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The Riccati Equation in Mathematical Finance

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This paper uses ideas from symbolic computation to classify solutions to an important class of problems in mathematical finance and thus provides a linkage between these two fields. We show that Kovacic's concept of *closed-form solutions* to the Riccati ordinary differential equation can be used to provide a precise mathematical definition that is useful in certain financial models. We extend this definition to a broader class of problems and discuss how these ideas can be usefully applied to practical problems in the finance area. We provide a specific application by developing a new implementation of the Cox–Ingersoll–Ross interest-rate model that may be of practical interest.

1. Introduction

The concept of a *closed-form solution* to the Riccati equation has been given a precise mathematical definition by Kovacic (1986). Furthermore Kovacic also gave an algorithm that can be used to determine the closed-form solutions to the Riccati ordinary differential equation. The Riccati equation plays an important role in the solution to a class of problems in mathematical finance. In this paper we note that Kovacic's definition provides a basis for classifying solutions to an important set of what are known as *interest-rate models*. It provides a precise meaning to the concept of a closed-form solution. We will also show that the Kovacic algorithm enables us to extend the class of solutions to a particular model that is widely used in the finance area. To provide some background we give a short overview of relevant developments in the finance field.

A key breakthrough in finance research can be traced to the seminal Black and Scholes (1973) paper which gave a methodology for valuing a particular type of financial contract known as a derivative security. In this context, the word derivative means that the value of the security depends on or is derived from some underlying security such as a common stock or a bond. Their key insight was to use an economic principle known as the *principle of no arbitrage* to value the related security. For an exposition of the no arbitrage principle and its role in modern finance, see Duffie (1996). This methodology has been applied to a wide range of problems in the finance area and it now provides the foundation for countless commercial transactions.

Broadly speaking, finance models assume that movements in prices stem from uncertainty in the economy. To model this uncertainty the price of the underlying asset is

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assumed to follow a stochastic differential equation. Often this equation is a continuous time Ito diffusion and the form of the equation depends on the variable or variables involved. By invoking the no arbitrage assumption one can construct a partial differential equation for the price of any derivative security whose value is a function of the basic underlying security. The contract provisions of the derivative security supply the boundary conditions for the differential equation. For example, if we assume that the underlying asset is a share of common stock which follows geometric Brownian motion the resulting equation is the Black–Scholes–Merton equation. If the derivative security is a so-called European option this equation has a simple solution which can be written in terms of the underlying variables and the cumulative normal distribution function. This solution is known as the Black–Scholes formula and it is generally described in the finance literature as being an analytical solution or a closed-form solution. However in the finance literature, the concept of a closed-form solution is often imprecise. The objective of this paper is to provide a more precise definition of this concept in finance applications. We will use ideas from symbolic computation in setting up this definition.

One of the most active areas of current research in the finance area is in the modeling of stochastic interest rates. There are additional complications when modeling interest rates, since at any given instant there is a vector of interest rates related to the maturities of the currently traded bonds. This contrasts with the case of a common stock where the current price is just a simple scalar. In the case of stochastic interest rates we assume the important sources of variation are captured by a small number of state variables. A common approach is to postulate a stochastic differential equation for the dynamics of these state variables. The Cox *et al.* (1985) model provides one of the best known models of this genre. It is quite tractable since analytical expressions exist for many quantities of interest such as bond prices and option prices. Furthermore the CIR process is based on plausible interest-rate dynamics since interest rates always remain positive and wander around a long-term mean.

One of the problems with the early implementations of stochastic Interest-rate models was that the theoretical model prices did not fit the existing observed market prices of bonds. The reason is that at any time there is a vector of current bond prices and a model with a few parameters simply cannot fit the entire set of bond prices. With the introduction of interest-rate derivatives the situation became worse since these derivatives depend critically on the volatility of interest rates. It is useful for trading[†] purposes to have a model which can fit exactly the prices of the currently traded instruments. One solution to this problem was to make the parameters of the model depend on time in a deterministic way. The actual nature of the time dependence could be computed by comparing the model prices with the observed market prices. This process of finding time-dependent parameters that ensures equality between model prices and market prices is known in the finance literature as *calibration*.

