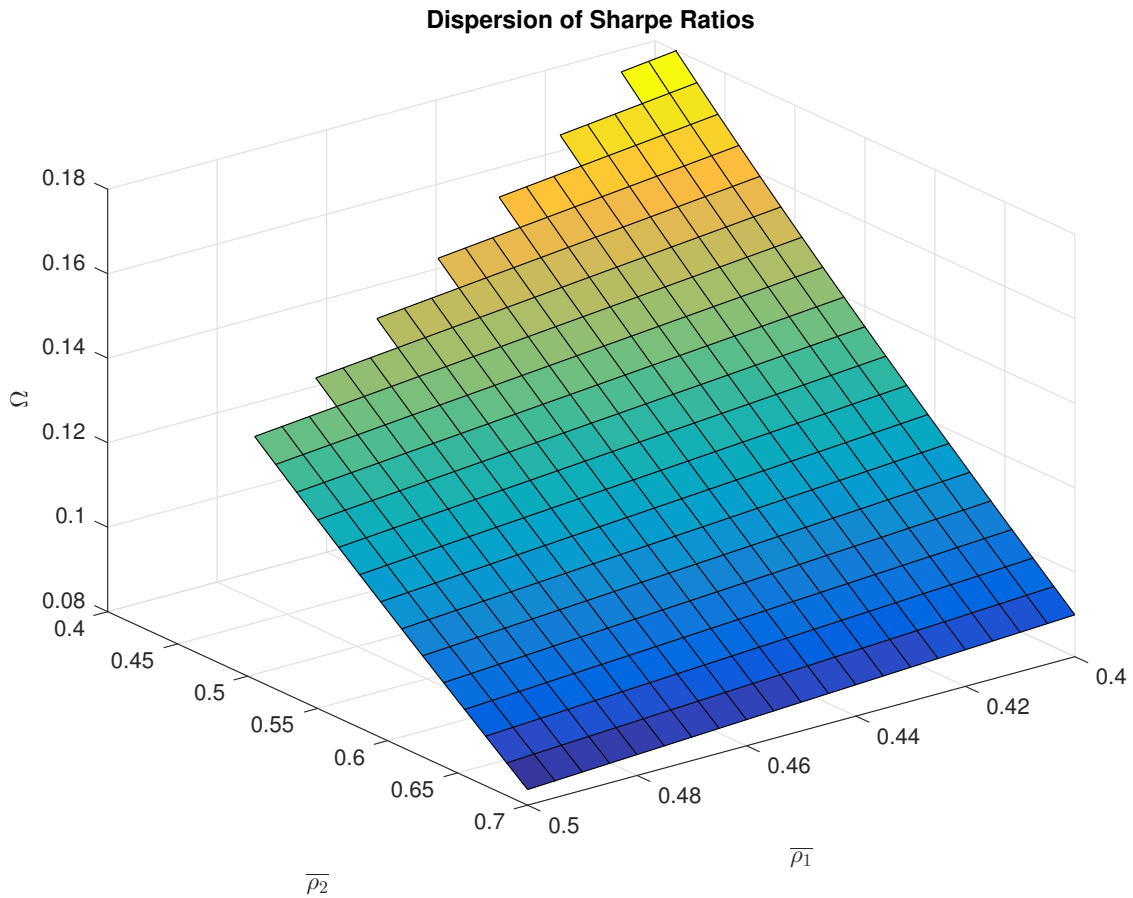


Figure 7: Sharpe ratios in a full participation equilibrium

This three dimensional figure shows the sensitivity of Sharpe ratios with respect to the highest plausible correlation coefficient $\bar{\rho}_1$ and $\bar{\rho}_2$ in a full participation equilibrium. The parameters are $N = 3$, $\nu = 0.3$, $\sigma_1 = 9\%$, $\bar{x}_1 = 1$; $\sigma_2 = 10\%$, $\bar{x}_2 = 5$; $\sigma_3 = 12\%$, $\bar{x}_3 = 10.5$. Notice that $\Omega(\sigma\bar{x}) = 0.56$ and $\frac{1}{N-1} \left\{ \frac{\nu}{1-\nu} \frac{\Omega(\sigma\bar{x})}{1-\Omega(\sigma\bar{x})} N - 1 \right\} = 0.3$. Both Asset 1 and Asset 2 are low eta assets and Asset 3 is a high eta asset.