In the case of the CIR model the calibration procedure is intimately related to the solution of the Riccati ordinary differential equation. For a discussion of the relationship, see Duffie and Kan (1996). Closed-form solutions of the Riccati equation have been studied in the differential algebra literature. In particular, Kovacic (1986) gave a precise mathematical definition of the concept of a closed-form solution in the case of the

[†]This is because many derivatives can be replicated by portfolios of more basic securities. The process of replicating a derivative by a portfolio of other securities is known as hedging. Hedging and pricing are very closely related. It is often important to use a model which faithfully reproduces the prices of the most commonly traded securities.

Riccati equation and also gave an algorithm that classifies the closed-form solutions. Kovacic's work on the Riccati equation is thus directly relevant to the calibration exercise in the CIR interest-rate model and in this paper we explain the significance of this connection. This connection enables us to develop new implementations of the so-called extended CIR model that may be useful in the finance area.

The remainder of this paper is as follows. In Section 2 we summarize the relevant literature on the closed-form solutions to the Riccati equation. We describe Kovacic's important contributions to this field. Next we summarize some of the relevant aspects of the Cox–Ingersoll–Ross model in Section 3. We describe in more detail the connection between the Riccati equation and the practical implementation of the CIR model. We also show that the basic intuition of an analytical solution that has been proposed in the finance literature by Jamshidian (1996) ties in closely with the definition we are proposing for closed-form solutions. Section 4 gives an explicit example of a new implementation of the extended CIR model that we have developed using the ideas in this paper. Section 5 concludes the paper and mentions a possible extension of this application.

2. Closed-form Solutions of the Riccati Equation

In this section we review some of the relevant literature on the existence of *closed-form* solutions of the Riccati ordinary differential equation. We recall the definition of a closed-form solution of the Riccati equation and present Kovacic's algorithm for finding closed-form solution of the Riccati equation. The Riccati ordinary differential equation is of the form

$$\frac{dy(x)}{dx} + a(x)y(x)^2 + b(x)y(x) + c(x) = 0.$$

In the next section we provide some definitions and recall some known results from differential algebra. We refer the reader to the excellent book by Bronstein (1997) on this topic.

2.1. Definition of closed-form solutions

A *differential field* (k, δ) is a field k endowed with a derivation δ. The field of constants, $Const_\delta(k)$, is the set of $f \in k$ with $\delta(f) = 0$. Consider the following Riccati ordinary differential equation (ODE) in (k, δ):

$$R_{a,b,c} : \delta(f) = af^2 + bf + c, \qquad a, b, c \in k. \tag{1}$$

The statement that the Riccati equation (1) has solution in (k, δ) means that there exists an element $f \in k$ such that $\delta(f) = af^2 + bf + c$. A *differential field extension* of (k, δ) is a differential field (K, Δ) such that $k \subseteq K$ and $\Delta(a) = \delta(a)$ for any $a \in k$.

Definition 1. A differential field extension (K, Δ) of (k, δ) is called *a Liouvillian extension* if there is a tower of fields

$$k = K_0 \subseteq K_1 \subseteq \cdots \subseteq K_m = K$$

where K_{i+1} is a simple field extension $K_i(t_i)$ of K_i, such that one of the following holds:

(i) t_i is algebraic over K_i, or
(ii) $\Delta(t_i) \in K_i$ (extension by an integral $\int u$ where $u = \Delta(t_i)$), or

(iii) $\frac{\Delta(t_i)}{t_i} \in K_i$ (extension by the exponential of an integral $e^{\int u}$ where $u = \frac{\Delta(t_i)}{t_i} \in K_i$).

We are now able to define a closed-form solution of the Riccati equation.

Definition 2. The Riccati equation $R_{a,b,c}$ has a *closed form* if there exists a Liouvillian extension (K, Δ) of (k, δ) and an element $t \in K$ such that

$$\Delta(t) = at^2 + bt + c \tag{2}$$

holds.

It is clear that any first-order linear ordinary differential equation can be solved in a Liouvillian extension. Throughout this paper we assume that $a \neq 0$. It will be convenient to reduce the Riccati equation to its standard form where the leading coefficient is equal to 1. Let $f = \frac{g}{a}$. Then the original Riccati equation in terms of f is equivalent to the following Riccati equation in terms of g:

$$\delta(g) = g^2 + (b + \frac{\delta(a)}{a})g + ac. \tag{3}$$

Given a (standard) Riccati equation $\delta(y) = y^2 + \alpha y + \beta$ and one solution y, the standard transformation property (see Hille, 1969, Appendix C) of the Riccati equation, implies that $y + \frac{1}{u}$, for some u, is another linearly independent solution of the same Riccati equation. Moreover, the function u satisfies the following linear ordinary differential equation

$$\delta(u) + [2y + \alpha]u + 1 = 0. \tag{4}$$

Therefore, any solution of the Riccati equation, of the form

$$c_1 y + \frac{c_2}{u}, \qquad c_1, c_2 \in \mathrm{Const}_\delta k$$

is a Liouvillian solution if there exists one Liouvillian solution y. The solution is determined uniquely using the boundary conditions.

2.2. Kovacic's algorithm

In this section we summarize Kovacic's algorithm for determining solutions of the Riccati equation.

By a well known result in differential algebra (see, for example, Rosenlicht, 1973), if the Riccati equation $R_{a,b,c}$ has a solution in some Liouvillian extension of k,then $R_{a,b,c}$ has an algebraic solution over k. Equivalently, $R_{a,b,c}$ has one solution in an algebraic extension of k.

To yield a complete algorithm for computing the algebraic (and thus all Liouvillian) solutions, it is critical to obtain an upper bound on the degree of the minimal polynomials of the algebraic function solutions. Fortunately, Kovacic's theorem provides the answers.[†]

Theorem. (Kovacic) *If there is an algebraic solution of the Riccati equation, then at least one solution has degree* $1, 2, 4, 6$ *or* 12. *Moreover, there exists an algorithm that*

[†]For the details see Kovacic (1986). An accessible presentation in textbook form is given in Chapter 9 of Cohen *et al.* (1999)

permits us to either (*i*) *show that there is no Liouvillian solution, or* (*ii*) *be able to find one* (*and hence all of them using the results of Section* 2.1).

Kovacic's algorithm has been implemented in the MACSYMA computer algebra system and in the MAPLE computer algebra system. In the MAPLE V Release 5.1 package *DETools*, the library function *kovacicsols* can be used to compute Liouvillian solutions of a Riccati equation if one exists. Then, given an arbitrary Riccati equation, we know, by symbolic computation, whether this Riccati equation has closed-form solutions or not; and if one exists, all solutions are closed form and can be given explicitly.

3. The Cox–Ingersoll–Ross Interest-rate Model

In this section we explain the connection between the Riccati equation and certain applications in finance. We describe the significance of having closed-form solutions to the Riccati equation in the practical implementation of certain stochastic interest-rate models.

3.1. links with Riccati equations

We first introduce a particular class of stochastic interest-rate models known as affine term structure models. Affine models have certain attractive properties in terms of their tractability. We need a couple of concepts from finance to proceed. The first concept is that of a *zero-coupon bond*. A zero-coupon bond is a contract which will pay one unit, with certainty, at a prespecified future date. If the current time is t and the future payment will occur at time $T > t$, then we denote the current price of the zero-coupon bond by $P(t, T)$. The market prices for these bonds can be obtained from traded market instruments. The *short rate* of interest can be defined in terms of zero-coupon bond prices. Let us first define the function $f(t, T)$ by

$$f(t, T) = -\frac{\partial \log P(t, T)}{\partial T}.$$

This function $f(t, T)$ is known as the forward rate. The short rate $r(t)$ is defined as the limiting value of $f(t, T)$ as T tends to t

$$r(t) = \lim_{T->t} f(t, T).$$

The short rate can be thought of as the limiting interest rate on a bond where we let the time to maturity tend to zero. For a more complete discussion of this concept and related matters, see Duffie (1996). The starting point for several popular stochastic interest-rate models is to assume that the short rate dynamics are governed by a stochastic differential equation involving one or more state variables and a number of structural parameters. In this paper, we assume that there is only one such state variable. Such models are known as *one factor models*. Under a special class of such models, known as the one factor affine class, the resulting model prices for the zero-coupon bonds are exponential-linear functions of the short rate of interest. We supply specific examples later.