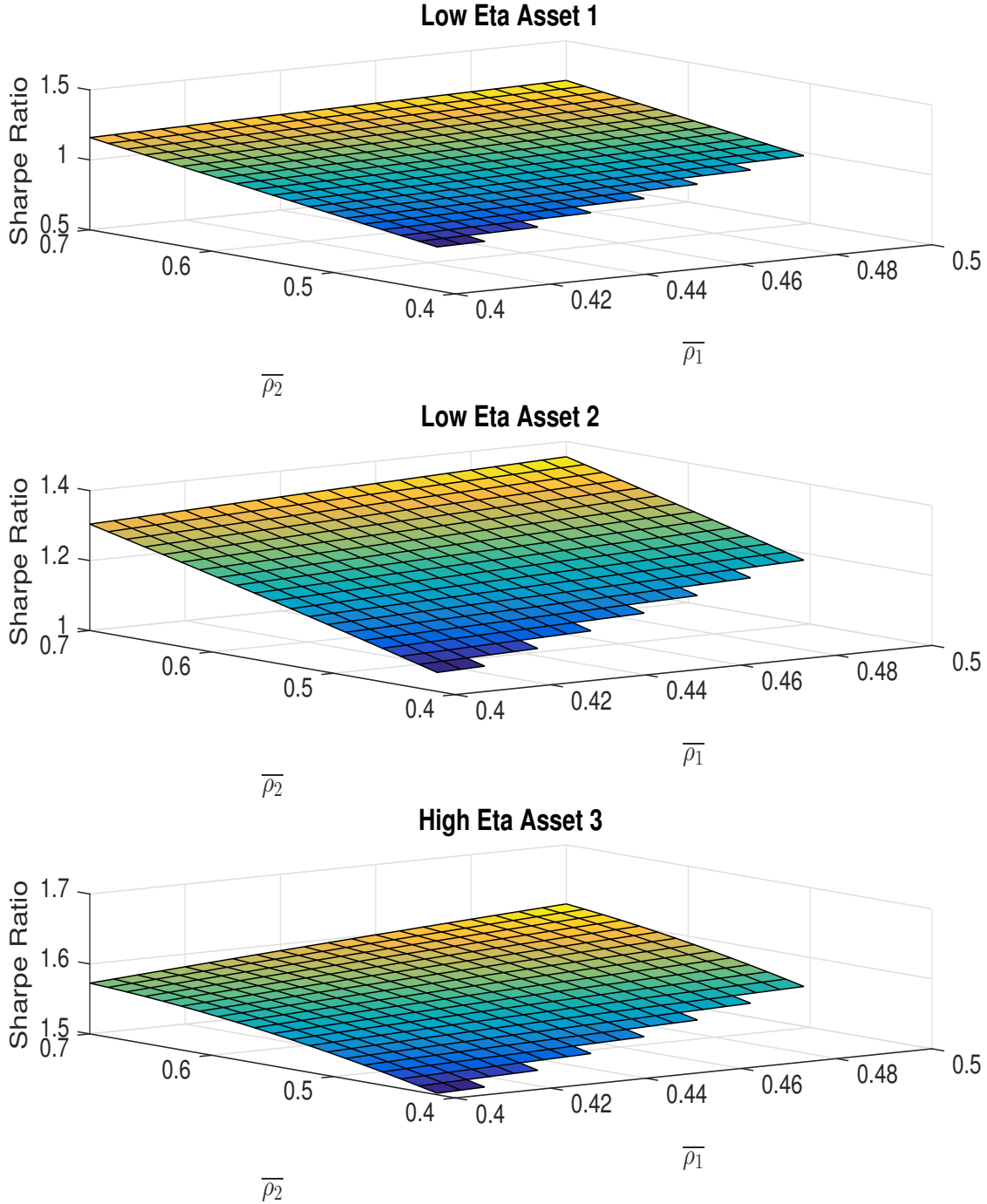


Figure 8: Dispersion of Optimal Portfolios of Agents

This figure demonstrates the dispersion of the optimal portfolio for the sophisticated and naive agent respectively in a full participation equilibrium. Clearly, the dispersion of the sophisticated agent's optimal portfolio is smaller than the dispersion of the naive agent's optimal portfolio for all possible values of $\bar{\rho}_1$ and $\bar{\rho}_2$. The same pattern is reported in Table 3 in a limited participation equilibrium. The parameters in this figure are $N = 3$, $\nu = 0.3$, $\sigma_1 = 9\%$, $\bar{x}_1 = 1$; $\sigma_2 = 10\%$, $\bar{x}_2 = 5$; $\sigma_3 = 12\%$, $\bar{x}_3 = 10.5$. Note that $\Omega(\sigma\bar{x}) = 0.56$ and $\frac{1}{N-1} \left\{ \frac{\nu}{1-\nu} \frac{\Omega(\sigma\bar{x})}{1-\Omega(\sigma\bar{x})} N - 1 \right\} = 0.3$.