Duffie and Kan (1996) derived necessary and sufficient conditions on the drift and diffusion of the stochastic differential equation for the short rate to ensure an affine term structure model. Assume the short-term interest rate is described by the following stochastic differential equation

$$dr = \mu(t,r)dt + \sigma(t,r)dW_t,$$

where W_t is a standard Brownian motion under the risk-neutral[†] equivalent measure. The definition of an affine term structure model is that $P(t,T)$ has the form

$$P(t,T) = \exp[A(t,T) - B(t,T)r(t)]$$

and thus the yield $y(t,T) = -\frac{\log(P(t,T))}{T-t}$ is a linear function of $r(t)$. Duffie–Kan's result is that $P(t,T)$ is exponential-affine if and only if the diffusion μ and the volatility σ have the form

$$\mu(t,r) = \alpha(t)r + \beta(t), \qquad \sigma(t,r) = \sqrt{\gamma(t)r + \delta(t)}.$$

Moreover, the coefficients $A(t,T), B(t,T)$ are determined by the following ordinary differential equations

$$B_t(t,T) = \frac{\gamma(t)}{2}B(t,T)^2 - \alpha(t)B(t,T) - 1, \qquad B(T,T) = 0 \tag{5}$$

and

$$A_t(t,T) = \beta(t)B(t,T) - \frac{\delta(t)}{2}B(t,T)^2, \qquad A(T,T) = 0. \tag{6}$$

The first equation for $B(t,T)$ is the Riccati equation and the second one is solved easily from the first one by integration. Since there is a general method (see, for example, Heston, 1993; Duffie *et al.*, 1999) making use of Laplace transforms to evaluate bond options and other interest-rate sensitive securities starting from the same Riccati equations, it suffices to study the Riccati equations.[‡]

To give some appreciation of their significance, we review some affine term structure models that have already been popularized in the finance literature. When $\gamma(t) = 0$, i.e. the volatility structure is deterministic, this yields the so-called Gaussian interest-rate model (see Jamshidian, 1992). Under this assumption, analytical formulae for bond and European bond pricing formulae are available. This is easy to see from the point of view of differential equations since (5) reduces to a linear ordinary differential equation, which can be readily solved using simple methods. If all the parameters are positive constants, this is the well known Vasicek interest-rate model (1977) which was one of the first stochastic interest-rate models.

Another well known affine term structure model has the property that $\gamma(t) \neq 0$ and $\delta(t) = 0$. When all the parameters $\gamma(t), \beta(t), \alpha(t)$ are positive constants, we have the square-root model of Cox *et al.* (1985). Hull and White (1990) discuss the model in the form

$$dr = (b(t) - ar)dt + \sigma\sqrt{r}dW.$$

[†]For an explanation of the risk neutral measure, see Duffie (1996).

[‡]To value all interest-rate sensitive securities, it suffices to compute what is termed the state-price density. By using Laplace transformations, we reduce this to the computation of the function

$$P(t,T;x) = E_t\left[\exp\left(-\int_t^T r_s ds\right)\exp(xr_T)\right]$$

where x is a parameter. In the extended CIR model, we have $P(t,T;x) = \exp[A(t,T;x) - B(t,T;x)r(t)]$, where $B(t,T;x)$ satisfies the same Riccati equation as $B(t,T)$ with different boundary condition $B(T,T;x) = -x$, and $A(t,T;x) = -\int_t^T \theta(s)B(s,T;x)ds$. See Duffie *et al.* (1999) for details.

They assume that $b(t)$ is time-dependent and then determine $b(t)$ (calibrate the model) using the current market prices of the zero-coupon bonds. In both the (constant parameter) version of the CIR model and in the Hull–White extension, analytical formulae for zero-coupon bonds and bond options exist. This is also evident from the ordinary differential equation's perspective, since the Riccati equation (5) has *constant coefficients*. A major goal of this paper is to analyze the extended versions of CIR where $\gamma(t), \beta(t), \alpha(t)$ are time-dependent functions. Hull and White (1990) claimed that no analytical expression for the bond prices is available when the model has time-dependent parameters. In fact, as we shall show later, the closed-form theory of the Riccati equation enables us to generate many new examples of the extended[§] CIR model with analytical expressions for the bond prices and interest-rate derivatives.