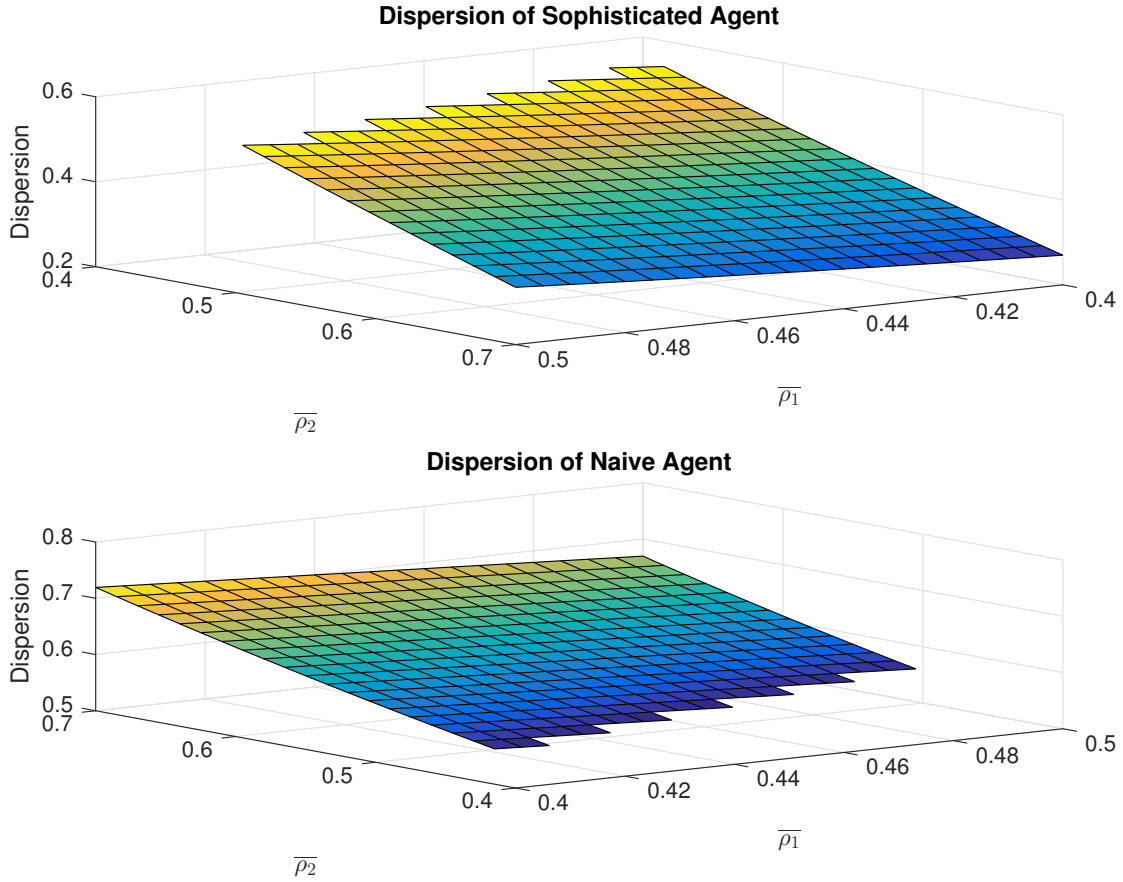


Figure 9: Optimal Portfolio Risks of Agents

This figure demonstrates that the variance (risk) of the optimal portfolio the sophisticated agent is always larger than the variance (risk) of the naive agent's optimal portfolio in a full participation equilibrium. It shows that a higher correlation uncertainty leads to a higher risk averse behavior. The same pattern is reported in Table 5 in equilibrium B. The parameters in this figure are $N = 3$, $\nu = 0.3$, $\sigma_1 = 9\%$, $\bar{x}_1 = 1$; $\sigma_2 = 10\%$, $\bar{x}_2 = 5$; $\sigma_3 = 12\%$, $\bar{x}_3 = 10.5$. In this situation $\Omega(\sigma\bar{x}) = 0.56$ and $\frac{1}{N-1} \left\{ \frac{\nu}{1-\nu} \frac{\Omega(\sigma\bar{x})}{1-\Omega(\sigma\bar{x})} N - 1 \right\} = 0.3$.