We assume the short interest rate $r(t)$ follows the process

$$dr(t) = [\theta(t) - \kappa(t)r(t)]dt + \sigma(t)\sqrt{r(t)}dW(t) \tag{7}$$

where $\kappa(t)$ denotes the speed of adjustment of the short rate process, $\frac{\theta(t)}{\kappa(t)}$ denotes its mean, and where $\sigma(t)\sqrt{r(t)}$ denotes its volatility. It is well known that, if $2\theta(t) > \sigma(t)^2$ for all t, then $r = 0$ is an inaccessible boundary, and the interest-rate model (7) is well-defined, and these dynamics drive the price movements of all interest-rate securities. In this case the corresponding Riccati equation is:

$$R_{\kappa,\theta,\sigma} : y(t)' = \frac{1}{2}\sigma(t)^2 y(t)^2 + \kappa(t)y(t) - 1 \tag{8}$$

and in standard form it becomes:

$$d(f) = f^2 + \left[\kappa(t) + \frac{2\sigma(t)'}{\sigma(t)}\right] f - \frac{\sigma(t)^2}{2}. \tag{9}$$

The corresponding second-order linear differential equation is

$$y(t)'' - \left[\kappa(t) + \frac{2\sigma(t)'}{\sigma(t)}\right] y(t)' - \frac{\sigma(t)^2}{2} y(t) = 0.$$

Definition 3. The extended CIR model (7) has a *closed-form* expression for the bond price if the Riccati equation (8) has a closed-form solution.

We shall show, in the next section, that this definition corresponds to the definition of an *analytical-form* expression in the finance literature. For the extended CIR model, if the *dimension* of the model: $\frac{2\theta(t)}{\sigma(t)^2}$ is a constant, this is the *simple square-root model* in the sense of Jamshidian (1995).[†] Although he did not solve the Riccati equation explicitly, Jamshidian showed that, for the purpose of calibration, it suffices to solve one Riccati equation and deduce the analytical expression of the bond options. Moreover, if the dimension is a positive integer, Maghsoodi (1996) proved that process (7) can be recovered from the Gaussian models in a certain sense and then the bond option valuation formula is available. Strictly speaking, the solutions derived by Jamshidian

[§]The extended CIR model corresponds to the case where the three structural parameters depend on time.

[†]Rogers (1996) showed that the simple square-root model can be derived from the standard CIR model by a time-scale transformation.

and Maghsoodi are not closed-form solutions because the Riccati equation has to be solved numerically.[‡] We note that the dimension does not appear directly in the Riccati equation (8). It is thus more natural and more general to begin with the closed-form solution of the Riccati equation (8).

3.2. ANALYTICAL SOLUTIONS IN FINANCE

It is often desirable in finance practice to have very efficient methods for the valuation of derivative securities. This property is useful in a trading environment where a trader often needs to give a quote in real time over the telephone. It can also be important in risk management since institutions now carry out complex calculations involving their entire portfolios which may include millions of derivative securities. In the finance literature the term *analytical solution* is often used to indicate that a particular expression can be computed quickly and efficiently. Although this concept is not precisely defined there is usually a general consensus on its meaning. In this section we relate the conventional finance interpretation to our previous discussion of closed-form solutions.

We first review what is meant by *analytical expression* in the finance literature. We first give the definition of a call option. A call option is a security that gives its owner the right to buy some underlying asset in the future for a fixed price. The fixed price is called the *strike price*. If the asset can only be bought at the maturity of the option contract, the option is known as a European option.

The best known analytical expression is the Black–Scholes formula for the price of a European call option (Black and Scholes, 1973)

$$c(S,t) = SN(d_1) - Ke^{r(T-t)}N(d_2) \tag{10}$$

where S is the asset price at time t, and the asset price S follows the diffusion process

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t.$$

The symbol K denotes the strike price of the option contract, r is the (assumed constant) interest rate over the period $[t,T]$ and

$$d_1 = \frac{\ln(S/K) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$

and

$$d_2 = \frac{\ln(S/K) + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}.$$

The $N(\cdot)$ denotes the standard cumulative normal distribution function.