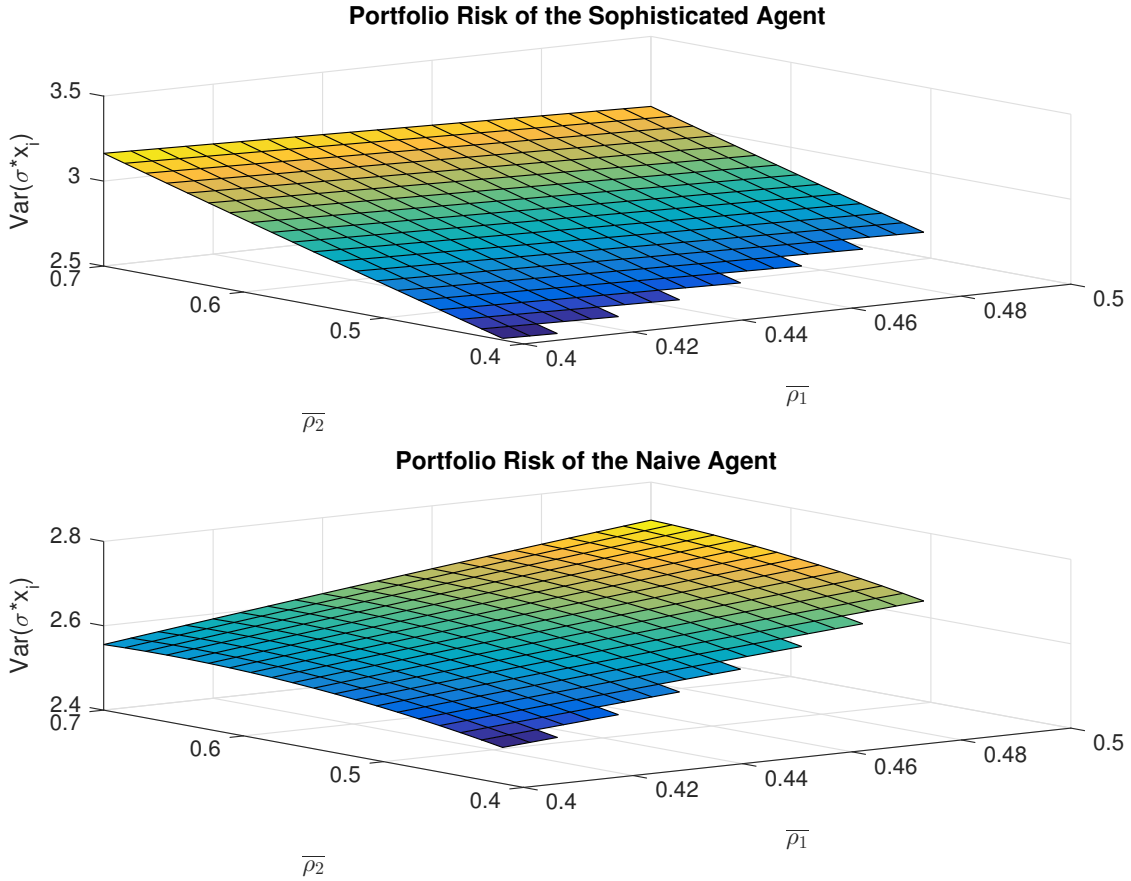


Figure 10: Optimal Holdings of Agents

This figure reports the optimal holdings of the sophisticated and naive agent on low eta asset 1 and high eta asset 3 respectively. The parameters in this figure are $N = 3$, $\nu = 0.3$, $\sigma_1 = 9\%$, $\bar{x}_1 = 1$; $\sigma_2 = 10\%$, $\bar{x}_2 = 5$; $\sigma_3 = 12\%$, $\bar{x}_3 = 10.5$. Notice that $\Omega(\sigma\bar{x}) = 0.56$ and $\frac{1}{N-1} \left\{ \frac{\nu}{1-\nu} \frac{\Omega(\sigma\bar{x})}{1-\Omega(\sigma\bar{x})} N - 1 \right\} = 0.3$. $\eta_1 = 0.048 < \frac{1}{N}$, $\eta_3 = 0.68 > \frac{1}{N}$.

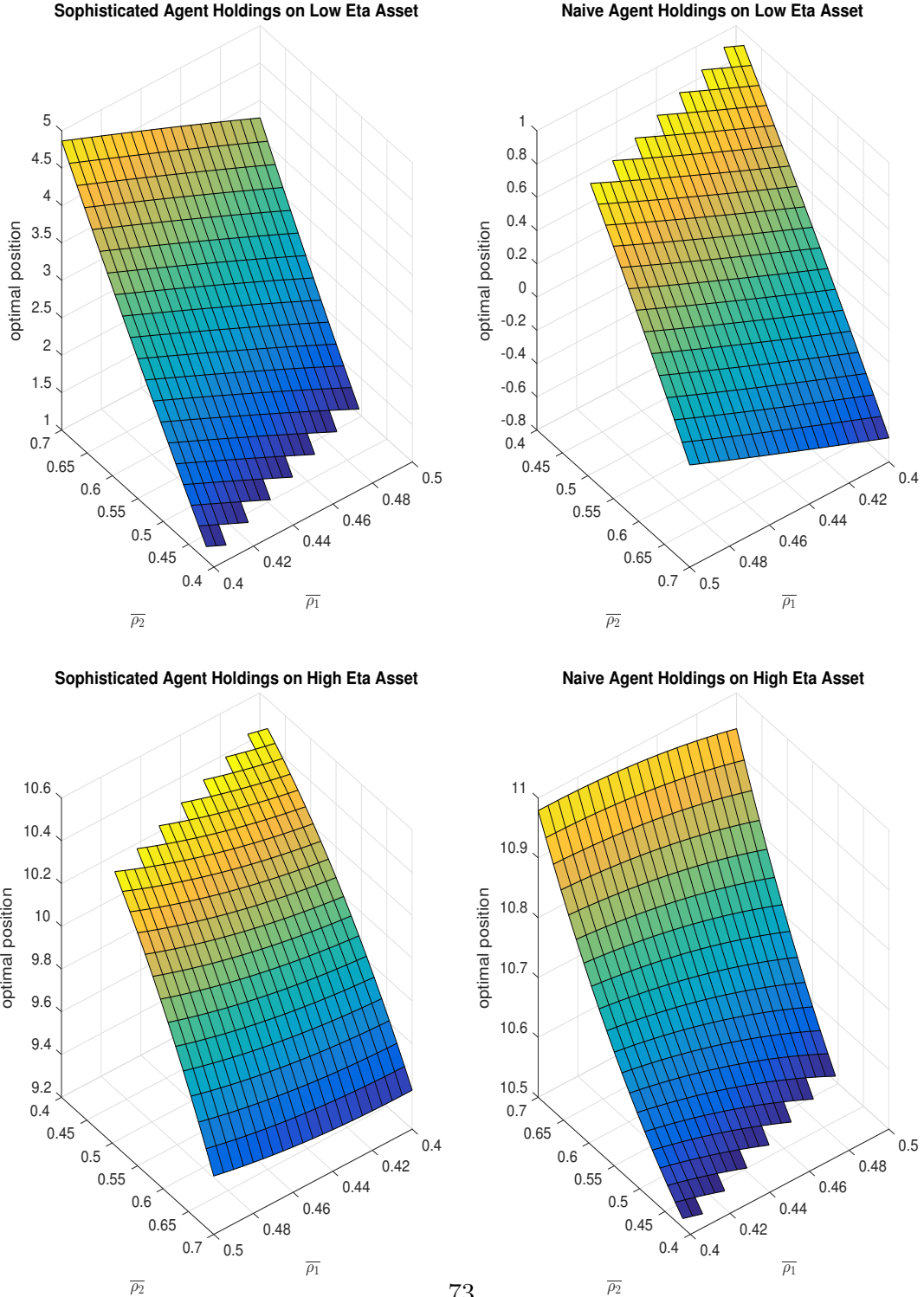


Table 1: Sharpe Ratios and Correlation Uncertainty Premium in a Homogeneous Model