This valuation formula is a benchmark for many other popular valuation formulae in finance. However, as far as we know there is, as yet, no mathematically precise definition of an analytical solution for a valuation formula in the finance literature. We see that the above Black–Scholes solution $c(S,t)$ belongs to a Liouvillian extension of the rational field $\mathcal{C}(S,t)$ over two variables S and t. We suggest that it is natural to give a definition of analytical solutions in finance in terms of Liouvillian extensions.

[‡]By using the method in this paper, we have implemented a simple class square model in which both the Riccati equation and the price-density have closed-form solutions, thus completing the missing part in Jamshidian's research.

Consider a derivative security whose value depends on n underlying variables $X_1, X_2, \ldots, X_n$. Suppose these variables follow a Markovian stochastic process. Then the price of the derivative security at time t is a function of the underlying(state) variables $X_1, X_2, \ldots, X_n$, and time t. Denote its price by

$$p(X_1, X_2, \ldots, X_n, t).$$

Our proposed definition is:†

DEFINITION 4. If the function $p(X_1, X_2, \ldots, X_n, t)$ is contained in a Liouvillian extension of the rational function field of the variables $X_1, X_2, \ldots, X_n$, and t, we say that the derivative security has an *analytical* or *closed-form* valuation formula.

In this paper we discussed the extended CIR model. Because of the exponential-linear expression (the exponential belongs to a Liouvillian extension) of the bond price, and the integral expression of the function of A in terms of B (the integral also belongs to a Liouvillian extension), to find the analytical expressions for the bond prices, it suffices to study the closed-form solution of the Riccati equation $R_{\kappa,\theta,\sigma}$.

In a series of investigations of the nature of solutions under one factor interest-rate models, Jamshidian defines *an analytical solution as a single integral of a function where the integrand does not contain the integral of another function.* The Black–Scholes formula (10) is then an analytical solution according to this definition. Since Jamshidian's definition is the clearest expression in the finance literature of an analytical solution and since his definition has been used by other researchers, it is interesting to note that our definition ties in nicely with Jamshidian's definition.

At first sight, our definition of the analytical solution looks different since we do not restrict the number of integral symbols in the Liouvillian extension, whereas there is at most one integral in Jamshidian's definition. For example, functions such as

$$\int \exp(x^2)\left(\int_{-\infty}^{x} \exp(u^2)du\right)dx$$

are not allowed in Jamshidian's framework but they are permitted in a Liouvillian extension. Hence, a natural abstraction of Jamshidian's concept is to define a *special* Liouvillian extension in which only one primitive extension, which corresponds to a single integral, should be used. But as we know from Section 2.2, in solving the Riccati equation and thus for the extended CIR model, these two concepts are equivalent. In fact, if there exists a Liouvillian extension of the Riccati equation, then there is an algebraic function solution f of the original Riccati equation. Thus, any other solution must be of the form

$$c_1 f + \frac{c_2}{u}$$

where u is a solution of one linear ordinary differential equation. It is known that u can be expressed by a single integral. This means that if we can solve a Riccati equation in a Liouvillian extension, we can also solve it in a special Liouvillian extension in the above sense as well. We have thus demonstrated that our definition is a suitable abstraction of Jamshidian's concept, at least for the extended CIR model.

†We plan to investigate, in future work, the extent to which all existing analytical solutions in finance conform to this definition.