This table reports the Sharpe ratios, the dispersion of all Sharpe ratios and the correlation uncertainty premium in homogeneous environment. The parameters are: $N = 3, \sigma_1 = 9\%, \bar{x}_1 = 1, \sigma_2 = 10\%, \bar{x}_2 = 5, \sigma_3 = 12\%, \bar{x}_3 = 10.5$. Notice that the dispersion of the risks is $\Omega(\sigma\bar{x}) = 0.56$. The reference correlation coefficient is $\rho^{avg} = 40\%$. By the “premium column” we mean the percentage of the correlation uncertainty premium over the Sharpe ratio without the correlation uncertainty, that is, for the reference correlation coefficient.

ϵ	s_1	s_2	s_3	$\Omega(s)$	Premium Asset 1	Premium Asset 2	Premium Asset 3
0	0.79	1.49	2.70	0.54	NA	NA	NA
0.02	0.83	1.55	2.80	0.51	4.43%	4.14%	3.95%
0.04	0.86	1.61	2.91	0.48	8.87%	8.27%	7.90%
0.06	0.90	1.67	3.02	0.46	13.30%	12.41%	11.85%
0.08	0.93	1.74	3.12	0.43	17.73%	16.54%	11.85%
0.1	0.97	1.80	3.23	0.40	22.17%	20.68%	19.76%
0.12	1.01	1.86	3.34	0.38	26.60%	24.81%	23.71 %
0.14	1.04	1.92	3.44	0.38	31.03%	28.95%	27.66%
0.16	1.08	1.98	3.55	0.34	35.47%	33.08%	31.61%
0.18	1.11	2.04	3.66	0.31	39.90%	37.22%	35.56%
0.2	1.15	2.11	3.76	0.29	44.33%	41.36%	39.52%

Table 2: Sharpe ratios in a limited participation equilibrium

This table reports all Sharpe ratios in a limited participation equilibrium including the Sharpe ratios, the dispersion of all Sharpe ratios, the agent's optimal position and the corresponding dispersion of the optimal portfolio. The parameters are: $N = 3$, $\nu = 0.6$, $\sigma_1 = 2\%$, $\bar{x}_1 = 0.5$, $\sigma = 8\%$, $\bar{x}_2 = 3$, $\sigma_3 = 12\%$, $\bar{x}_3 = 10.5$. Notice that the dispersion of the risks is $\Omega(\sigma\bar{x}) = 0.7631$. The reference correlation coefficient is $\rho^{avg} = 40\%$.

$\bar{\rho}_1$	$K(\bar{\rho}_1)$	s_1	s_2	s_3	S	s_1/S	s_2/S	s_3/S	$\Omega(s)$
0.400	0.557	0.70	0.82	1.36	2.89	0.244	0.285	0.471	0.210
0.405	0.561	0.71	0.83	1.36	2.90	0.245	0.286	0.469	0.207
0.410	0.566	0.72	0.84	1.36	2.92	0.246	0.287	0.467	0.204
0.415	0.570	0.73	0.84	1.36	2.93	0.247	0.287	0.465	0.201
0.420	0.575	0.73	0.85	1.37	2.95	0.249	0.288	0.463	0.198
0.425	0.579	0.74	0.86	1.37	2.96	0.250	0.289	0.461	0.195
0.430	0.583	0.75	0.86	1.37	2.98	0.251	0.289	0.460	0.192
0.435	0.588	0.75	0.87	1.37	2.99	0.252	0.290	0.458	0.190
0.440	0.592	0.76	0.87	1.37	3.01	0.253	0.291	0.456	0.187
0.445	0.596	0.77	0.88	1.37	3.02	0.255	0.291	0.454	0.184
0.450	0.601	0.78	0.89	1.37	3.04	0.256	0.292	0.452	0.181
0.455	0.605	0.78	0.89	1.37	3.05	0.257	0.292	0.452	0.179
0.460	0.609	0.79	0.90	1.38	3.06	0.258	0.293	0.449	0.176
0.465	0.613	0.80	0.90	1.38	3.08	0.259	0.294	0.447	0.174
0.470	0.618	0.80	0.91	1.38	3.09	0.260	0.294	0.446	0.171
0.475	0.622	0.81	0.92	1.38	3.11	0.261	0.295	0.444	0.169
0.480	0.626	0.82	0.92	1.38	3.12	0.262	0.295	0.442	0.166
0.485	0.630	0.83	0.93	1.38	3.14	0.263	0.296	0.441	0.164
0.490	0.635	0.83	0.93	1.38	3.15	0.264	0.297	0.439	0.161
0.495	0.639	0.84	0.94	1.38	3.16	0.265	0.297	0.438	0.159
0.500	0.643	0.85	0.95	1.39	3.18	0.266	0.298	0.436	0.156

Table 3: Optimal portfolios in a limited participation equilibrium

This table reports a limited participation equilibrium including the agent's optimal position and the corresponding dispersion of the optimal portfolio. The parameters are: $N = 3, \nu = 0.6, \sigma_1 = 2\%, \bar{x}_1 = 0.5, \sigma_2 = 8\%, \bar{x}_2 = 3, \sigma_3 = 12\%, \bar{x}_3 = 10.5$. Notice that the dispersion of the risks is $\Omega = 0.7631$. The reference correlation coefficient is $\bar{\rho} = 40\%$. For each $i = 1, 2, 3$, $x_i^{(n)}$ and $x_i^{(s)}$ denote the optimal holding of the naive agent and the sophisticated agent on asset i . $\Omega^{(n)}$ represents the dispersion of the naive agent's optimal portfolio while $\Omega^{(s)}$ is the dispersion of the sophisticated agent's optimal portfolio.

$\bar{\rho}_1$	$K(\bar{\rho}_1)$	$(x_1^{(n)}, x_1^{(s)})$	$(x_2^{(n)}, x_2^{(s)})$	$(x_3^{(n)}, x_3^{(s)})$	$\Omega^{(n)}$	$\Omega^{(s)}$
0.400	0.557	(-6.478, 5.152)	(11.277, 9.982)	(1.791, 3.806)	1.000	0.629
0.405	0.561	(-6.484, 5.156)	(11.287, 9.976)	(1.792, 3.805)	1.000	0.629
0.410	0.566	(-6.489, 5.159)	(11.297, 9.969)	(1.794, 3.804)	1.000	0.628
0.415	0.570	(-6.495, 5.163)	(11.306, 9.962)	(1.796, 3.803)	1.000	0.628
0.420	0.575	(-6.500, 5.167)	(11.316, 9.956)	(1.797, 3.802)	1.000	0.628
0.425	0.579	(-6.506, 5.171)	(11.326, 9.949)	(1.799, 3.801)	1.000	0.628
0.430	0.583	(-6.512, 5.175)	(11.336, 9.943)	(1.800, 3.800)	1.000	0.628
0.435	0.588	(-6.517, 5.178)	(11.346, 9.936)	(1.802, 3.799)	1.000	0.627
0.440	0.592	(-6.523, 5.182)	(11.356, 9.930)	(1.803, 3.798)	1.000	0.627
0.445	0.596	(-6.529, 5.186)	(11.365, 9.923)	(1.805, 3.797)	1.000	0.627
0.450	0.601	(-6.534, 5.190)	(11.375, 9.917)	(1.807, 3.796)	1.000	0.627
0.455	0.605	(-6.540, 5.193)	(11.385, 9.910)	(1.808, 3.795)	1.000	0.627
0.460	0.609	(-6.546, 5.197)	(11.395, 9.903)	(1.810, 3.794)	1.000	0.626
0.465	0.613	(-6.551, 5.201)	(11.405, 9.897)	(1.811, 3.793)	1.000	0.626
0.470	0.618	(-6.557, 5.205)	(11.415, 9.890)	(1.813, 3.791)	1.000	0.626
0.475	0.622	(-6.563, 5.208)	(11.424, 9.884)	(1.814, 3.790)	1.000	0.626
0.480	0.626	(-6.568, 5.212)	(11.434, 9.877)	(1.816, 3.789)	1.000	0.626
0.485	0.630	(-6.574, 5.216)	(11.444, 9.871)	(1.817, 3.788)	1.000	0.626
0.490	0.635	(-6.580, 5.220)	(11.454, 9.864)	(1.819, 3.787)	1.000	0.626
0.495	0.639	(-6.585, 5.223)	(11.464, 9.858)	(1.821, 3.786)	1.000	0.625
0.500	0.643	(-6.591, 5.227)	(11.474, 9.851)	(1.822, 3.785)	1.000	0.625