4. An Application

In this section we shall show how our analysis of closed-form solutions can be applied to develop a new implementation of the extended CIR model. We have discovered that this implementation appears to provide a promising approach to the calibration of actual interest-rate data.‡

We now give a functional form of the extended CIR model that we have found very useful in our empirical work. We emphasize that this is just one of the many solutions that we can generate using the ideas of this paper. In the extended CIR model the short rate satisfies:

$$dr(t) = (\theta(t) - \kappa(t)r(t))dt + \sigma(t)\sqrt{r(t)}dW(t). \tag{11}$$

In our implementation† the functional forms for the time-dependent parameters are:

$$\frac{\sigma(t)^2}{2} = \theta_1^2 - \theta_1\theta_3 + \frac{2\theta_1\theta_2 - \theta_1\theta_4 - \theta_2\theta_3}{t+\theta_5} + \frac{\theta_2^2 + \theta_2 - \theta_2\theta_4}{(t+\theta_5)^2} \tag{12}$$

and the speed of adjustment, $\kappa(t)$ is given by

$$\begin{aligned} &-\theta_3 - \frac{\theta_4}{t+\theta_5} \\ &+ \frac{2(\theta_2^2 + \theta_2 - \theta_2\theta_4) + (2\theta_1\theta_2 - \theta_1\theta_4 - \theta_2\theta_3)(t+\theta_5)}{(\theta_2^2 + \theta_2 - \theta_2\theta_4)(t+\theta_5) + (2\theta_1\theta_2 - \theta_1\theta_4 - \theta_2\theta_3)(t+\theta_5)^2 + (\theta_1^2 - \theta_1\theta_3)(t+\theta_5)^3}. \end{aligned}$$

There are five constant parameters $\theta_1, \ldots, \theta_5$ in this model in addition to the (as yet) unspecified drift term $\theta(t)$. By selecting different values for these parameters, a rich variety of volatility structures can be generated.‡ By letting the parameters θ_2 and θ_4 be zero, the interest-rate process reverts to the classical CIR interest-rate process. As the time parameter, t, tends to infinity, the time-dependence of the parameters vanishes, that is, the adjustment speed $\kappa(t)$ and the volatility $\sigma(t)$ become constant.

One advantage of this version over some other versions of the extended CIR model is that the drift term can be chosen *independent* of other parameters. Moreover, the Riccati equation has a closed-form solution in our model. In particular, by choosing the drift term according to Jamshidian (1995), we thus obtain a simple-class square model in which the bond price has a closed-form expression and the bond option has an analytical expression in terms of a chi-square density function. The parameters can be estimated by fitting the market prices of bonds and various interest-rate derivatives to their corresponding model prices. This is the so-called *calibration procedure* in financial practice.

‡We shall demonstrate the implementation and calibration in detail in another paper.

†Our implementation comes from a modified Riccati equation based on the closed form in Bronstein (1997, pp. 98, 99). The Riccati equation given there is

$$df = f^2 - \frac{3}{2x}f - \frac{1}{2x}$$

with solution

$$f = \frac{1}{2x} \mp \sqrt{\frac{1}{2x}}.$$

By comparing this equation with equation (9) we see that the choice of $\sigma(t)$ and $\kappa(t)$ are not suitable for our application when $t \to \infty$. However, by modifying the solution we obtain the parameters in the current form and this works well.

‡Although the volatility structure $\sigma(t)$ and the speed of adjustments $\kappa(t)$ may appear complicated at first sight, they can be used to match the movements of the real data better than the constant assumption such as in Cox *et al.* (1985) and Hull and White (1990).

The Riccati equation corresponding to our implementation is

$$d(f) = f(t)^2 - \left[\theta_3 + \frac{\theta_4}{t+\theta_5}\right] f(t) - \frac{\sigma(t)^2}{2} \tag{13}$$

and the corresponding second-order linear equation is

$$y(t)'' + \left[\theta_3 + \frac{\theta_4}{t+\theta_5}\right] y(t)' - \frac{\sigma(t)^2}{2} y(t) = 0. \tag{14}$$

By Kovacic's algorithm one Liouvillian solution of this second-order linear differential equation is

$$f_1(t) = \theta_1 + \frac{\theta_2}{t+\theta_5} \tag{15}$$

and another independent solution is

$$f_2(t) = f_1(t) + \frac{1}{u(t)} \tag{16}$$

where

$$u(t)' + m(t)u(t) + 1 = 0 \tag{17}$$

and

$$m(t) = 2\theta_1 - \theta_3 + \frac{2\theta_2 - \theta_4}{t+\theta_5}. \tag{18}$$

We now determine the time t-price $P(t,T)$ of the zero-coupon bond maturing at T. Write

$$P(t,T) = \exp[A(t,T) - B(t,T)r(t)]$$

where $f(t) = \frac{B(t,T)\sigma(t)^2}{2}$ is one solution of the Riccati equation (13) with the boundary condition $f(T) = 0$. The usual way in (mathematical) finance is to choose $u(t)$ such that $f_1(T) + \frac{1}{u(T)} = 0$, as the other boundary condition, and thus uniquely determine the function $u(t)$. We thus obtain

$$B(t,T) = \frac{2}{\sigma(t)^2}\left[\theta_1 + \frac{\theta_2}{t+\theta_5} + 1\Big/\left\{\frac{-1}{f_1(T)}\exp\left[\int_t^T m(s)ds\right] + \int_t^T \exp\left[\int_t^u m(s)ds\right]du\right\}\right]$$

and

$$A(t,T) = -\int_t^T \theta(s)B(s,T)ds. \tag{19}$$

The price of any interest-rate sensitive derivative can be given explicitly. As an application, we might assume that all parameters $(\theta(t), \theta_1, \ldots, \theta_5)$ are constants and are calibrated to match market yield curves. These yield curves can assume a wide range of shapes and it is well known that the standard CIR model does a poor job in reproducing them. Our preliminary analysis indicates that even when the parameters are constant, these yield curves can be well matched with our implementation.†

†Moreover, our implementation results also show that it is possible to calibrate the volatility curve as well. See Tian *et al.* (2000).

5. Conclusions and Further Research

In this paper we have shown that concepts from symbolic computation are useful in developing new solutions for an important class of stochastic interest-rate models. These interest-rate models are known as affine models and we discussed the one-factor extended CIR versions. If the corresponding Riccati equation has a closed-form (Liouvillian) solution, we show that there exists an analytical expression for any interest-rate derivative in the extended CIR interest-rate model. To the best of our knowledge, these analytical expressions for derivatives under the extended CIR represent a new implementation of the extended interest-rate model. Moreover, we plan to show, using actual data, how our model can be used to match the market data much better than the classic CIR model.

The interest-rate process developed in this paper is *time-inhomogeneous.* Time-inhomogeneity implies that, looking forward, security prices might be different for similar contracts starting from different times in the future. Many important market variables such as forward volatilities are indeed time-inhomogeneous. Consequently, such interest-rate processes have advantages for practitioners who wish to have a very good fit to market prices.

A natural extension of these ideas would be to apply them to multifactor affine models. We briefly outline the structure of these models. For more details, see Duffie and Kan (1996). Suppose that there exist n state variables $X_1, \ldots, X_n$ such that

$$dX_t = (K_0 + K_1 X_t)dt + \sigma(X_t)dW_t \tag{20}$$

where τ will denote the matrix transpose, $X_t = (X_1(t), \ldots, X_n(t))^\tau, K_0 \in R^n, K_1 \in R^{n\times n}$, $(\sigma(x)\sigma(x)^\tau)_{ij} = H_{ij}(x), H = (H_{ij}) \in R^{n\times n}, \sigma : R^n \to R^{n\times n}$. In addition $W_t = (W_1(t), \ldots, W_n(t))^\tau$ is an n-dimensional Brownian motion.

The short rate is given by $r(t) = X_1(t) + \cdots + X_n(t)$. With this set-up the time t price of the zero-coupon bond maturing at time T is

$$P(t,T) = \exp[A(t,T) - B(t,T)X_t]$$

where B satisfies the matrix Riccati equation:

$$B_t(t,T) = \frac{1}{2}B(t,T)^\tau HB(t,T) - K_1^\tau B(t,T) - a \tag{21}$$

$$A_t(t,T) = K_0 B(t,T) \tag{22}$$

where $a = (1, \ldots, 1)^\tau$.

We see that the matrix Riccati equation now plays the same role as the scalar Riccati equation did in the case of the one factor model. An interesting extension of this work would be to classify the closed-form solutions of the matrix Riccati equation. Such an analysis may assist us in deriving new solutions in the multifactor case along the lines used in this paper for the one factor model. We leave this task for future research.

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